Chapter 5

Finite Element Method for Plate Bending Problems

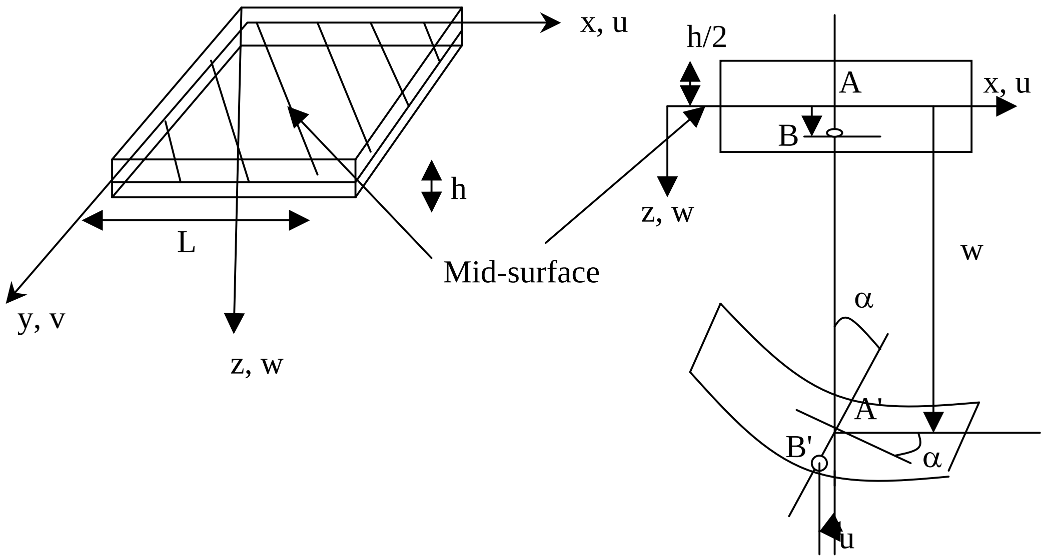
1- Plate Bending (Review of theory)

Linear elastic small deflections

Assumptions:

1. Plate is thin (h<<L where L=typical length)
2. Normals perpendicular to the mid-surface remain normal to the deflected mid-surface.
3. Small deflections (normal to the plate) so that the mid-surface remains unstretched

Also note that $\tau_{zz} << \tau_{xx}$ and τ_{yy}



Deflections:

Displacements of B to B'

u=-z sin α≅-zα=-z$\frac{\partial w}{\partial x}$ similarly, v=-z$\frac{\partial w}{\partial y}$

Strains:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

1.1- Stresses (Isotropic Case)

Plane stress in xy plane

$$\tau_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}) = -\frac{Ez}{1-\nu^2}(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2})$$

$$\tau_{yy} = \frac{E}{1-\nu^2}(\nu\varepsilon_{xx} + \varepsilon_{yy}) = -\frac{Ez}{1-\nu^2}(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2})$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x \partial y}$$

Section properties (resultants stresses as bending and twisting moments)

$$M_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\tau_{xx}dz = -\frac{E}{1-\nu^2}(w_{xx} + \nu\, w_{yy)}\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 dz = -\frac{Eh^3}{12(1-\nu^2)}(w_{xx} + \nu\, w_{yy)} = -D(w_{xx} + \nu\, w_{yy)}$$

similarly,

$$M_{yy} = -\frac{Eh^3}{12(1-\nu^2)}(\nu\, w_{xx} + w_{yy)} = -D(\nu\, w_{xx} + w_{yy)}$$

$$M_{xy} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} z\tau_{xy}dz = \frac{Eh^3}{12(1+\nu)} w_{xy} = D(1-\nu)w_{xy} = M_{yx}$$

Where D is the bending rigidity and M_x, M_y are bending moments per unit length about x and y axes, respectively. M_{xy} and M_{yx} are the twisting moments per unit length about x and y axes, respectively.

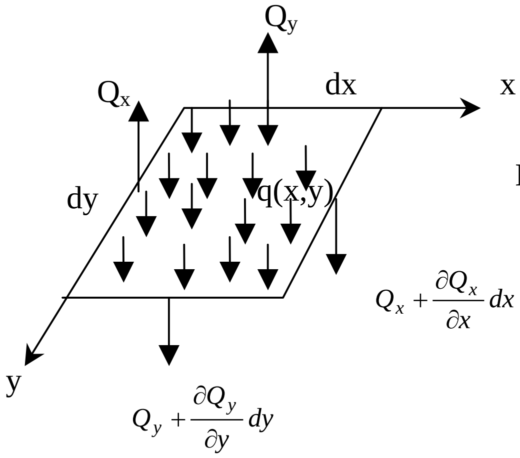
1.2- Shear Forces

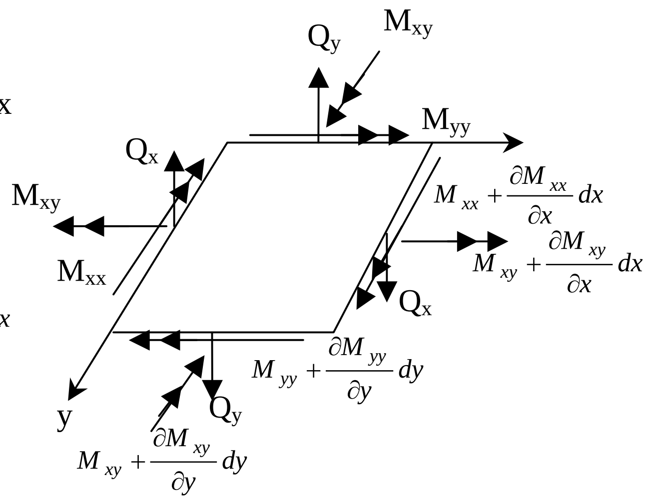
$$Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz}dz \qquad Q_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz}dz$$

Where Q_x and Q_y are the shear forces per unit length on edges whose normals are x and y axes, respectively.

1.3- Equilibrium Equations

q(x,y) is the transverse load per unit area:





$$\sum F_z = -Q_x dy - Q_y dx + (Q_x + \frac{\partial Q_x}{\partial x} dx)dy + (Q_y + \frac{\partial Q_y}{\partial y} dy)dz + q(x,y)dxdy = 0$$

cancelling terms gives :

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x,y) = 0$$

Moment Equilibrium ignoring the second order effects :

$$\sum M_x = M_{yy} dx - (M_{yy} + \frac{\partial M_{xx}}{\partial x} dx)dy + Q_x dxdy - M_{xy} dy + (M_{xy} + \frac{\partial M_{xy}}{\partial x} dx)dy = 0$$

or :

$$\frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x} + Q_y = 0$$

$$\sum M_y = -M_{xx} dy + (M_{xx} + \frac{\partial M_{xx}}{\partial x} dx)dy - Q_x dxdy + M_{xy} dx - (M_{xy} + \frac{\partial M_{xy}}{\partial x} dy)dx = 0$$

or :

$$\frac{\partial M_{xx}}{\partial x} - \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

Eliminating Q_x and Q_y and substituting inequations :

$$\frac{\partial Q_y}{\partial y} = \frac{\partial^2 M_{yy}}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial y \partial x}$$

$$\frac{\partial Q_x}{\partial x} = \frac{\partial^2 M_{xx}}{\partial x^2} - \frac{\partial^2 M_{xy}}{\partial y \partial x} \quad then:$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x,y) = \frac{\partial^2 M_{xx}}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial y \partial x} + \frac{\partial^2 M_{yy}}{\partial y^2} + q(x,y) = 0$$

Substitute for M_{xx}, M_{yy} and M_{xy} :

$$-D(w_{xxxx} + \nu w_{xxyy}) - 2D(1-\nu)w_{xxyy} - D(\nu w_{xxyy} + w_{yyyy}) + q(x,y) = 0$$

$$D(w_{xxxx} + 2w_{xxyy} + w_{yyyy}) = q(x,y)$$

$$D\nabla^4 w = q(x,y) \text{ where } \quad \nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \text{ is called biharmonic operator}$$

Also encountered in many other engineering problems, e.g. very viscous or creeping incompressible flow, stress analysis using Airy's stress function ϕ or $\nabla^4\phi=0$ (compatibility equation)

1.4- Strain Energy

This is analogous to beam bending.
Energy stored=Elastic work done by moments

$$dU = -\frac{1}{2}(M_{xx}dy)w_{xx}dx - \frac{1}{2}(M_{yy}dx)w_{yy}dy + \frac{1}{2}(M_{xy}dy)w_{xy}dx + \frac{1}{2}(M_{yx}dx)w_{yx}dy$$

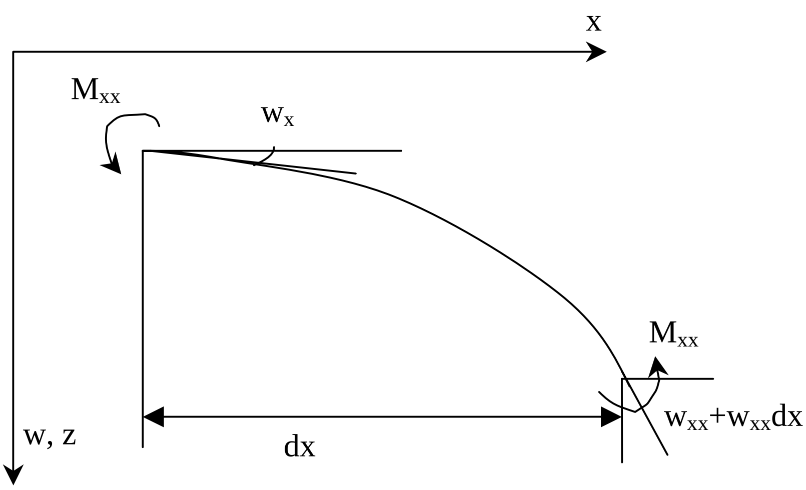
$$note: M_{xy} = M_{yx} \qquad w_{xy} = w_{yx}$$

$$U = \frac{1}{2}\iint_A \left(-M_{xx}w_{xx} - M_{yy}w_{yy} + 2M_{xy}w_{xy}\right)dxdy$$

