Chapter 4

Finite Element Analysis of Steady-State Field problems

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1- Steady-State Field Problems (Quasi-Harmonic Equations)

1.1- Quasi-harmonic Steady State Field Problem

Quasi-harmonic steady state field eqn is given by:

Equation 1

$$\frac{\partial}{\partial x}(K_x \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}(K_y \frac{\partial \phi}{\partial y}) + \frac{\partial}{\partial z}(K_z \frac{\partial \phi}{\partial z}) + f(x,y,z) = 0$$

where $\phi(x,y,z)$ is the field variable to be determined in a three dimensional domain Ωbounded by surface Γ. K_x, K_y and K_z are given functions of space coordinates only and are independent of ϕ (i.e. linear problem).

The description of the field problem is not complete until boundary conditions are specified. Let these be:

Equation 2

$$\phi = \phi_B \qquad on\ \Gamma_B$$

$$K_x \frac{\partial \phi}{\partial x} n_x + K_y \frac{\partial \phi}{\partial y} n_y + K_z \frac{\partial \phi}{\partial z} n_z + g(x,y,z) + h(x,y,z)\phi = 0 \quad on\ \Gamma_B$$

Where g(x,y,z) and h(x,y,z) are known a priori and n_x, n_y and n_z are the direction cosines of the unit outward normal to the surface. Γ_A and Γ_B are parts of the boundary, i.e. $\Gamma_A+\Gamma_B=\Gamma$, the total boundary.

Boundary condition in above equation is known as the dirichlet condition and ϕ_B, the dirichlet data. Equation 2(b) represents the Cauchy boundary condition.

If g=h=0, the Cauchy condition reduces to the Neumann boundary condition, also called the Natural boundary condition.

A field problem is said to have mixed boundary conditions when some portions of the boundary have Dirichlet boundary conditions while the other portions have Cauchy or Neumann boundary conditions.

Physical interpretation of the parameters in eqn 1 depends upon the particular physical problem and listed in the table below:

Identification of Physical Parameters

Problem	**ϕ**	**K_x, K_y and K_z**	**f**	**g**	**H**
Diffusion flow in porous media	**Hydraulic head**	**Hydraulic conductivity**	**Internal sources flow**	**Boundary flow**	**-**
Heat conduction	**Temperature**	**Thermal conductivity**	**Internal heat generation**	**Boundary heat generation**	**Convective heat transfer coefficient**
Irrotational flow	**Velocity potential or stream function**	**-**	**0**	**Boundary velocity**	**0**
Torsion	**Stress Function**	**Reciprocal of shear modulus**	**Angle of twist per unit length**	**-**	**-**
Seepage	**Pressure**	**Permeability**	**Internal flow**	**-**	**-**

1.2- Variational Principle

Variation principle for equation 1 and 2 is given by:

Equation 3

$$J(\phi)=\frac{1}{2}\int_{\Omega}\left[K_x(\frac{\partial\phi}{\partial x})^2+K_y(\frac{\partial\phi}{\partial y})^2+K_z(\frac{\partial\phi}{\partial z})^2-2f\phi\right]d\Omega+\int_{\Gamma_a}(g\phi+\frac{1}{2}h\phi^2)d\Gamma$$

It can be shown that $\delta J(\varphi)=0$ yields the Euler equations which are the same as the equations 1 and 2.

Note: there are some slight modifications involved when eqns1 to 3 are applied to a particular physical problem.

2- Two Dimensional Steady-State Heat Flow

Governing differential equation:

$$\frac{\partial}{\partial x}(k_x \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}(K_y \frac{\partial \phi}{\partial y}) + Q(x, y) = 0 \qquad in \quad \Omega$$

subject to boundary conditions :

Equation 4

$$\phi = \phi_B \qquad on\, \Gamma_B$$

$$K_x \frac{\partial \phi}{\partial x} n_x + K_y \frac{\partial \phi}{\partial y} n_y + \overline{q}_A = 0 \quad on\, \Gamma_A$$

