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Spectral Controllability of Some Singular Hyperbolic Equations on Networks

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Abstract The purpose of this paper is to address the question of well-posedness and spectral controllability of the wave equation perturbed by potential on networks which may contain unbounded potentials in the external edges. It has been shown before that in the absence of any potential, there exists an optimal time T^* (which turns out to be simply twice the sum of all length of the strings of the network) that describes the spectral controllability of the system. We will show that this holds in our case too, i.e., the potentials have no effect on the value of the optimal time T^*. The proof is based on the famous Beurling-Malliavin's Theorem on the completeness interval of real exponentials and on a result by Redheffer who had shown that under some simple condition the completeness interval of two complex sequences are the same.

Keywords Network · Spectral controllability · Approximate controllability · Spectrum of an operator

Mathematics Subject Classification (2010) 35K65 · 93B05 · 93B07

1 Introduction

In the recent years, considerable efforts have been devoted to the mathematical study of mechanical systems constituted by coupled flexible or elastic elements as strings, beams, membranes, or plates. These systems are known as multi-link or multi-body structures.

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Their practical relevance is huge. However, the mathematical models describing their evolution are generally quite complex. They can be viewed as systems of partial differential equations (PDE) on networks or graphs.

This paper is mainly devoted to analyze the vibrations of a simplified 1-d model of a multi-body structure consisting of a finite number of flexible strings distributed along a planar graph. Deformations are assumed to be perpendicular to the reference plane.

For a general graph, if one considers wave equation on each edge of the graph, it was shown in [6] that when the time is greater than twice the total lengths, i.e., $T > 2L$, one can deduce that there exist some Fourier weights so that the observation property holds in the corresponding weighted norm if and only if all the eigenfunctions of the network are observable. Furthermore, for times smaller than $2L$ system is not spectrally controllable.

The main goal of this paper is to give a necessary and sufficient condition for the spectral controllability of the wave equation perturbed by potential on networks. The potentials could be singular on exterior vertices. The importance of this singularity emanates form the fact that in 1-d case, the so-called inverse-square potential arises, for example, in the context of combustion theory [3, 5, 8, 10] and quantum mechanics [1, 7, 15]. We consider two cases:

(1) All of the potentials are bounded on each edge, but they do not need to be positive.

(2) We let the potentials on the external edges unbounded but positive. For simplicity, we assume that they behave like $1/x^{\alpha}$ for $\alpha \in (0, 1]$. We consider the bounded potentials on the internal edges too.

It will be shown that in two cases above, there is an optimal time T^* that describe the spectrally controllability of the system. That is for every $T > T^*$ system is spectral controllable if and only if all the eigenfunctions of the system are observable. Whereas for $T < T^*$ system is not spectrally controllable. This optimal time depends only on the length of the edges and is equal to the optimal time for the spectral controllability of the corresponding system without any potential. In other words, the potentials (even the singular ones in the external edges) have no effect on the value of the optimal time for spectral controllability. In fact, this is a consequence of the celebrated Buerling-Malliavin's Theorem [4] and a result proved by Haraux and Jaffard [12] that relates the spectral controllability of the system to the asymptotic behavior of the eigenvalues of the network.

The paper is organized as follows. In Section 2, we state some elements of modeling and introduce the problem. Well-posedness of the system will be discussed in Section 3. In Section 4, we study the spectrum of the operator A corresponding to the system and as a consequence, we derive a necessary and sufficient condition for the spectral controllability for large enough time T.

2 Networks of Strings

2.1 Elements on Graphs

A graph G is a pair $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set, whose elements are called vertices of G, and $\mathcal{E}$ is a family of non-ordered pairs v, w of vertices, which we denote by $v\hat{w}$. The elements of $\mathcal{E}$ are called edges of G. When G does not contain edges of the form $v\hat{v}$, it is said that the graph is simple.