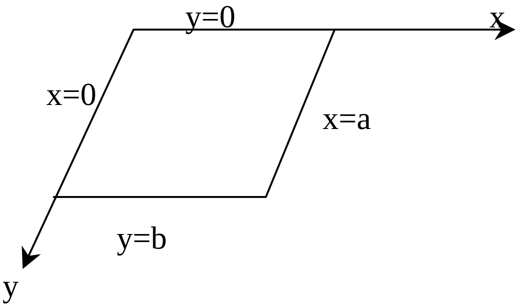
Substituting for M_{xx} M_{yy} M_{xy} *from above equations*:

$$U = \frac{1}{2}\iint_A \left[D(w_{xx} + \nu w_{yy})w_{xx} + D(\nu w_{xx} + w_{yy}) + 2D(1-\nu)w_{xy}^2\right]dxdy$$

$$U = \frac{D}{2}\iint_A \left[w_{xx}^2 + w_{yy}^2 + 2\nu w_{xx}w_{yy} + 2(1-\nu)w_{xy}^2\right]dxdy$$



1.5- Boundary Conditions



a) Simply Supported Edge

1) along x=0 and x=a edges

$w=0$ and $M_{xx}=0$

But if $w=0$ along y on x=0 then $w_y=w_{yy}=0$

Therefore $w_y=w_{yy}=0$

$M_{xx}=-D(w_{xx}+\nu w_{yy})=0 \therefore w_{xx}=0$

2)along y=0 and y=b

$w=0$ and $M_{yy}=0$ or $w_{yy}=0$

for $w=0$ along y=0 and y=b, $w_x=0$ and $w_{xx}=0$

b) Clamped or built-in edge

1) along x=0 and x=a

$w=0$ and $\frac{\partial w}{\partial x}=w_x=0$

2) along y=0 and y=b

$w=0$ and $\frac{\partial w}{\partial y}=w_y=0$

c) Free edge

There are no restrictions on displacements- no edge forces.
One is tempted to say that along x=a

$Q_x=0$, $M_{xx}=0$ and $M_{yy}=0$

This is wrong because only two independent conditions are allowed. According to kirchhoff, should use only two, i.e. $M_{xx}=0$ and $T_{xx}=0$ (effective shear force) where, $T_x=Q_x-\frac{\partial M_{xy}}{\partial y}=0$ is effective shear force along the edge.

1.6- Potential Energy

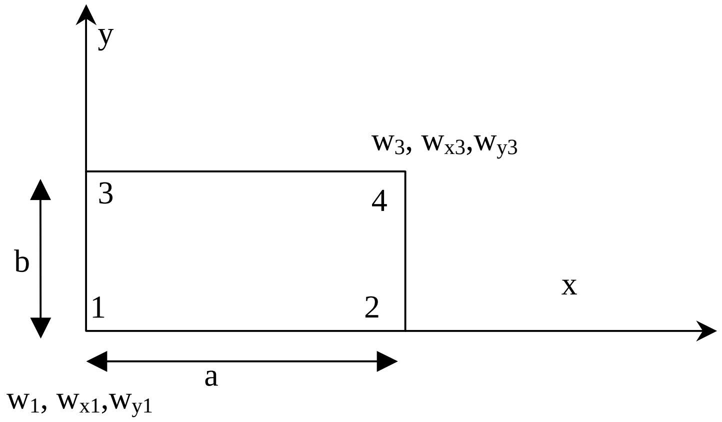
Potential energy of the plate bent to transverse load q(x,y) is given by:

$$\pi = U - W$$

$$\pi = \frac{D}{2}\iint_A \left[w_{xx}{}^2 + w_{yy}{}^2 + 2\nu w_{xx} w_{yy} + 2(1-\nu) w_{xy}{}^2 \right] dxdy - \iint_A qwdxdy$$

2- Rectangular Plate Bending Elements

2.1- Non-conforming Rectangular finite element use deflection and two slopes as generalized displacements at each node i.e. use w, w_x, w_y as nodal degrees of freedom. This element has wide use application and performs very well.



With three dof per nodes, we have 12 dof per element, therefore, require a twelve term polynomial

$$w(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + a_{11} x^3 y + a_{12} xy^3$$

i.e. complete cubic plus two terms (x^3y and xy^3) polynomial of above equation satisfies the homogeneous plate equation $D\nabla^4 w=0$, this fact is of little significance in the finite element formulation.

Rigid Body Modes

W=constant (translation) and also need two rotations. Three rigid body modes required are included through $a_1+a_2x+a_3y$ in the polynomial of above equation.

Constant Strain

In plate bending, the strains are curvatures and twist i.e. w_{xx}, w_{yy} and w_{xy}. This is provided by the second degree terms i.e. $a_4x^2+a_5xy+a_6y^2$ which are also included.

Continuity

The polynomial in above equation has been chosen carefully and for a very good reason we included x^3y and xy^3 terms instead of x^4 and y^4. For constant y, w(x,y) is cubic in x and vice-versa. Now a cubic polynomial in one dimension contains four independent parameters or coefficient which may be specified uniquely by two conditions at each end point (i.e. the end nodes. This particular feature leads to ensuring displacement continuity between adjacent elements. We will look into it in more detail later.

Generalized Displacement

The element formulation begins by first solving for generalized displacements from displacement function. This yields the following matrix equation;

$$\begin{Bmatrix} w_1 \\ w_{x1} \\ w_{y1} \\ w_2 \\ w_{x2} \\ w_{y2} \\ w_3 \\ w_{x3} \\ w_{y3} \\ w_4 \\ w_{x4} \\ w_{y4} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & a & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & za & 0 & 0 & 3a^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a & 0 & 0 & a^2 & 0 & 0 & a^3 & 0 \\ 1 & a & b & a^2 & ab & b^2 & a^3 & a^2b & ab^2 & b^3 & ab^3 & ab^3 \\ 0 & 1 & 0 & za & b & 0 & 3a^2 & zab & b^2 & 0 & 3a^{2b} & b^3 \\ 0 & 0 & 1 & 0 & a & zb & 0 & a^2 & 2ab & 3b^2 & a^3 & 3ab \\ 1 & 0 & b & 0 & 0 & b^2 & 0 & 0 & 0 & b^3 & 0 & 0 \\ 0 & 1 & 0 & 0 & b & 0 & 0 & 0 & b^2 & 0 & 0 & b^3 \\ 0 & 0 & 1 & 0 & 0 & 2b & 0 & 0 & 0 & 3b^2 & 0 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \end{Bmatrix}$$

or $\{\underline{w}\}=[T]\{A\}$

where $\{A\}=<a_1,a_2,\ldots\ldots,a_{12}>$

The matrix [T] in equations above can be inverted in order to solve for {A} as functions of the generalized displacements. Once a_1, a_2 etc. are substituted back into displacement function, we obtain deflection in terms of the generalized displacements w_1, w_{x1}, w_{y1},….etc. as:

Equation 1

$$w(\xi,\eta)=\{W\}^T\left\{\begin{array}{c} a(1-\xi)^2\xi(1-\eta) \\ b(1-\xi)\eta(1-\eta)^2 \\ a\left[1-\xi\eta-(3-2\xi)\xi^2(1-\eta)-(1-\xi)(3-2\eta)\eta^2\right] \\ -a(1-\xi)\xi^2(1-\eta) \\ b(1-\eta)^2\xi\eta \\ a\left[(3-2\xi)\xi^2(1-\eta)+\xi\eta(1-\eta)(1-2\eta)\right] \\ -a(1-\xi)\xi^2\eta \\ -b(1-\eta)\xi\eta^2 \\ a\left[(3-2\xi)\xi^2\eta-\xi\eta(1-\eta)(1-2\eta)\right] \\ a(1-\xi)^2\eta\xi \\ -b(1-\xi)(1-\eta)\eta^2 \\ a\left[(1-\xi)(3-2\eta)\eta^2+\xi(1-\xi)(1-2\xi)\eta\right] \end{array}\right\}$$

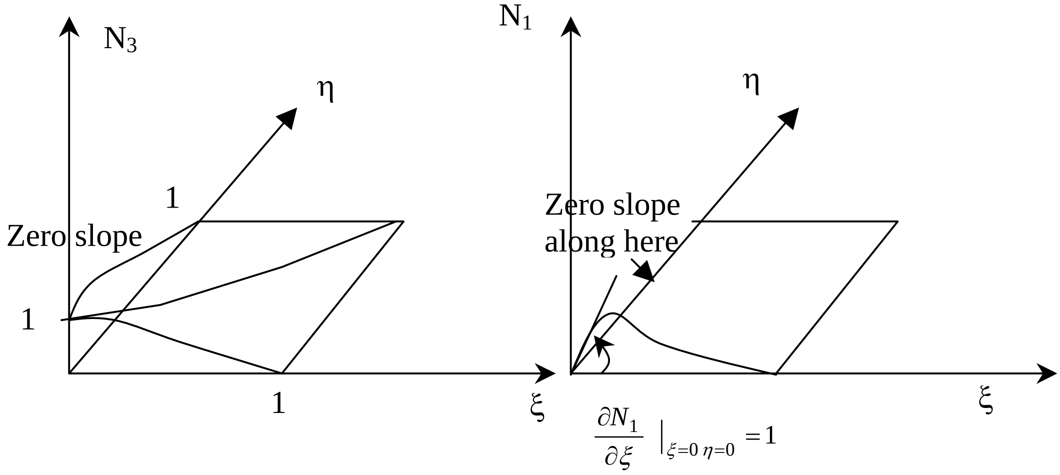
where {w} is the column vector of nondimensional generalized displacements:

$$\{w\}^T=\left[w_{x1}\quad w_{y1}\quad w_1/a\quad w_{x2}\quad w_{y2}\quad w_2/a\quad w_{x3}\quad .\quad .\quad w_{x4}\quad .\quad .\right]$$

$$\xi=\frac{x}{a}\quad \eta=\frac{y}{b}\quad \textit{are non-dimensional coordinates}$$

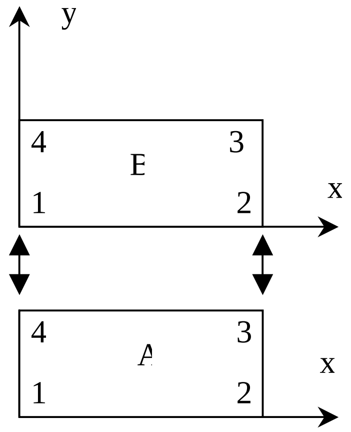
Each row in equation of $w(\xi,\eta)$ represent a shape function or interpolation function N_i. We may write it as:

$$w(\xi,\eta)=\sum_{i=1}^{12} N_i(\xi,\eta)\,\delta_i \quad where\ \delta_1=w_{x1}\quad \delta_2=w_{y1}\quad \delta_3=\frac{w_1}{a}\quad etc.$$



Now that we have the displacement distribution within the element defined by displacement equation, we may ask what continuity will be provided by the element. Consider for example, the joining of two elements A and B together as illustrated in the Figure.

$$w = w_1 H_{01}(x) + w_2 H_{02}(x) + w_{x1} H_{11}(x) + w_{x2} H_{12}(x)$$



Joining two elements along edges parallel to x

For element A, the displacement along edge 3-4 is obtained by putting η=1 in equation 1.

$$w_A(\xi) = w_4(1-3\xi^2+2\xi^3) + a w_{x4}\xi(1-\xi)^2 + w_3(3-2\xi)\xi^2 - a w_{x3}\xi^2(1-\xi)$$

for element B, the displacement along edge $1-2$ $(\eta = 0)$ *is* :

$$w_B(\xi) = w_1(1-3\xi^2+2\xi^3) + a w_{x1}\xi(1-\xi)^2 + w_2(3-2\xi)\xi^2 - a w_{x2}\xi^2(1-\xi)$$

It may be seen that the same function of ξ occur in these two equations. Therefore, if w_1, w_{x1}, w_2, w_{x2} of element B are equated to w_4, w_{x4}, w_3, w_{x3}, respectively of element A, w will have to be continuous between the elements. Similar arguments are easily made for edges parallel to the y-axis.