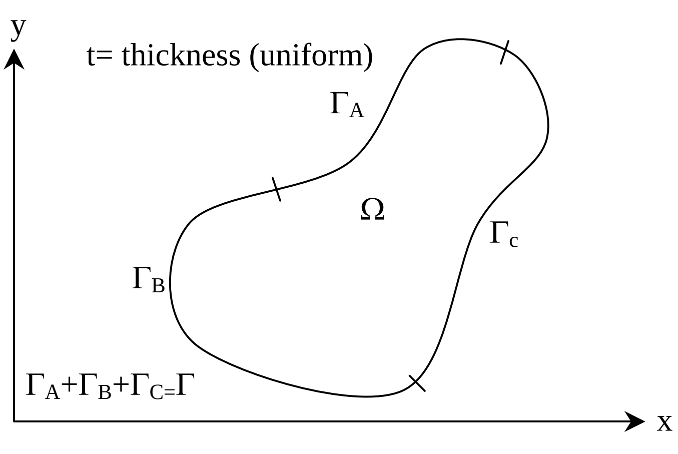
$$K_x \frac{\partial \phi}{\partial x} n_x + K_y \frac{\partial \phi}{\partial y} n_y + \overline{q}_c + \alpha(\phi - \overline{\phi}_c) = 0 \quad on \quad \Gamma_c$$

where ϕ = temperature
K_x=Thermal conductivity in x-direction
K_y= Thermal conductivity in y-direction
Q=Heat input per unit volume
$\underline{q}_a$ and $\underline{q}_c$ = specified heat input per unit area on Γ_A and Γ_C, respectively.
α=Convective heat transfer coefficient
$\underline{\phi}_c$=ambient temperature of the environment

Variational principle in two dimensions with thickness "t" take the following form:

$$J(\phi) = t\int_{\Omega}\left[\frac{1}{2}\left[K_x(\frac{\partial \phi}{\partial x})^2 + K_y(\frac{\partial \phi}{\partial y})^2\right] - Q\phi\right]dxdy + t\int_{\Gamma_A}\overline{q}_A\phi\, d\Gamma + t\int_{\Gamma_C}\left[\overline{q}_C + \alpha(\frac{\phi}{2} - \overline{\phi}_c)\right]\phi\, d\Gamma$$

Equation 4-b for boundary condition on $\Gamma_A$is valid only for transfer of heat through conduction.



Comment: Since temperature ϕ is a scalar quantity, no transformation of matrices (computed in local coordinates to global coordinates) is necessary before assembling the global matrix.

2.1- Heat Transfer Matrix

Assume finite element approximation for ϕ as:

Equation 5

$$\phi = \sum_{i=1}^{n} N_i \phi_i$$

Where N_i are the shape functions, ϕ_I are the nodal values of ϕ, n is the number of nodes per element.

Finite element approximation is required to have only C_0 continuity. That is only ϕ needs to be continuous and no derivatives of it are required to be continuous.

For the interior elements we do not have to consider the boundary integrals. Hence, for interior elements:

Equation 6

$$J_e(\phi) = t\int_A \left[\frac{1}{2}\left[K_x (\frac{\partial \phi}{\partial x})^2 + K_y (\frac{\partial \phi}{\partial y})^2 \right] - Q\phi \right] dxdy$$

Substitution of eqn 5 into eqn 6 yields:

$$J_e(\phi_i^e) = t\int_A \left[\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \left[K_x N_{i,x} N_{j,x} \phi_i \phi_j + K_y N_{i,y} N_{j,y} \phi_i \phi_j \right] - \sum_{i=1}^{n} QN_i \phi_i \right] dxdy$$

$$J_e(\phi_i^e) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} H_{ij}{}^e \phi_i \phi_j - \sum_{i=1}^{n} f_i^e \phi_i$$

$where:$

$$H_{ij}{}^e = t\iint_A \left[K_x N_{i,x} N_{j,x} + K_y N_{i,y} N_{j,y} \right] dxdy$$

$$f_i^e = t\iint_A QN_i dxdy$$

$$\delta(J_e(\phi_i^e)) = 0 \ \ for\ stationary\ then\ leads\ to:$$

$$[H^e]\{\phi^e\} - \{f^e\} = \{0\}$$

$where:$

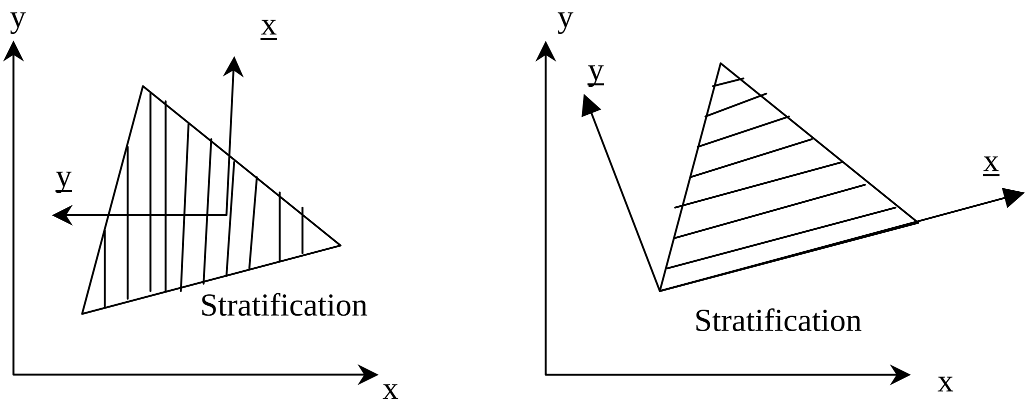
$$\{\phi^e\}^T = \left[\phi_1^e \quad \phi_2^e \quad . \quad . \quad \phi_n^e \right]$$