Let us suppose that G has a finite number of $N \geq 2$ vertices and $M \geq 1$ edges:

$$\mathcal{V} = \{v_1, \dots, v_N\}, \qquad \mathcal{E} = \{e_1, \dots, e_M\}.$$

We denote $deg(v)$ as the degree of the vertex v in the graph G. We also define the sets

$$\mathcal{V}_{\mathcal{S}} := \{v \in \mathcal{V} : deg(v) = 1\}, \quad \mathcal{V}_{\mathcal{M}} := \mathcal{V} \setminus \mathcal{V}_{\mathcal{S}},$$

where $\mathcal{V}_{\mathcal{S}}$ is the set of those vertices that belong to a single edge, the exterior ones, while $\mathcal{V}_{\mathcal{M}}$ contains the remaining vertices, the interior ones, i.e., those that belong to more than one edge.

For a vertex v, we denote by

$$I_v := \{i : v \text{ is a vertex of} e_i\},$$

the set of indices of all those edges of G which are incident to v. If the vertex v_j is exterior, I_{v_j} contains a single index; it will be denoted by $i(j)$ and, if this does not lead to misunderstanding, simply by i. Also let us define

$$I_{ext} := \{i : \exists v \in \mathcal{V}_{\mathcal{S}}, i \in I_v\}, \qquad I_{int} := I \setminus I_{ext}.$$

In this paper, we consider only simple finite and connected graphs whose edges are viewed as rectilinear segments joining some of those points. The length of the segment corresponding to the edge e_i is called length of e_i and is denoted by ℓ_i. On every edge of G, we choose an orientation (that is, one of the vertices has been chosen as the initial one). Then, e_i may be parametrized as a function of its arc length by means of the functions $x_i : [0, \ell_i] \to e_i$. For $i \in I_{ext}$, we choose the orientation x_i such that $x_i(0)$ shows the exterior vertex. We define the incidence matrix of G

$$\epsilon_{ij} = \begin{cases} -1 \ \text{if} \, x_i(0) = v_j, \\ \\ +1 \ \text{if} \, x_i(\ell_i) = v_j. \end{cases}$$

Given functions $u^i : [0, \ell_i] \to \mathbb{R}$, $i = 1, \dots, M$, we will denote by $\bar{u} : G \to \mathbb{R}$ the function defined for $x \in e_i$ by

$$\bar{u}(x) = u^i(x_i^{-1}(x)).$$

In this case, we say that $\bar{u}$ is a function defined on the graph G with components u^i. Frequently, we indicate this fact just by writing $\bar{u} = (u^1, \dots, u^M)$.

2.2 Equations of Motion for Networks

Now, we consider a network of elastic strings that undergo small perpendicular vibrations. Let us suppose that the function $u^i = u^i(t, x) : \mathbb{R} \times [0, \ell_i] \to \mathbb{R}$, describes the transversal displacement in time t of the string that coincides at rest with the edge e_i. Then, for every $t \in R$, the functions u^i, $i = 1, \dots, M$, define a function $\bar{u}(t, x)$ on G with components u^i defined by $u^i(t, x) = u^i(t, x_i^{-1}(x))$ for $x \in e_i$. This function allows to identify the network with its rest graph.

As a model of the motion of the network, we assume that the displacements u^i satisfy the following non-homogeneous system with some conditions on the exterior vertices

$$\begin{cases} u^i_{tt} - u^i_{xx} + b_i(x)u^i = 0, & (t, x) \in \mathbb{R} \times [0, \ell_i], i = 1, \dots, M \\ u^{i(j)}(t, v_j) = h_j(t), & t \in \mathbb{R}, v_j \in \mathcal{C}, \\ u^{i(j)}(t, v_j) = 0, & t \in \mathbb{R}, v_j \in \mathcal{V}_{\mathcal{S}} \setminus \mathcal{C}, \\ u^i(t, v) = u^j(t, v), & t \in \mathbb{R}, v \in \mathcal{V}_{\mathcal{M}}, i, j \in I_v, \\ \sum_{i \in I_v} \partial_n u^i(t, v) = 0, & t \in \mathbb{R}, v \in \mathcal{V}_{\mathcal{M}}, \\ u^i(0, x) = u^i_0(x), u^i_t(0, x) = u^i_1(x), & x \in [0, \ell_i]. \end{cases} \tag{1}$$

Indeed, conditions in second and third lines in Eq. 1 reflect the fact that over some of the exterior nodes, precisely over those corresponding to the vertices contained in $\mathcal{C} = \{v_1, \ldots, v_r\}$ (i.e., the set of controlled nodes), some controls act to regulate their displacements, while the remaining nodes are fixed. Also, $\partial_n u^i(t, v) = \epsilon_{ij} u^i_x(t, x_i^{-1}(v))$ is the exterior normal derivative of u_i at the node v.