What about slopes normal to the edges?

There is no continuity of slopes normal to the element edges. This can be shown by taking derivatives of w with respect to η and substituting η=1 for element A and η=0 for element B.

It will be found that $\frac{\partial w}{\partial \eta}$ terms along those edges are cubic and there is no way we can make normal slopes continuous by equating w_{y3} and w_{y4} of element A to w_{y2} and w_{y3} of element B, respectively.

Therefore, the element is called non-conforming

2.1.1- Stiffness Matrix

Calculate the stiffness matrix for the non-conforming plate bending element by substituting equation 1 into expression for strain energy. After, carrying out integration over the area of the element, we obtain the quadratic form in term of generalized displacements (as expected) for strain energy:

$$U_e = \frac{1}{2}\{W\}^T [K]\{W\}$$

Here, [K] is the 12 by 12 stiffness matrix for the element and is given in the following page.

Note, this matrix has been derived for {W} as given in equation of δ_3=w_1/a, δ_6=w2/a, δ_9=w_3/a and δ_{12}=w_4/a i.e. in dimensionless displacements. To allow w's to take on dimensionless displacements, the 3rd, 6th, 9th and 12th row should be multiplied by a again. Further, if the degrees of freedom are desired to be arranged as:

$$\{\overline{w}\}^T = \begin{bmatrix} w_1 & w_{x1} & w_{y1} & w_2 & w_{x2} & w_{y2} & w_3 & w_{x3} & w_{y3} & w_4 & w_{x4} & w_{y4} \end{bmatrix}$$

Then the rows and columns should be rearranged accordingly, e.g. 1st and 2nd rows should be moved into second and 3rd rows. And 3rd row should be placed into 1st row, etc, etc., etc.

The stiffness matrix for the plate bending element may also be derived following the alternative method we discussed for beam element.

Figure 1 Stiffness Matrix for 12 parameter Rectangular Element (non-conforming)

$$m=\frac{a}{b} \qquad SYMMETRIC \quad \nu = Poisson's Ratio$$

$$\begin{array}{ccccccc}
\frac{2}{3m}+\frac{2(1-\nu)m}{15} & & & & & & \\
\frac{\nu}{2} & \frac{2m}{3}+\frac{2(1-\nu)}{15m} & & & & & \\
\frac{1}{m}+\frac{(1+4\nu)}{10} & 2m^3+\frac{2}{m}+\frac{(7-2\nu)m}{5} & & & & & \\
\frac{1}{3m}-\frac{(1-\nu)m}{30} & 0 & \frac{1}{m}+\frac{(1-\nu)m}{10} & \frac{2}{3m}+\frac{2(1-\nu)m}{15} & & & \\
0 & \frac{m}{3}-\frac{2(1-\nu)}{15m} & \frac{m^2}{2}-\frac{(1+4\nu)}{10} & -\frac{\nu}{2} & \frac{2m}{3}+\frac{2(1-\nu)}{15m} & & \\
\frac{-1}{m}-\frac{m(1-\nu)}{10} & \frac{m^2}{2}-\frac{(1+4\nu)}{10} & m^3-\frac{2}{m}-\frac{(7-2\nu)m}{5} & -\frac{1}{m}-\frac{(1+4\nu)m}{10} & m^2+\frac{(1+4\nu)}{10} & 2m^3+\frac{2}{m}+\frac{(7-2\nu)m}{5} & \\
\frac{1}{6m}+\frac{m(1-\nu)}{30} & 0 & \frac{1}{2m}-\frac{m(1-\nu)}{10} & \frac{1}{3m}-\frac{2m(1-\nu)}{15} & 0 & \frac{-1}{2m}+\frac{m(1+4\nu)}{10} & \frac{2}{3m}+\frac{2m(1-\nu)}{15} \\
0 & \frac{m}{6}+\frac{(1-\nu)}{30m} & \frac{m^2}{2}-\frac{(1-\nu)}{10} & 0 & \frac{m}{3}-\frac{(1-\nu)}{30m} & m^2+\frac{(1-\nu)}{10} & \frac{\nu}{2} \\
\frac{-1}{2m}-\frac{m(1-\nu)}{10} & \frac{-m^2}{2}+\frac{(1-\nu)}{10} & -m^3-\frac{1}{m}+\frac{(7-2\nu)m}{5} & \frac{-1}{2m}+\frac{(1+4\nu)m}{10} & -m^2-\frac{(1-\nu)}{10} & -2m^3+\frac{1}{m}-\frac{(7-2\nu)m}{5} & \frac{-1}{m}-\frac{m(1+4\nu)}{10} \\
\frac{-1}{m}-\frac{2m(1-\nu)}{15} & 0 & \frac{1}{2m}-\frac{(1+4\nu)m}{10} & \frac{1}{6m}+\frac{(1-\nu)m}{30} & 0 & \frac{-1}{2m}+\frac{(1-\nu)m}{10} & \frac{1}{3m}-\frac{m(1-\nu)}{30} \\
0 & \frac{m}{3}-\frac{(1-\nu)}{30m} & m^2+\frac{(1-\nu)m}{10} & 0 & \frac{m}{6}+\frac{(1-\nu)}{30m} & \frac{m^2}{2}-\frac{(1-\nu)}{10} & 0 \\
\frac{1}{2m}-\frac{m(1+4\nu)}{10} & -(m^2+\frac{(1-\nu)}{10}) & -2m^3+\frac{1}{m}-\frac{(7-2\nu)m}{5} & \frac{1}{2m}-\frac{(1-\nu)m}{10} & \frac{-m^2}{2}+\frac{(1-\nu)}{10} & -m^3-\frac{1}{m}+\frac{(7-2\nu)m}{5} & \frac{1}{m}+\frac{m(1-\nu)}{10}
\end{array}$$

CONTINUE

$$\begin{array}{ccccc}
\frac{2m}{3}+\frac{2(1-\nu)}{15m} & & & & \\
-m^2+\frac{(1+4\nu)}{10} & 2m^3+\frac{2}{m}+\frac{m(7-2\nu)}{5} & & & \\
0 & -\frac{1}{m}-\frac{(1-\nu)m}{10} & \frac{2}{3m}+\frac{2(1-\nu)m}{15} & & \\
\frac{m}{3}-\frac{2(1-\nu)}{15m} & \frac{-m^2}{2}+\frac{(1+4\nu)}{10} & -\frac{\nu}{2} & \frac{2m}{3}+\frac{2(1-\nu)}{15m} & \\
\frac{-m^2}{2}+\frac{(1+4\nu)}{10} & m^3-\frac{2}{m}-\frac{m(7-2\nu)}{5} & \frac{1}{m}+\frac{(1+4\nu)m}{10} & -m^2-\frac{(1+4\nu)}{10} & 2m^3+\frac{2}{m}+\frac{(7-2\nu)m}{5}
\end{array}$$

]

2.1.2- Consistent Load Vector

Assume uniform pressure q_0. Recall from equation of potential energy π, the work done W is given by:

$$W = \iint_{A_e} q_0 w dxdy = \{p\}^T \{W\}$$

where A_e is the element area, and {w} is given by equation 1, when equation 1 is substituted into above equation and integrating, the load vector for the element in dimensional form:

$$\{p\}^T = abq_e \begin{bmatrix} \frac{a}{24} & \frac{b}{24} & \frac{1}{4} & -\frac{a}{24} & \frac{b}{24} & \frac{1}{4} & -\frac{a}{24} & -\frac{b}{24} & \frac{1}{4} & \frac{a}{24} & \frac{b}{24} & \frac{1}{4} \end{bmatrix}$$

When nonconforming elements are used to obtain an approximate solution for some loading, generally we use reasonably large number of elements and can obtain reasonable answer by using lumped load i.e. $q_0ab/4$ at each corner node. However, for very refined elements, we must use consistent load vector since much fewer elements are used. In such cases, we may be introducing an undesirable error through lumped loads.

2.1.3- Stresses

Bending and twisting moments

Define:

$$\{\varepsilon\} = \begin{Bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{Bmatrix} \quad \textit{strain and curvature}$$

$$\{\tau\} = \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} \quad \textit{stresses and moments}$$

$$\{\tau\} = \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} -1 & -\nu & 0 \\ -\nu & -1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{Bmatrix}$$

$$\{\tau\} = [D]\{\varepsilon\}$$

$$[D] = D \begin{bmatrix} -1 & -\nu & 0 \\ -\nu & -1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

From the shape functions in equation 1, we can obtain w_{xx}, w_{yy} and w_{xy}. Further, these can be evaluated at various points (x_i, y_i) or (ξ_I, η_i) and hence M_{xx}, M_{yy} and M_{xy} can be evaluated at specified points.

We must know {w} before we can compute $\{\tau\}$.