$$\{f^e\}^T = \left[f_1^e \quad f_2^e \quad . \quad . \quad f_n^e \right]$$

We can use the shape functions developed for Isoparametric elements earlier in order to compute $[H^e]$ and $\{f^e\}$ above.

2.2- Anisotropic and Non-homogeneous Media

The material properties K_x and K_y can vary from element to element in a discontinuous manner. Also the material properties are known only with respect to principle axes (or axes of symmetry) which can change direction from element to element as well. If these properties and direction are reasonably constant within the element, then the element heat transfer matrix can be formulated in local axes which is coincide with the principle (or symmetry) axes shown in the figure.



Then:

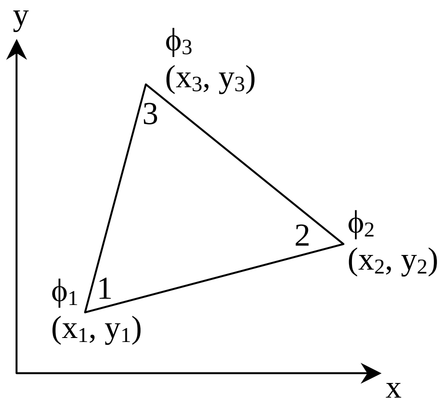
$$H_{ij}^{e} = t \iint_{A} \left[K_{\bar{x}} N_{i,\bar{x}} N_{j,\bar{x}} + K_{\bar{y}} N_{i,\bar{y}} N_{j,\bar{y}} \right] d\bar{x} d\bar{y}$$

The only dufference is that the derivatives of N_i are now taken with respect to $\underline{x}$ and $\underline{y}$, the local coordinates.

Again as commented before, there is no transformation needed from $\underline{x}$ and $\underline{y}$ axes to x-y axes because ϕ is a scalar quantity. This then leads to a considerable economy in computations.

2.3-Formulation of Linear Temperature Triangular Elements

For now assume we know the material properties K_x and K_y along x and y axes as shown.



For linear temperature variation within the element, using area coordinatea:

Equation 7

$$N_1 = L_1 \qquad N_2 = L_2 \qquad N_3 = L_3$$

$$\phi = L_1\phi_1 + L_2\phi_2 + L_3\phi_3$$

$$N_{i,x} = \frac{b_i}{2A} \qquad N_{i,y} = \frac{a_i}{2A} \quad \textit{where } A = \textit{area of the triangular element in the figure}$$

$$b_1 = y_2 - y_3 \qquad b_2 = y_3 - y_1 \qquad b_3 = y_1 - y_1$$

$$a_1 = x_3 - x_1 \qquad a_2 = a_1 - a_3 \qquad a_3 = x_2 - x_1$$

$$H_{ij}{}^{e} = \frac{t}{4A^2}\iint_A \left[K_x b_i b_j + K_y a_i a_j\right] dxdy$$

For isotropic material properties $K=K_x=K_y$

$$H_{ij}{}^{e} = \frac{Kt}{4A}\left[b_i b_j + a_i a_j\right]$$

Further, if we are dealing with an isotropic and nonhomogeneous material, then the coordinates in system are used in equation 7(d,e).