For the functions b_i, we consider one of these cases:
(1) For every $i = 1, \ldots, M$, we have $b_i \in L^\infty(0, \ell_i)$.
(2) For every $i \in I_{int}$, we have $b_i \in L^\infty(0, \ell_i)$ and for every $i \in I_{ext}$ the non-negative potentials b_i are unbounded near the exterior vertices. Here, we consider a singularity of the order $1/x^\alpha$ on the exterior vertices for some $\alpha \in (0, 1]$, i.e. $b_i(x) = O(\frac{1}{x^\alpha})$ near the exterior vertices.

In order to study system (1), we need a proper functional setting. We define the Hilbert spaces

$$V := \{\bar{u} \in \prod_{i=1}^{M} H^1(0, \ell_i) : u^i(v) = u^j(v) \text{ if } i, j \in I_v \text{ for } v \in \mathcal{V}_{\mathcal{M}},\ u^i(v) = 0 \text{ if } i \in I_v \text{ for } v \in \mathcal{V}_{\mathcal{S}}\},$$

$$H := \prod_{i=1}^{M} L^2(0, \ell_i),$$

endowed with the Hilbert structures

$$< \bar{u}, \bar{w} >_V := \sum_{i=1}^{M} < u^i, w^i >_{H^1(0,\ell_i)} = \sum_{i=1}^{M} \int_0^{\ell_i} u^i_x w^i_x dx,$$

and

$$< \bar{u}, \bar{w} >_H := \sum_{i=1}^{M} < u^i, w^i >_{L^2(0,\ell_i)} = \sum_{i=1}^{M} \int_0^{\ell_i} u^i w^i dx,$$

respectively. Obviously, the space

$$\mathcal{C}_0 := \{\bar{u} \in \prod_{i=1}^{M} C^1(0, \ell_i) : u^i(v) = u^j(v) \quad \text{if} \quad v \in \mathcal{V}_{\mathcal{M}}, \quad supp\, \bar{u} \cap \mathcal{V}_{\mathcal{S}} = \emptyset\},$$

is dense in H and V. Besides, we will denote

$$U = (L^2(0, T))^r,$$

the space of controls.

Remark 2.1 Since the elements of V are zero at the exterior vertices, we can write on a path on graph starting from an exterior vertex

$$|u(x)| \le \int_0^x |u_x| dx \le \int_0^\ell |u_x| dx.$$

Then, the Poincare inequality will be valid on different paths starting from an exterior vertex and one can easily deduce that the following Poincare inequality for the elements of V (note that the graph is connected and the paths may have intersection)

$$\sum_1^M \int_0^{\ell_i} u_{i,x}^2 dx \ge C \sum_1^M \int_0^{\ell_i} u_i^2 dx, \qquad \forall \bar{u} \in V,$$

which shows that the embedding $V \subset H$ is continuous and compact.

3 Well-posedness

The existence of the solution of system (1) could be studied in the standard way by the classical transposition method. Let us describe the main steps. First, we study the homogeneous problem ($h_j \equiv 0$ for all $j = 1, \dots, r$)

$$\begin{cases} \phi^i_{tt} - \phi^i_{xx} + b_i(x)\phi^i = 0, & (t,x) \in \mathbb{R} \times [0, \ell_i], i = 1, \dots, M \\ \phi^{i(j)}(t, v_j) = 0, & t \in \mathbb{R}, v_j \in \mathcal{V}_{\mathcal{S}}, \\ \phi^i(t,v) = \phi^j(t,v), & t \in \mathbb{R}, v \in \mathcal{V}_{\mathcal{M}}, i, j \in I_v, \\ \sum_{i \in I_v} \partial_n \phi^i(t,v) = 0, & t \in \mathbb{R}, v \in \mathcal{V}_{\mathcal{M}}, \\ \phi^i(0,x) = \phi^i_0(x), \phi^i_t(0,x) = \phi^i_1(x), & x \in [0, \ell_i]. \end{cases} \tag{2}$$