2.1.4- Boundary Conditions (Kinematic)

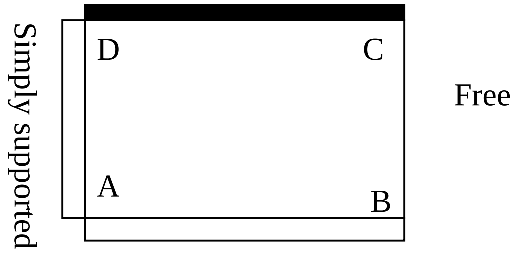
Along AB and AD, the plate is simply supported,

AB: $w=0$ and $w_x=0$

AD: $w=0$ and $w_y=0$

Along cd, the plate is clamped $w=0$ and $w_x=0$ and $w_y=0$

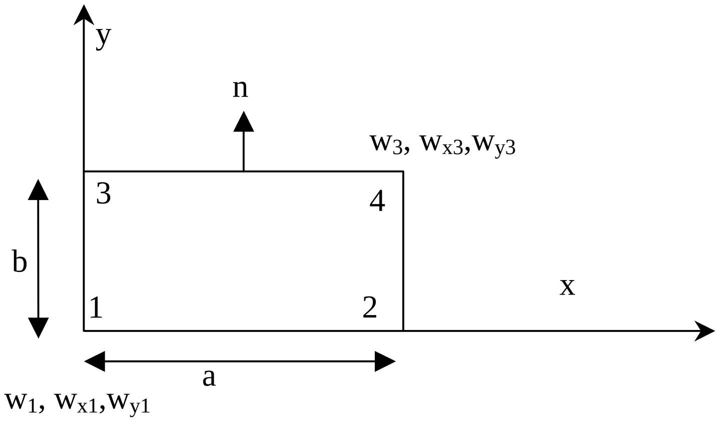
Nothing specified on free boundary.



2.2- Note on Continuity

Both w and its normal derivatives or normal slope must be uniquely determined by values along an interface or edge of an element in order to ensure, C_1 continuity.

Consider edge 3-4 of the rectangular element shown.



Here, $w_n=w_y$, the normal slope. It is desired that w and w_y be uniquely determined by the values of w and w_x and w_y at the nodes lying along edge 3-4.

$$w = a_1 + a_2 x + a_3 x^2 + ...$$

$$\frac{\partial w}{\partial y} = b_1 + b_2 x + b_3 x^2 + ...$$

along edge 3-4 with the number of constants a_i and b_i in each expression just sufficient to determine the expressions by nodal parameters or dof associated with the line.

- With w and w_x as nodal dof at each node i.e. two nodes, we can allow only four a_i (a_1, a_2 , a_3 and a_4) or at most cubic variation in x along 3-4.

Similarly only a linear variation can be allowed i.e. two terms (b_1 and b_2) for w_{yi}. In the same manner, w_x can be made continuous along the edge parallel to the y axis ($w_x=c_1+c_2y$ along 2-3)

Therefore, along edge 3-4

-w_y depends on nodal dof of edge 3-4

and along edge 2-3

-w_x depends on nodal dof of edge 2-3

Differentiate w_y along edge 3-4 wrt x $\rightarrow W_{xy}$

Differentiate w_x along edge 2-3 wrt y $\rightarrow W_{yx}$

The first depends on nodal dof of edge 3-4 and the second depends on nodal dof of edge 2-3.

At common node 3: $w_{xy}\big|_{3\text{-}4} \neq w_{yx}\big|_{2\text{-}3}$

Because of arbitrary nodal dof at nodes 2 and 4 where as for continuous functions $w_{xy}=w_{yx}$ ($b_2 \neq c_2$)

Assertion: It is therefore, impossible to use simple polynomials for shape functions ensuring full compatibility when only w and its slopes are used as dof at nodes.

If any functions satisfying compatibility are found with the three nodal variables, they must be such that at corner nodes they are not continuously differentiable and the cross derivative is not unique.

So far we have applied the argument to a rectangular element, we can extend this for any two arbitrary directions of interfaces or common edges at node 3 (triangular or quadrilaterals).

Unfortunately, this extension requires continuity of cross derivatives in several sets of orthogonal directions, which in fact implies a specification of all second derivatives at a node. This leads to excessive continuity that violates the continuity requirement of potential energy theorem, also the physical requirements. If the plate stiffness varies abruptly from element to element then equality of moments normal to the interface cannot be maintained.

3- Elements for C^1 Problems

Constructing two-dimensional elements that can be used for problems requiring continuity of the field variable ϕ as well as its normal derivative ϕ_n along element boundaries is far more difficult than constructing elements for C^o continuity alone. To preserve C^1 continuity, we must be sure that ϕ and ϕ_n are uniquely specified along the element boundaries by the degrees of freedom assigned to the nodes along a particular boundary. The difficulties arise from the following principles:

1. The interpolation functions must contain at least some cubic terms because the three nodal values ϕ, ϕ_x, and ϕ_y must be specified at each corner of the element.

2. For non-rectangular elements, C^1 continuity requires the specification of at least the six nodal values ϕ, ϕ_x, ϕ_y, ϕ_{xx}, ϕ_{yy}, and ϕ_{xy} at the corner nodes. For a rectangular element with sides parallel to the global axes, we need to specify at the corners nodes only ϕ, ϕ_x, ϕ_y and ϕ_{xy}.

It is sometimes very convenient to specify only ϕ, ϕ_x and ϕ_y at corners, but when this is done, it is impossible to have continuous second derivatives at the corner nodes. In general, the cross derivative ϕ_{xy} will be directionally dependent and hence, nonunique at intersections of the sides of the element.
Analysts first began to encounter difficulties in formulating elements for C^1 problems when they attempted to apply FE techniques to plate-bending problems. For such problems, the displacement of the mid plane of the plate for Kirchhoff plate bending theory is the field variable in each element, and interelement continuity of the displacement and its slope is a desirable physical requirement. Also, since the functional for plate bending involves second order derivatives, continuity of slope at element interface is a mathematical requirement because it ensures convergence as element size is reduced. For these reasons, analysts have labored to find elements giving continuity of slope and value.

Rectangular Elements

Whereas triangulars are the simplest element shapes to establish C^0 continuity in two dimensions, rectangles with sides parallel to the global axes are the simplest element shapes of C^1 continuity in 2 dimensions. The reason is that the element boundaries meet at right angles, and imposing continuity of the cross derivatives ϕ_{xy} at the corners quarantees continuity of the derivatives that otherwise might be nonunique.

A four-node rectangle with ϕ, ϕ_x, ϕ_y and ϕ_{xy} specified at the corner nodes assigns a 16-dof element.

4- Triangular Elements

For C^1 continuity, by assigning 21 dof to element, we can make a complete quintic polynomial to represent the field variable ϕ. When ϕ and all first and second derivatives are specified at the corner nodes. There are only 18 dof, so 3 more are needed to specify the 21-term quintic polynomial.The 3 dof are obtained by specifying the normal derivatives ϕ_n at the midside nodes. This element quarantees continuity of ϕ along element boundaries because, along a boundary where s is the linear coordinate, ϕ varies in s as aquintic function, which is uniquely determined by six nodal values, normal, ϕ, ϕ_s and ϕ_{ss} at each end node.

Slope continuity is also assured because the normal slope along each edge varies as a quartic function which is uniquely determined by five nodal variables, namely ϕ_n and ϕ_{nn} at each end node plus ϕ_n at the midside node.

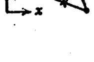
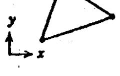
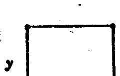
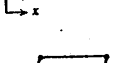
The presence of midside nodes is undesirable because they require special bookkeeping in the coding process, and they increase the bandwidth of the final matrix.

Apparantly, C^1 continuity is not always a necessary condition for convergence in C^1 problems. Experience has indicated that convergence is more dependent on the completeness than on the compatibility property of the element. The following table shows a sample of incompatible elements. Any of these elements can be used in the solution of continuum problems involving functionals containing up to second-order derivatives.

The analysts may ask, which element should I use to sole my problem? Unfortunately, no general answer can be given because the answer is problem dependent.

Some Incompatible Elements for C^1 Problems

Table 5.5. Some incompatible elements for C^1 problems

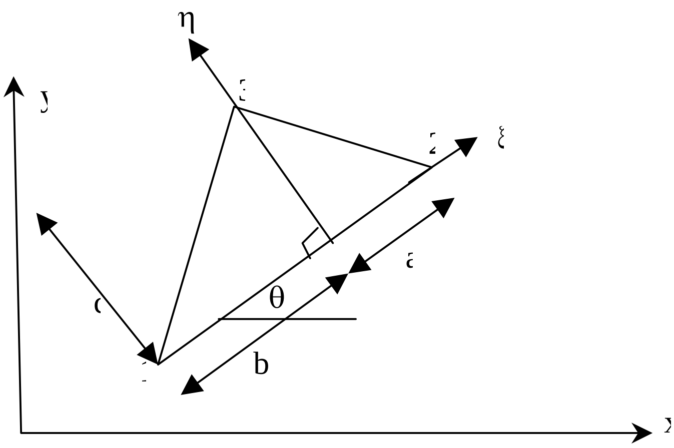
Element	Nodal Variables	Order of Polynomial	Degrees of freedom per element	References	Comments
	ϕ specified at ● $\frac{\partial\phi}{\partial n}$ specified at •	Complete quadratic	6	39	Simplest possible plate-bending element. Gives convergent answers comparable to those for more complex triangular elements.
	$\phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}$ specified at ●	Incomplete cubic: either xy^2 or x^2y term omitted	9	38	Geometric isotropy is not preserved. For certain orientations of the element $[G]^{-1}$ may not exist. Area coordinates can be used to express the interpolation functions and thus avoid $[G]^{-1}$ problem. Gives poor results
	$\phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}$ specified at ●	Incomplete quartic: x^4, x^2y^2, and y^4 terms omitted	12	34, 36	Geometric isotropy is preserved. $[G]^{-1}$ given explicitly in ref. 34. Gives satisfactory results when the rectangular elements can fit the given geometry.
	$\phi, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}$ specified at ●	Incomplete quartic	12	42	Sometimes a more convenient element for plate bending.

5- Nonconformin Triangular Plate Bending Elements

- we need an element of more general shape
- Triangular elements fit curved edges more appropriately than the rectangular elements
- Again consider local coordinates ξ and η. We shall use transformation matrix to go back to x-y system.
- Consider w, w_ξ, w_η as the dof at each node.
- A cubic has 10 generalized parameters:

$$w = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3$$

- for the element we have 9 dof but 10 generalized parameters in above equation. Therefore, must delete one of a_i (i=1,2,…,10) or add a dof.