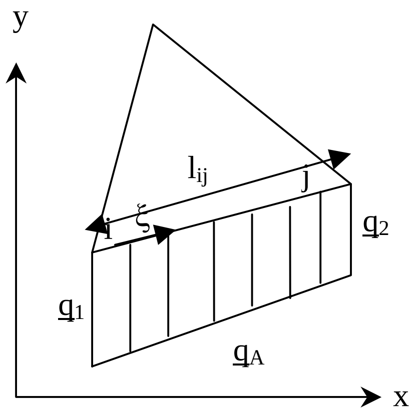
2.4- Heat Input Load Vector

From equation of potential energy, heat input load vector consists of internal heat generated Q, heat input on Γ_A given by $\underline{q}_A$, on Γ_C the amount of $\underline{q}_C$ and $-\alpha\underline{\phi}_c$, i.e.

$$-t\iint_A Q\phi dxdy + t\int_{\Gamma_A} \bar{q}_A \phi d\Gamma + t\int_{\Gamma_c} \left[\bar{q}_C - \alpha\bar{\phi}_c)\right]\phi \, d\Gamma$$

Obviously, the contribution from line integrals comes only from that part of boundary Γ i.e. Γ_A and Γ_C where $\underline{q}_A$, $\underline{q}_C$ and $\underline{\phi}_c$ are specified.
For Q constants:

$$\{f_1^e\} = -QA\begin{Bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{Bmatrix}$$



for elements with edge along Γ_A :

along $i-j$ *edge* $\phi(\xi) = (1-\xi)\phi_1 + \xi\phi_2$

$$\bar{q}_A(\xi) = \bar{q}_1(1-\xi) + \bar{q}_2\xi$$

$$t\int_{\Gamma_A} \bar{q}_A \phi d\Gamma = tl_{ij}\int_0^1 \bar{q}_A(\xi)\,\phi(\xi)\,d\xi$$

on integration

$$\{f_2^e\} = \frac{tl_{ij}}{6}\begin{Bmatrix} 2\bar{q}_1 + \bar{q}_2 \\ \bar{q}_1 + 2\bar{q}_2 \\ 0 \end{Bmatrix}$$

Similarly, for elements with edge along Γ_C :

$$\{f_3^e\} = \frac{tl_{ij}}{6}\begin{Bmatrix} 2g_1 + g_2 \\ g_1 + 2g_2 \\ 0 \end{Bmatrix}$$

$$g(\xi) = \bar{q}_C(\xi) - \alpha\bar{\phi}_c(\xi)$$

where both $\bar{q}_C$ *and* $\alpha\bar{\phi}_c$ *are assumed to have linear variation along edge* $i-j$:

$$g_1 = g(0) \quad g_2 = g(0)$$

Note if $\alpha=0$ then $\{f_2^e\}$ and $\{f_3^e\}$ are the same.
Also along Γ_C, we have to calculate $\delta\{\frac{t}{2}\int_{\Gamma} \alpha\phi^2 d\Gamma\} = \alpha t\int_{\Gamma_C} \phi\delta\phi d\Gamma$. This term yields contribution to H_{ij}. Then:

$$H_{ij} = tl_{ij}\int_0^1 \alpha N_i^C N_j^C d\xi$$

where N_i^C are shape function along edge i-j of the element that coincides with Γ_C.

$$[H^C]_{3\times3} = \frac{\alpha t l_{ij}}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In deriving above equation, $N_1=1-\xi$, $N_2=\xi$ and $N_3=0$

Above matrix is equivalent to having a line spring boundary in plane elasticity problem. $[H^C]$ is added to[H] for the element on boundary Γ_C to obtain the complete Heat Transfer Matrix, just as we did for plane elasticity case with spring boundaries in obtaining the complete stiffness matrix.

Once the heat transfer matrices and input load vectors have been determined, these can be assembled in exactly the same manner as stiffness and load vector matrices in plane elasticity. The kinematic boundary conditions or fixed boundary conditions on ϕ can be easily incorporated.

Number of constraints option can be incorporated for both zero and non-zero ϕ on the boundary, i.e. $\phi=\phi_B$ on Γ_B.

2.5- Advantages of Finite Element Method for Field Problem

1. It can deal simply with non-homogeneous and anisotropic situations (particularly when the direction of anisotropy is variable)
2. The elements can be graded in shape and size to follow arbitrary boundaries and to allow for regions of rapid variation of the function sought.
3. Specified gradient or radiation boundary condition are introduced naturally and with a better accuracy than in standard finite difference procedures.
4. Higher order elements can be readily used to improve accuracy without complicating boundary condition- a difficulty always arising with finite difference approximations of a higher order.
5. Finally, but of considerable importance in computer age, standard (structural) programs may be used for assembly and solution.