Define the operator $A : V \to V'$ by

$$< A\bar{u}, \bar{v} >= \sum_{i=1}^{M} \int_0^{\ell_i} u^i_x v^i_x + b_i(x) u^i v^i dx.$$

We have two following Lemmas.

Lemma 3.1 *The operator $A : V \to V'$ is bounded.*

Proof Note that in case 1, boundedness of A is clear since all b_i's are bounded. But in case 2, we first need to prove a Hardy inequality on external edges:
For every $i \in I_{ext}$ and $\bar{u} \in \mathcal{C}_0$, one can write

$$\int_0^{\ell_i} (u^i_x - \frac{1}{2}\frac{u^i}{x})^2 dx \geq 0,$$

thus

$$\int_0^{\ell_i} (u^i_x)^2 - \frac{1}{2}\frac{((u^i)^2)_x}{x} + \frac{1}{4}\frac{(u^i)^2}{x^2} dx \geq 0.$$

Then by using the integration by parts, we will have

$$\int_0^{\ell_i} (u^i_x)^2 - \frac{1}{4}\frac{(u^i)^2}{x^2} \geq \frac{1}{2}\frac{(u^i)^2}{x}|_0^{\ell_i}.$$

Note that u^i is equal to zero near the exterior vertices, so

$$\int_0^{\ell_i} (u^i_x)^2 dx \geq \frac{1}{4}\int_0^{\ell_i} \frac{(u^i)^2}{x^2} dx. \tag{3}$$

Thus for every $\bar{u}, \bar{v} \in V$ and every $i \in I_{ext}$, we have

$$\int_0^{\ell_i} |\frac{u^i v^i}{x^2}| dx \leq (\int_0^{\ell_i} \frac{(u^i)^2}{x^2} dx)^{\frac{1}{2}} (\int_0^{\ell_i} \frac{(v^i)^2}{x^2} dx)^{\frac{1}{2}} \leq C (\int_0^{\ell_i} (u^i_x)^2 dx)^{\frac{1}{2}} (\int_0^{\ell_i} (v^i_x)^2 dx)^{\frac{1}{2}}, \tag{4}$$

Now for every $i \in I_{ext}$ consider $C_i > 0$ such that $0 \leq b_i(x) \leq \dfrac{C_i}{x^2}$. Consequently,

$$\begin{aligned} | < Au, v > | &\leq \sum_{i \in I_{int}} \int_0^{\ell_i} |u^i_x v^i_x| + |b_i(x) u^i v^i| dx + \sum_{i \in I_{ext}} \int_0^{\ell_i} |u^i_x v^i_x| + \frac{C_i}{x^2}|u^i v^i| dx \\ &\leq C\|u\|_V \|v\|_V. \end{aligned}$$

□

Lemma 3.2 *There exists $K \geq 0$ such that the operator $A + KI : V \to V'$ is coercive.*

Proof **Case 1:** $b_i \in L^{\infty}(0, \ell_i)$ for every i. We have

$$< (A + KI)u, u >= \sum_{i=1}^{M} \int_0^{\ell_i} [u_x^2 + b_i(x)(u^i)^2 + K(u^i)^2]dx \geq \|u\|_V^2,$$

for

$$K > Max_{1 \leq i \leq M} \|b_i\|_{\infty}.$$

Case 2: Clearly, for $K > 0$ sufficiently large, one can write

$$< (A + KI)u, u >= \sum_{1}^{M} \int_0^{\ell_i} (u_x^i)^2 dx + \sum_{I_{ext}} \int_0^{\ell_i} b_i(x)(u^i)^2 + K(u^i)^2 dx$$

$$+ \sum_{I_{int}} \int_0^{\ell_i} (b_i(x)(u^i)^2 + K(u^i)^2)dx \geq C \sum_{1}^{M} \int_0^{\ell_i} [(u_x^i)^2 + (u^i)^2] = C\|u\|_V^2,$$

for some $C > 0$.