Possibilities:

a) use w at centroid as an extra dof
   -this element doesnot work sometimes and also exhibit poor convergence
   -Certain orientations may lead to less than a cubic along one of the edges and violates w continuity requirement

b) Throwaway one term- say a_5=0
   This violates constant curvature or constant strain energy requirement i.e. will not work since $w_{\xi\eta}$=constant not present

c) combine two terms, i.e. equate a_8=a_9
   -we get $a_8(\xi^2\eta + \xi\eta^2)$ which keeps some symmetry.
   -in general, ruins isotrophy of the polynomial so we expect orientation problems.

Recall:

$$\begin{Bmatrix} \{\overline{w}\} \\ 0 \end{Bmatrix}_{10\times 1} = [T]_{10\times 10}\{A\}_{10\times 1}$$

$$\{A\}^T = \begin{bmatrix} a_1 & a_2 & . & . & . & . & . & . & . & . & a_{10} \end{bmatrix}$$

$$\{\overline{w}\} = \begin{bmatrix} w_1 & w_{\xi 1} & w_{\eta 1} & . & . & . & . & . & . & . & w_{\eta 3} \end{bmatrix}$$

[T] matrix becomes singular sometimes. This happens when two of edges are parallel to the global axes (x,y).

d) Use area coordinates (Zienkiewics, 9dof triangular element)
-explain lack of full cubic because of only 9 dof. Let us look at (c) in more detail. [T] matrix

$$[T] = \begin{bmatrix} & & & & & & & \overset{8}{.} & \overset{9}{.} & \\ 1 & -b & 0 & b^2 & . & . & . & . & . & . \\ 0 & 1 & 0 & -2b & . & . & . & . & . & . \\ & & & & & & & . & . & \\ & & & & & & & . & . & \\ & & & & & & & . & . & \\ & & & & & & & . & . & \\ & & & & & & & . & . & \\ & & & & & & & . & . & \\ & & & & & & & . & . & \\ & & & & & & & 1 & -1 & \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ . \\ \\ a_8 \\ a_9 \\ a_{10} \end{Bmatrix}$$

The last equation is a constraint equation i.e. a_8-a_9=0
This is a more elegant way of doing it.
Det[T]=$c^5(a+b)^5(c+b-a)$
If a=c+b or c+b-a=0 then det[T]=0 and we cannot invert [T] to formulate the element.
If this situation is avoided then:

$$\{A\} = [T]^{-1}\begin{Bmatrix}\{\overline{w}\} \\ 0\end{Bmatrix}$$

this can be written as :

$$\{A\}_{10\times1} = [T_2]_{10\times9}\{\overline{w}\}_{9\times1}$$

$[T_2]$ *contains first* 9 *columns of* $[T]^{-1}$

then

$$w(\xi,\eta) = \begin{bmatrix}1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 & \xi^3 & \xi\eta^2 & \xi\eta^2 & \eta^3\end{bmatrix}[T_2]\{\overline{w}\}$$

$$w(\xi,\eta) = [p]^T{}_{1\times10}[T_2]_{10\times9}\{\overline{w}\}_{9\times1}$$

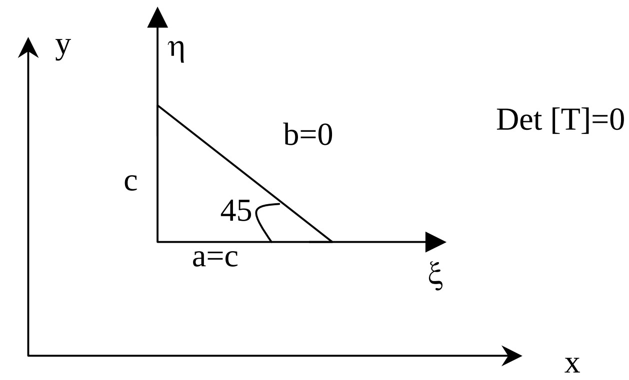
to transform from $\{\overline{w}\}$ *to* $\{w\} = \begin{bmatrix}w_1 & w_{x1} & w_{y1} & . & . & . & . & . & . & w_{y3}\end{bmatrix}$

$$\{\overline{w}\} = [R]\{w\}$$

where $[R]$

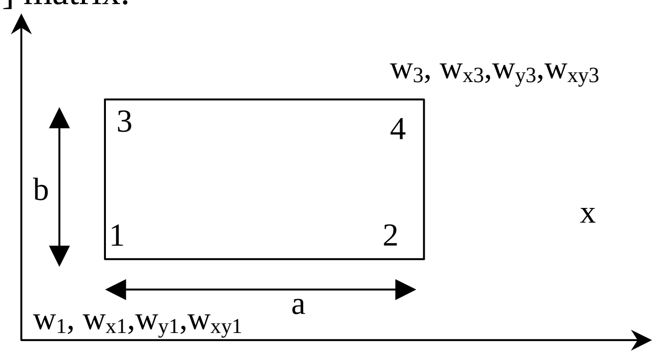
$$[R] = \begin{bmatrix}[R_1] & [0] & [0] \\ [0] & [R_1] & [0] \\ [0] & [0] & [R_1]\end{bmatrix} \qquad [R_1] = \begin{bmatrix}1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta\end{bmatrix}$$

where θ *is the angle between* (ξ,η) *and* (x,y) *axes.*



6- Conforming Rectangular Element (16 dof)

Nodal degrees of freedom at each node are w, w_x, w_y and w_{xy}. Extra dof w_{xy} is permissible as it does not involve excessive continuity. Thus, we have 16 dof per element and a polynomial expression involving 16 constants could be used. We retain terms which do not produce a higher order variation of w or its normal slope than cubic along the sides. There are many alternatives as far as choosing the polynomial is concerned. But some of these alternatives may not produce invertible [T] matrix.



An alternative is to use Hermitian polynomials. These are one dimensional polynomials and possess certain properties. A Hermitian polynomial
$H^n_{mi}(x)$ is a polynomial of order 2n+1 which gives, where $x=x_i$:

Equation 2

$$\frac{d^k H}{dx^k} = 1 \quad k = m \quad for \quad m = 0 \; to \; n$$

and

$$\frac{d^k H}{dx^k} = 0 \quad k \neq m \quad or \quad when \;\; x = x_j$$

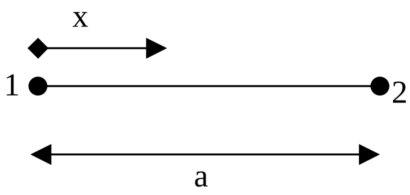
A set of first order Hermitian polynomaols is thus a set of cubics giving shape functions for a line element ij and at the ends, slopes and values of the function are used as nodal degrees of freedom along 1-2

$$H^1_{01}(x) = \frac{1}{a^3}(2x^3 - 3ax^2 + a^3)$$

$$H^1_{02}(x) = -\frac{1}{a^3}(2x^3 - 3ax^2)$$

$$H^1_{11}(x) = \frac{1}{a^2}(x^3 - 2ax^2 + a^2 x)$$

$$H^1_{12}(x) = \frac{1}{a^2}(x^3 - ax^2)$$



These polynomials are plotted in the following figure.
Note these polynomials provide unit values of displacements and slopes at one end and zero at the other as was implies by equation 2. assume w(x,y) of the following form:

$$\begin{aligned}
w(x,y) = {} & H_{01}(x)H_{01}(y)w_1 + H_{02}(x)H_{01}(y)w_2 + H_{02}(x)H_{02}(y)w_3 + \\
& H_{01}(x)H_{02}(y)w_4 + H_{11}(x)H_{01}(y)w_{x1} + H_{12}(x)H_{01}(y)w_{x2} + \\
& H_{12}(x)H_{02}(y)w_{x3} + H_{11}(x)H_{02}(y)w_{x4} + H_{01}(x)H_{11}(y)w_{y1} \\
& H_{02}(x)H_{11}(y)w_{y2} + H_{02}(x)H_{12}(y)w_{y3} + H_{01}(x)H_{12}(y)w_{y4} + \\
& H_{11}(x)H_{11}(y)w_{xy1} + H_{12}(x)H_{11}(y)w_{xy2} + H_{12}(x)H_{12}(y)w_{xy3} + H_{11}(x)H_{12}(y)w_{xy4}
\end{aligned}$$

The superscript for H has been dropped since all H_{mi} are 2x1+1=3^{rd} degree polynomials (n=1). Further for $H_{mi}(y)$, just replace x with y and a with b.

Checks

1. we can show that w(x,y) has three rigid body modes (can be shown by performing an eigenvalue analysis)
2. we can also show that w(x,y) has constant strain modes.

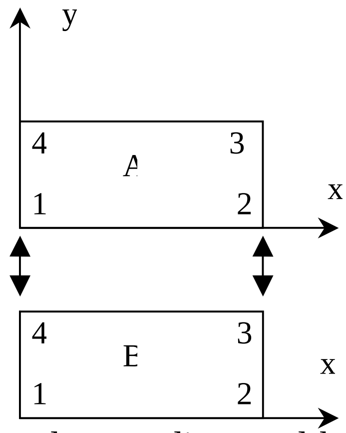
3. continuity: look at edge 1-2 of the element:

$$w = w_1 H_{01}(x) + w_2 H_{02}(x) + w_{x1} H_{11}(x) + w_{x2} H_{12}(x)$$

w_y: only those terms having $H_{11}(y)$ will have non-zero values

$$w_y = w_{y1} H_{01}(x) + w_{y2} H_{02}(x) + w_{xy1} H_{11}(x) + w_{xy2} H_{12}(x)$$

from above two equations, we note w and w_y depends on nodal dof at nodes 1 and 2 for edge 1-2.
Similarly, we can show that we get the same expressions for w and w_y along edge 3-4 except w_4 replaces w_1, w_3 replaces w_2, etc.
Therefore, equating the nodal variables along edge 1-2 of element A in the figure to nodal variables along edge 3-4 of element B will ensure continuity of w and w_y as requitred. In exactly the same manner we can show continuity of w and w_x along edges parallel to y axis.



Thus, the plate bending element discussed here is conforming in the sense that displacements and normal slopes are continuous so that the potential energy theorem does apply. We expect monotonic convergence of potential energy as well as strain energy. Potential energy will converge to the exact value from above where as strain energy from below as was shown for the beam problem, i.e. potential energy is bounded above and strain energy is bounded below.

7- Alternative Method for Plate Bending Element

The alternative method for deriving the stiffness matrix and the consistent load vector is presented for the conforming element discussed in the previous section. However, the approach is general enough to apply to any rectangular or triangular elements.