3- Transient two-Dimensional Heat Flow

The time dependent governing differential equation is:

$$\frac{\partial}{\partial x}(k_x \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial y}(K_y \frac{\partial \phi}{\partial y}) + Q(x,y) - C\frac{\partial \phi}{\partial t} = 0 \qquad in \quad \Omega$$

where ϕ is a function of x, y and time t. Boundary conditions are still given by equations 4, except these can vary with time. Equivalent steady state variational principle for any time t is then given by:

$$J(\phi) = t\int_{\Omega}\left[\frac{1}{2}\left[K_x(\frac{\partial \phi}{\partial x})^2 + K_y(\frac{\partial \phi}{\partial y})^2\right] - Q\phi + \frac{C}{2}\dot{\phi}\phi\right]dxdy + t\int_{\Gamma_A}\bar{q}_A\phi\, d\Gamma + t\int_{\Gamma_C}\left[\bar{q}_C + \alpha(\frac{\phi}{2} - \bar{\phi}_c)\right]\phi\, d\Gamma$$

$$note: \dot{\phi} = \frac{\partial \phi}{\partial t}$$

finite element approximation within an element is now chosen as:

$$\phi(x,y,t) = \sum_{i=1}^{n} N_i(x,y)\phi_i(t)$$

where nodal variable $\phi_i(t)$ are now functions of time. Since we look for stationary of J(ϕ) at any time t, i.e.:

$$\delta\ J(\phi) = 0 \quad at\ any\ time\ t$$

Therefore matrices $[H^e]$ and $[H^C]$, the element heat transfer matrix and contribution to it from boundary integral on Γ_C, are still the same. Load vector $\{f_1^e\}$, $\{f_2^e\}$ and $\{f_3^e\}$ may vary with time. However a new matrix needs to be derived, i.e. element heat capacity matrix $[C^e]$ from $\frac{t}{2}\iint_A C\dot{\phi}\phi dxdy$

$$c_{ij}^{\ e} = t\iint_A CN_i(x,y)N_j(x,y)dxdy$$

In terms of shape function:

$$N_1 = L_1 \qquad N_2 = L_2 \qquad N_3 = L_3$$

$$c_{ij}^{\ e} = t\iint_A CL_iL_j dA$$

$$finally\ after\ integration(\ for\ constant\ C)$$

$$[C^e] = \frac{CAt}{12}\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}$$

After assembling individual element matrices, the final discretized global equation take the following form:

$$[H]\{\phi\} + [C]\{\dot{\phi}\} + \{F\} = \{0\}$$

[H]=global or master conductivity or heat transfer matrix
[C]=global heat capacity matrix
{F}=global heat input load vector

Assume the material properties involved K_x, K_y, α, C do not change when temperature changes with time, i.e. we have a linear problem. Further, at t=0 the initial conditions are generally given, i.e.:

$$\phi(x, y, 0) = \phi_0(x, y)$$

A numerical recurrence process is now required to find the solution at subsequent times. Finite differences in time are employed to obtain such a recurrence formula.

Approximate of above equation by finite differences in interval t to t+Δt can be written for mid interval as:

Equation 8

$$[H]\{\phi\}_{t+\Delta t} + [C]\underbrace{\left(\{\phi\}_{t+\Delta t} - \{\phi\}_t\right)/\Delta t}_{\{\dot{\phi}\}_{t+\Delta t}} + \{F\}_{t+\Delta t} = \{0\}$$

Where [H], [C] (if variable with ϕ) and {F} are assigned their mid interval values, and $\{\dot{\phi}\}$ has been replaced by:

Equation 9

$$\{\dot{\phi}\}_{t+\Delta t} = \frac{\{\phi\}_{t+\Delta t} - \{\phi\}_t}{\Delta t}$$

Also note for linear variation within the time interval

$$\{\phi\}_{t+\frac{\Delta t}{2}} = \frac{1}{2}\left(\{\phi\}_{t+\Delta t} + \{\phi\}_t\right)$$

i.e. as an average value

$$\{\phi\}_{t+\Delta t} = 2\{\phi\}_{t+\frac{\Delta t}{2}} - \{\phi\}_t$$

Substituting equation 9c in equation 8:

$$\left([H] + \frac{2}{\Delta t}[C]\right)\{\phi\}_{t+\frac{\Delta t}{2}} = \frac{2}{\Delta t}[C]\{\phi\}_t - \{F\}_{t+\Delta t}$$

$$\{\phi\}_{t+\frac{\Delta t}{2}} = \left([H] + \frac{2}{\Delta t}[C]\right)^{-1}\left(\frac{2}{\Delta t}[C]\{\phi\}_t - \{F\}_{t+\Delta t}\right)$$

And from equation 9c, we can calculate $\{\phi\}_{t+\Delta t}$. Equation 8 and 9c provide the recurrence process sought.

If K_x, K_y, C and α depend on temperature, then the problem becomes nonlinear and some special iterative techniques are required for a solution.