Although, we used Hermitian polynomials in deriving the displacement approximation, one can multiply out these polynomials in eq1 of the previous section and obtain the following expression:

$$w(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + a_{11} x^3 y + a_{12} x^2 y^2 + a_{13} xy^3 + a_{14} x^3 y^2 + a_{15} x^2 y^3 + a_{16} x^3 y^3$$

In this equation, the polynomial is complete only upto cubic terms. Using Taylo series approach error in w is f(h^4) where h =typical element dimension

Error in strain f(h^2) (strain are second derivatives)

Error in strain energy is f(h^4)

For h=L/N, the strain energy error is f(N^{-4}) where n=number of elements along a side of length L. Generally for convergence rate study, use square elements. Asymptotic convergence rate is N^{-4}. When w is given in the form of above equation, it is obvious that w(x,y) contains rigid body modes and constant strains.

We can write the polynomial in the following form:

$$w(x,y) = \sum_{i=1}^{16} a_i x^{m_i n_i}$$

$$\{m\}^T = \begin{bmatrix} 0 & 1 & 0 & 2 & 1 & 0 & 3 & 2 & 1 & 0 & 3 & 2 & 1 & 3 & 2 & 3 \end{bmatrix}$$

$$\{n\}^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 3 & 1 & 2 & 3 & 2 & 3 & 3 \end{bmatrix}$$

Let us first obtain the stiffness matrix in terms of a_i,s and later transform to obtain [K] in terms of w_i,s.

$$\{w\}^T = \begin{bmatrix} w_1 & w_{x1} & w_{y1} & w_{xy1} & . & . & . & . & w_4 & w_{x4} & w_{y4} & w_{xy4} \end{bmatrix}$$

$$\{w\}_{16\times 1} = [T]_{16\times 16}\{A\}_{16\times 1}$$

$$\{A\}^T = \begin{bmatrix} a_1 & a_2 & . & . & . & a_{16} \end{bmatrix}$$

$$w_{xx} = \sum_{i=1}^{16} m_i(m_i - 1)a_i x^{m_i - 2} y^{n_i}$$

$$w_{yy} = \sum_{i=1}^{16} n_i(n_i - 1)a_i x^{m_i} y^{n_i - 2}$$

$$w_{xy} = \sum_{i=1}^{16} m_i n_i a_i x^{m_i - 1} y^{n_i - 1}$$

$$U = \frac{D}{2}\iint_A \left[w_{xx}{}^2 + w_{yy}{}^2 + 2\nu w_{xx} w_{yy} + 2(1-\nu) w_{xy}{}^2 \right] dxdy$$

$$U_e = \frac{D}{2}\int_0^b \int_0^a \left\{ \begin{array}{l} m_i m_j (m_i - 1)(m_j - 1) x^{m_i + m_j - 4} y^{n_i + n_j} + n_i n_j (n_i - 1)(n_j - 1) x^{m_i + m_j} y^{n_i + n_j - 4} + \\ \nu\left[m_i n_j (m_i - 1)(n_j - 1) + m_j n_i (m_j - 1)(n_i - 1) \right] x^{m_i + m_j - 2} y^{n_i + n_j - 2} + \\ 2(1-\nu) m_i m_j n_i n_j x^{m_i + m_j - 2} y^{n_i + n_j - 2} \end{array} \right\} dxdy a_i a_j$$

Define :

$$G(m,n) = \int_0^b \int_0^a x^m y^n dxdy = \frac{a^{m+1} b^{n+1}}{(m+1)(n+1)}$$

Note that w_{xx}, w_{yy} term has been split into two terms tp preserve symmetry i.e. if we change I with j U_e is still the same.

It is obvious that this integration is not valid when m=-1 or n=-1 and blows up for m≤-1 or n≤-1 at lower limit i.e. x=0 (m=0,1,2,… and n=0,1,2,…).

Strain Energy Can be written as:

$$U_e = \frac{1}{2}\{A\}[\overline{K}]\{A\}$$

$$\overline{K}_{ij} = D\left\{ \begin{array}{l} m_i m_j (m_i - 1)(m_j - 1)G(m_i + m_j - 4, n_i + n_j) + n_i n_j (n_i - 1)(n_j - 1)G(m_i + m_j, n_i + n_j - 4) + \\ \left[\nu m_i n_j (m_i - 1)(n_j - 1) + \nu m_j n_i (m_j - 1)(n_i - 1) + 2(1-\nu) m_i m_j n_i n_j \right] G(m_i + m_j - 2, n_i + n_j - 2) \end{array} \right\}$$

next the $[T]$ *matrix has to be computed* :

$$[T] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & a & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 2a & 0 & 0 & 3a^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & a & 0 & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2a & 0 & 0 & 3a^2 & 0 & 0 & 0 & 0 & 0 \\
1 & a & b & a^2 & ab & b^2 & a^3 & a^2b & ab^2 & b^3 & a^3b & a^2b^2 & ab^3 & a^3b^2 & a^2b^3 & a^3b^3 \\
0 & 1 & 0 & 2a & b & 0 & 3a^2 & 2ab & b^2 & 0 & 3a^2b & 2ab^2 & b^3 & 3a^2b^2 & 2ab^3 & 3a^2b^3 \\
0 & 0 & 1 & 0 & a & 2b & 0 & a^2 & 2ab & 3b^2 & a^3 & 2a^2b & 3ab^2 & 2a^3b & 3a^2b^2 & 3a^3b^2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2a & 2b & 0 & 3a^2 & 4ab & 3b^2 & 6a^2b & 6ab^2 & 9a^2b^2 \\
1 & 0 & b & 0 & 0 & b^2 & 0 & 0 & 0 & b^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & & b & 0 & 0 & 0 & b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2b & 0 & 0 & 0 & 3b^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2b & 0 & 0 & 0 & 3b^2 & 0 & 0 & 0
\end{bmatrix}$$

-Either we can program the matrix above or determine in a more general form as follows:

Define : $x_i, y_i \quad i = 1,2,3,4$ *as the nodal coordinates*

$(x_1, y_1) = (\varepsilon, \varepsilon) \quad , \quad (x_2, y_2) = (a, \varepsilon)$

$(x_3, y_3) = (a, b) \quad , \quad (x_4, y_4) = (\varepsilon, b)$

Where ε is a very small number, $\varepsilon = 10^{-13}$, instead of zero.

This helps retaining some more accuracy and some times makes the inversion possible especially for triangular elements which may exhibit some orientation preferences.

Then:

$$T_{ij} = x_k^{m_j} y_k^{n_j} \qquad for \quad i = 1,5,9,13$$

$$T_{ij} = m_j x_k^{m_j - 1} y_k^{n_j} \qquad for \quad i = 2,6,10,14$$

$$T_{ij} = n_j x_k^{m_j} y_k^{n_j - 1} \qquad for \quad i = 3,7,11,15$$

$$T_{ij} = m_j n_j x_k^{m_j - 1} y_k^{n_j - 1} \qquad for \quad i = 4,8,12,16$$

$$where \quad j = 1,2,3,....,16 \quad k = 1 \quad for \quad i = 1,2,3,4$$

$$k = 2 \quad for \quad i = 5,6,7,8$$

$$k = 3 \quad for \quad i = 9,10,11,12$$

$$k = 4 \quad for \quad i = 13,14,15,16$$

$[T]$ *matrix can be programmed*

```
      Do 59 k = 1,4
      I = 4*(k-1)+1
      Do 60 j = 1,16
      T(I,J) = x(k)**M(J)*y(k)**N(J)
      T(I+1,J) = M(J)*x(k)**(M(J)-1)*y(k)**N(J)
      T(I+2,J) = N(J)*x(k)**M(J)*y(k)**(N(J)-1)
60    T(I+3,J) = M(J)*N(J)*x(k)**(M(J)-1)*y(k)**(N(J)-1)
59    continue
```

The matrix [T] is then inverted and the stiffness matrix is the global coordinates is calculated:

$$[K]_{16\times16} = [T]^{-1}_{16\times16}[\bar{k}]_{16\times16}[T]^{-1}_{16\times16}$$

one has to be cautious when computing G(m,n) or [G] matrix. This is because some of the terms (lower order) in the polynomial of equation 1 may lead to negative or zero m and n i.e. terms like m_i+n_j-4, etc. For example, $m_1=0$, $n_1=0$ then $m_1+m_1-4=-4$ and $n_1+n_1-4=-4$. These are the smallest possible indecises for G(m,n) or [G]. This can be avoided by taking a matrix [F] such that:

F(m+5,n+5)=G(m,n)

Where [F] has dimensions at least 4 larger than [G] would require.

$$F(m+5,n+5) = G(m,n) = \int_0^b \int_0^a x^m y^n dxdy = \frac{a^{m+1} b^{n+1}}{(m+1)(n+1)}$$

$$where \quad F_{i,j} = 0 \quad for \quad i = 1,2,3,4 \quad and \quad j = 1,2,3,4$$

Load Vector

Assume constant load q_0/unit area applied to the plate. Therefore, work done is given by:

$$W_e = \iint_A q_0 w\,dxdy = \int_0^a\int_0^b q_0 \sum_{i=1}^{16} a_i x^{m_i n_i}\,dxdy$$

$$W_e = \{A\}^T\{\bar{f}\} = \{w\}^T\left([T]^{-1}\right)^T\{\bar{f}\}$$

$$W_e = q_0 \sum_{i=1}^{16} a_i G(m_i, n_i)$$

$$\bar{f}_i = q_0 G(m_i, n_i) \qquad i = 1,2,..,16$$

$$\{f\} = \left([T]^{-1}\right)^T_{16\times16}\{\bar{f}\}_{16\times1} \quad \textit{load} \quad \textit{vector in global coordinates}$$

Stress Matrix for Obtaining Moments (for an element)

Recall:

$$\{\varepsilon\} = \begin{Bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{Bmatrix} = [S]\{A\} \quad \textit{strain and curvature}$$

$$\{\tau\} = \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} \quad \textit{stresses and moments}$$

$$\{\tau\} = \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = D\begin{bmatrix} -1 & -\nu & 0 \\ -\nu & -1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}\begin{Bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{Bmatrix}$$

$$\{\tau\} = [D]\{\varepsilon\}$$

$$[D] = D\begin{bmatrix} -1 & -\nu & 0 \\ -\nu & -1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \qquad D = \frac{Eh^3}{12(1-\nu^2)}$$

$$w = \sum_{i=1}^{16} a_i x^{m_i n_i}\,dxdy$$

$$w_{xx} = \sum_{i=1}^{16} m_i(m_i - 1)a_i x^{m_i - 2} y^{n_i}$$

$$w_{yy} = \sum_{i=1}^{16} n_i(n_i - 1)a_i x^{m_i} y^{n_i - 2}$$

$$w_{xy} = \sum_{i=1}^{16} m_i n_i a_i x^{m_i - 1} y^{n_i - 1}$$

$$S_{1j} = m_j(m_j - 1)x^{m_j - 2} y^{n_j}$$

$$S_{2j} = n_j(n_j - 1)x^{m_j} y^{n_j - 2}$$

$$S_{3j} = m_j n_j x^{m_j - 1} y^{n_j - 1}$$

$$\{A\} = [T]^{-1}\{w\}$$

$$\{\varepsilon\} = [S][T]^{-1}\{w\}$$

where w is the displacement vector for element under consideration and is extracted from the global displacement vector.

$$\{\tau\} = \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = [D]_{3\times3}[S]_{3\times16}[T]^{-1}_{16\times16}\{w\}_{16\times1}$$

Note that the matrix [S] is function of x and y and has to be evaluated at the points (x_i,y_i) where bending moments and twisting moment are desired to be evaluated.
The $[T]^{-1}$ matrices can be stored away e.g. on a file so that these can be used for determining moments later, i.e. after displacements have been calculated.
As mentioned earlier, the procedure is general and only changes need to be made are integration routine, different data for m_i and n_i and changing sizes of various matrices. The logic does not change at all.

8- Triangular Element for Conforming C^1 Continuity

Using quintic polynomial for the displacement field:

Equation 3

$$w(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + a_{11} x^4 + a_{12} x^3 y + a_{13} x^2 y^2 + a_{14} xy^3 + a_{15} y^4 + a_{16} x^5 + a_{17} x^4 y + a_{18} x^3 y^2 + a_{19} x^2 y^3 + a_{20} xy^4 + a_{21} y^5$$

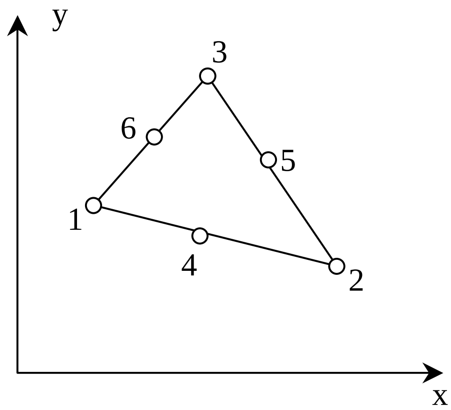
$$w(x,y) = \sum_{i=1}^{21} a_i x^{m_i} y^{n_i}$$

$$\{m_i\}^T = \begin{bmatrix} 0 & 1 & 0 & 2 & 1 & 0 & 3 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 0 & 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

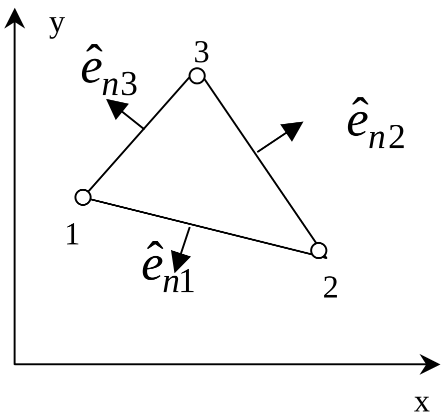
$$\{n_i\}^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

There are 21 generalized parameters a_i's, therefore either 21 dof are required or 21 indepemdent equations to relates a_i's to the dof.

$\hat{e}_{ni}$



Option One



Option Two

Option One:

Six dof at corner nodes (1,2,3), i.e. w, w_x, w_y, w_{xx}, w_{xy} and w_{yy} and one dof at the mid side nodes (4,5,6) i.e. w_n($w_n = \nabla w.\hat{e}_n$)

Option two:

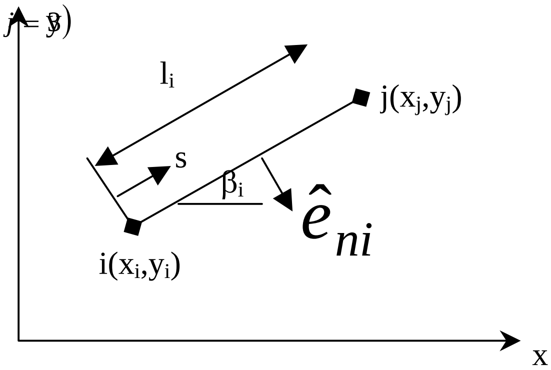
Only six dof at corner nodes (1,2,3), i.e. w, w_x, w_y, w_{xx}, w_{xy} and w_{yy} for a total of 18 dof per element. Additional three equations come from constraining the normal slope w_n to vary cubically.

$$\{w\} = [T]\{a\}$$
$$\{a\}^T = \begin{bmatrix} a_1 & a_2 & . & . & . & a_{21} \end{bmatrix}$$
$$\{w\}^T = \begin{bmatrix} w_1 & w_{x1} & w_{y1} & w_{xx1} & w_{xy1} & w_{yy1} & . & . & . & w_{n4} & w_{n5} & w_{n6} \end{bmatrix}$$

Edge Geometry

Consider the ith edge defined by nodes i and j as shown. Let s be the running coordinate along the edge and $\hat{e}_{ni}$ be the unit outward normal to the ith edge:

$$l_i = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \qquad j = i + 1 \quad \left(\, for \;\; j > 3 \, , \; j = 3 \right)$$
$$\hat{e}_{si} = \cos \beta_i \hat{i} + \sin \beta_i \hat{j}$$
where
$$\cos \beta_i = \frac{x_j - x_i}{l_i} \qquad \sin \beta_i = \frac{y_j - y_i}{l_i}$$
$$\hat{e}_{ni} = \sin \beta_i \hat{i} - \cos \beta_i \hat{j}$$
also along the i^{th} *edge* :
$$x = s \cos \beta_i \quad and \quad y = s \sin \beta_i$$



ith edge

Option One

Equation 4

$$T_{k,j} = x_i^{m_j} y_i^{n_j} \qquad k = 1 + 6(i-1) \quad i = 1,2,3 \; and \;\; j = 1,2,3,..,21$$
$$T_{k+1,j} = m_j x_i^{m_j - 1} y_i^{n_j}$$
$$T_{k+2,j} = n_j x_i^{m_j} y_i^{n_j - 1}$$
$$T_{k+3,j} = m_j (m_j - 1) x_i^{m_j - 2} y_i^{n_j}$$
$$T_{k+4,j} = m_j n_j x_i^{m_j - 1} y_i^{n_j - 1}$$
$$T_{k+5,j} = n_j (n_j - 1) x_i^{m_j} y_i^{n_j - 2}$$
$i = 1,2,3$ *takes care of* 18 *dof at the corner nodes.*

At mid – side nodes :

$$\frac{\partial w}{\partial n} = w_n = \overline{\nabla} w . \hat{e}_n = w_x \sin \beta - w_y \cos \beta$$
$$T_{i+18,j} = m_j x_{i+3}^{m_j - 1} y_{i+3}^{n_j} \sin \beta_i - n_j x_{i+3}^{m_j} y_{i+3}^{n_j - 1} \cos \beta_i \qquad j = 1,2,3,..,21 \; and \; i = 1,2,3 \left(at \; nodes 4,5,6 \right)$$
$$\{w\}_{21 \times 1} = [T]_{21 \times 21} \{a\}_{21 \times 1} \qquad invert \, [T]$$

Option Two

Equation4 (a-f) for corner nodes still apply. These yields 18 eqns and therefore three more equations are still to be accounted for. Note that for a quintic polynomial, the normal slope along all three edges vary as quartic (4^{th} degree polynomial).

"additional three equations arise from constraining the normal slope to vary cubically along each edge."

Consider only the 5^{th} degree term in equation 3 and denote this partial w(x,y) as w_p i.e.:

$$w_p = a_{16}x^5 + a_{17}x^4 y + a_{18}x^3 y^2 + a_{19}x^2 y^3 + a_{20}xy^4 + a_{21}y^5$$

also along an edge:

$$x = s\cos\beta_i \quad and \quad y = s\sin\beta_i$$

$$\frac{\partial w_p}{\partial n} = \overline{\nabla} w_p . \hat{e}_{ni} = \frac{\partial w_p}{\partial x}\sin\beta_i - \frac{\partial w_p}{\partial y}\cos\beta_i$$

$$\frac{\partial w_p}{\partial n} = \begin{bmatrix} a_{16}(s\cos\beta_i{}^4 \sin\beta_i) + a_{17}(4\cos\beta_i{}^3\sin\beta_i{}^2 - \cos\beta_i{}^5) + a_{18}(3\cos\beta_i{}^2\sin\beta_i{}^3 - 2\cos\beta_i{}^4\sin\beta_i) + \\ a_{19}(2\cos\beta_i\sin\beta_i{}^4 - 3\cos\beta_i{}^3\sin\beta_i{}^2) + a_{20}(\sin\beta_i{}^5 - 4\cos\beta_i{}^2\sin\beta_i{}^3) + a_{21}(-5\cos\beta_i\sin\beta_i{}^4) \end{bmatrix} S^2$$

Note the bracked term […] is the combined coefficient of s^4. For w_n to be cubic along an edge […] must be set equal to zero and hence yields three more equations, from each edge.

Hence,

$$T_{i+18,j} = m_j(\cos\beta_i)^{m_j-1}(\sin\beta_i)^{n_j}\sin\beta_i - n_j(\cos\beta_i)^{m_j}(\sin\beta_i)^{n_j-1}\cos\beta_i = 0$$

$$along\ edge\ 1,2,3 \quad j = 16,17,18,19,20,21 \quad and \quad i = 1,2,3$$

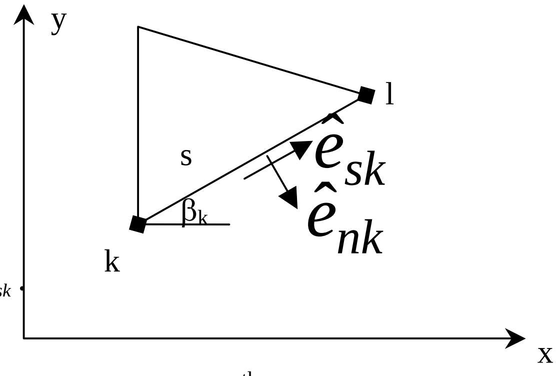
$$\begin{Bmatrix} \{w\}_{18\times1} \\ \{0\}_{3\times1} \end{Bmatrix} = [T]_{21\times21}\{a\}$$

invert $[T]$ *and ignore the last three columns of* $[T]^{-1}$ *to obtain*

$$\{a\}_{21\times1} = [T]^{-1}{}_{21\times18}\{w\}_{18\times1}$$

8.2- Transformation of Nodal DOF along an Inclined Edge

Before any boundary conditions can be applied along an inclined edge, all first and second derivatives must be transformed to perpendicular and parallel to the edge.



k^th edge

For the first derivatives:

$$w_{,n} = \lambda_{ni} w_{,i} \quad i = 1,2 \quad or \quad w_{,i} = \lambda_{in} w_{,n}$$

$$w_{,s} = \lambda_{si} w_{,i} \quad i = 1,2 \quad or \quad w_{,j} = \lambda_{jn} w_{,n}$$

where λ_{ni} are direction cosines of the unit outward normal $\hat{e}_{nk}$ and λ_{si} are the direction cosines of the unit tangential vector $\hat{e}_{sk}$.

For second derivatives:

$$w_{,nn} = \lambda_{ni}\lambda_{nj} w_{,ij} \quad w_{,ns} = \lambda_{ni}\lambda_{sj} w_{,ij} \qquad w_{,ss} = \lambda_{si}\lambda_{sj} w_{,ij}$$

$$w_{,ii} = \lambda_{in}\lambda_{is} w_{,ns} \quad w_{,ij} = \lambda_{in}\lambda_{js} w_{,ns} \qquad w_{,jj} = \lambda_{jn}\lambda_{js} w_{,ns}$$

$$\lambda_{ni} = [\cos\theta \quad \sin\theta] \quad \lambda_{si} = [-\sin\theta \quad \cos\theta]$$

$$to\ obtain\ \lambda_{si},\ replace\ \theta\ by\ \theta + 90\ in\ \lambda_{ni}$$

$$\lambda_{in} = [\cos\theta \quad -\sin\theta] \qquad n = 1\ for\ n,\ n = 2\ for\ s$$

$$\lambda_{jn} = [\sin\theta \quad \cos\theta] \quad i\ for\ x\ j\ for\ y$$

$$\begin{Bmatrix} w_{,x} \\ w_{,y} \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} w_{,n} \\ w_{,s} \end{Bmatrix} = [T_1] \begin{Bmatrix} w_{,n} \\ w_{,s} \end{Bmatrix}$$

$$\begin{Bmatrix} w_{,xx} \\ w_{,xy} \\ w_{,yy} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & -2\sin\theta\cos\theta & \sin^2\theta \\ \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta & -\sin\theta\cos\theta \\ \sin^2\theta & 2\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} \begin{Bmatrix} w_{,nn} \\ w_{,ns} \\ w_{,ss} \end{Bmatrix} = [T_2] \begin{Bmatrix} w_{,nn} \\ w_{,ns} \\ w_{,ss} \end{Bmatrix}$$

$$note: \cos\theta = \sin\beta_i \quad \sin\theta = -\cos\beta_i$$

$$for\ option 2,\ for\ edge\ 1-2\ (k-l)\ as\ an\ inclined\ edge,\ the\ followin\ gransformation\ will\ apply$$

$$\underbrace{\{w\}_{xy}}_{18\times1} = \begin{bmatrix} [Q] & [0] & [0] \\ [0] & [Q] & [0] \\ [0] & [0] & [I] \end{bmatrix} \underbrace{\{w\}_{ns}}_{18\times1} \quad [Q]_{6\times6} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & [T_1] & [0] \\ 0 & [0] & [T_2] \end{bmatrix}_{6\times6} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \end{bmatrix}$$

$$[\overline{K}]_{18\times18} = [Q_B]^T{}_{18\times18}[K]_{18\times18}[Q_B]_{18\times18}$$

9- Two-Dimensional Creeping Flow

$$I = \frac{\nu}{2}\iint_\Omega \left(\nabla^2\Psi\right)^2 d\Omega \qquad \Psi = streamline\ function$$

$$\delta I = \nu\iint_\Omega \nabla^2\Psi\,\delta\left(\nabla^2\Psi\right)d\Omega = \nu\iint_\Omega \nabla^2\Psi\left(\delta\psi_{xx}+\delta\psi_{yy}\right)d\Omega$$

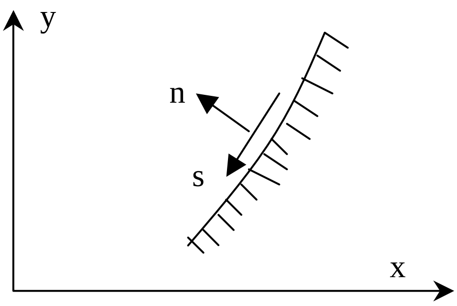
$$\delta I = \nu\oint \nabla^2\Psi\,\delta\psi_n ds - \nu\iint_\Omega \left[\left(\nabla^2\Psi\right)_{,x}\delta\psi_x + \left(\nabla^2\Psi\right)_{,y}\delta\psi_y\right]d\Omega$$

$$\delta I = \nu\oint \nabla^2\Psi\,\delta\psi_n ds - \nu\oint \left(\nabla^2\Psi\right)_{,n}\delta\psi ds + \nu\iint_\Omega \left[\left(\nabla^2\Psi\right)_{,xx} + \left(\nabla^2\Psi\right)_{,yy}\right]\delta\psi d\Omega$$

field eqn

$$\nu\left[\left(\nabla^2\Psi\right)_{,xx} + \left(\nabla^2\Psi\right)_{,yy}\right] = \nu\left[\Psi_{xxxx} + 2\Psi_{xxyy} + \Psi_{yyyy}\right] = 0 \quad or \quad \nu\nabla^4\psi = 0$$

Boundary conditions:



$either: \quad \nu\nabla^2\psi = 0 \quad or \quad \delta\psi_n = 0 \quad on\ S$

$either: \quad \nu\left(\nabla^2\psi\right)_{,n} = 0 \quad or \quad \delta\psi = 0 \quad on\ S$

$$u_n = \frac{\partial\psi}{\partial s} \quad u_s = -\frac{\partial\psi}{\partial n}$$

on solid boundary

$u_n = u_s = 0$

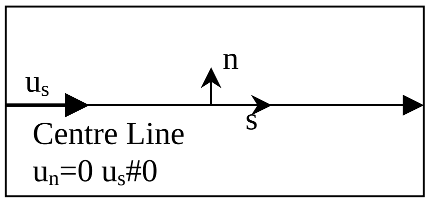
$\therefore \quad \delta(\psi_n) = 0$

$$also \quad \nu\nabla^2\psi = \nu(\psi_{nn} + \psi_{ss}) \quad \psi_{ss} = 0 \quad for\ straight\ edge \therefore \quad \nu\nabla^2\psi = \mu\Big(-\frac{\partial u_s}{\partial n} + \underbrace{\frac{\partial u_n}{\partial s}}_{=0}\Big) = -\tau_{ns} \neq 0$$

$$Along\ a\ centreline:\ u_n = 0 \quad u_s \neq 0 \quad \therefore \quad \delta(\psi_n) = 0 \quad hence \quad \nu\nabla^2\psi = 0 \quad \nabla^2\psi = w = 0 (vorticity\ is\ zero)$$

look at:

$\nu\left(\nabla^2\Psi\right)_{,n} \quad or \quad \mu\left(\nabla^2\Psi\right)_{,n}$



from momentum equations:

$p_x = \mu\nabla^2 u \qquad p_y = \mu\nabla^2 v \quad (u = u_x \quad and \quad v = u_y)$

$\hat{e}_n = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\hat{e}_s = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$$\mu\nabla^2\psi_n = \mu\nabla^2(\psi_x\cos\theta + \psi_y\sin\theta) = \mu\nabla^2(-v\cos\theta + u\sin\theta) = \mu\nabla^2 v\cos\theta + \mu\nabla^2 u\sin\theta$$

$$\mu\nabla^2\psi_n = -p_x\cos\theta + p_y\sin\theta = \overline{\nabla}p.\hat{e}_s = \frac{\partial p}{\partial s} \therefore \quad \mu\nabla^2\psi_n = \frac{\partial p}{\partial s}$$

$\frac{\partial p}{\partial s} =$ P*ressure drop across a wake, s normal to the wake or free surface*

the second boundary integral reveals that if ψ *is not fixed along a boundary then* $\delta\psi \neq 0$ *along s:*

for $\delta\psi \neq 0$ *then* $\frac{\partial p}{\partial s} = \mu\nabla^2\psi_n = 0$

9.1- Fully Developed Parallel Flow

$$U = \frac{\partial \Psi}{\partial y} = 6y(1-y)$$

$$V = 0$$

$$\psi = 3y^2 - 2y^3 \quad \psi(0) = 0 \quad \psi(1) = 1$$

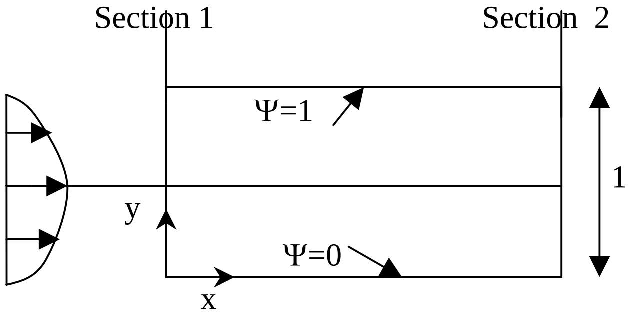
Bc's

$\psi = 3y^2 - 2y^3$ *and* $\Psi_x = 0$ *on section one*

$\psi = 0$ *and* $\Psi_y = 0$ *at bottom edge*

$\psi = 1$ *and* $\Psi_y = 0$ *on top edge*

$\psi_x = 0$ *on section two*



9.2- Flow Past a Cylinder

Computational domain=20xR away, flow can be assumed uniform
Bc,s:

Bc's

$\psi = u_0 y$ *and* $\Psi_x = 0$ *on section one*

$\psi = 0$ *at bottom along x*

$\psi_{,y} = u_0$ *and* $\Psi = 10\,u_0$ *on top edge*

$\psi_x = 0$ *on section two*