□

Now let $K > 0$ be as defined in Lemma 3.2, then $A + KI : V \to V'$ is a self-adjoint, bounded and coercive operator, thus by Lax-Milgram Theorem $A + KI : V \to V'$ is an isomorphism. Also, the injection $V \subset H$ is compact, so the restriction $(A+KI)^{-1}|_H$ (when H is identified with its dual H' by means of the Riesz-Frechet isomorphism) is a compact operator . Note that the spectrum of a compact operator, either is finite, or is a sequence that converges to zero. On the other hand, by Lemma 3.2, all of the eigenvalues of $A + KI$ are non-negative, therefore one can deduce that the number of negative eigenvalues of the operator A is finite. For some technical reasons, we assume all of the eigenvalues of A are nonzero.

Remark 3.3 The eigenvalues of A can be represented in the form $\{\xi_n\}_{n \in I^+ \cup I^-}$ where I^+ and I^- are indices with respect to the positive and negative eigenvalues, respectively, and the cardinality of I^- is finite. The corresponding eigenfunctions $\{\theta_n\}_{\{I^+ \cup I^-\}}$ may be chosen to form an orthonormal basis of H.

In what follows, in order to simplify the notations, we suppose that $r = 1$, that is, only one node of the network is controlled, the one corresponding to the index $i = 1$. Obviously, this is the most delicate situation for controllability to hold. When the number of controls increases, the controllability properties of the system are enhanced. Now, the corresponding system of Eq. 1 is

$$\begin{cases} u_{tt}^i - u_{xx}^i + b_i(x)u^i = 0, & (t, x) \in \mathbb{R} \times [0, \ell_i], i = 1, \ldots, M \\ u^1(t, v_1) = h(t), & t \in \mathbb{R}, \\ u^{i(j)}(t, v_j) = 0, & t \in \mathbb{R}, v_j \in \mathcal{V}_{\mathcal{S}} \setminus \{v_1\}, \\ u^i(t, v) = u^j(t, v), & t \in \mathbb{R}, v \in \mathcal{V}_{\mathcal{M}}, i, j \in I_v, \\ \sum_{i \in I_v} \partial_n u^i(t, v) = 0, & t \in \mathbb{R}, v \in \mathcal{V}_{\mathcal{M}}, \\ u^i(0, x) = u_0^i(x), u_t^i(0, x) = u_1^i(x), & x \in [0, \ell_i]. \end{cases} \tag{5}$$

Now for every $t \in (0, T]$ define the operator $A_t : H \times V \to L^2(0, T)$, which associate to every $(\bar{\phi}_1, -\bar{\phi}_0) \in H \times V$ the normal derivative $\partial_n \phi^1(\cdot, v_1)$ in the controlled node of the solution of the system (2).

Proposition 3.4 *The operator $A_t : H \times V \to L^2(0,T)$ is continuous.*

Proof Consider the C^1 function $q : G \to \mathbb{R}$ such that $q(v_1) = -1$ and $q(v_j) = 0$ for other vertices. Now multiply the first equation in Eq. 2 by $-2q\phi_x^i$ and integrate by parts:

$$\begin{aligned}
2\int_0^T\int_0^{\ell_i} b_i q\phi^i\phi_x^i dxdt &= -2\int_0^T\int_0^{\ell_i}(\phi_{tt}^i-\phi_{xx}^i)q\phi_x^i dxdt\\
&= -2\int_0^{\ell_i}\phi_t^i q\phi_x^i dx|_0^T + 2\int_0^T\int_0^{\ell_i}(\phi_t^i q\phi_{tx}^i+\phi_{xx}^i q\phi_x^i)dxdt\\
&= -2\int_0^{\ell_i}\phi_t^i q\phi_x^i dx|_0^T + \int_0^T\int_0^{\ell_i} q[(\phi_t^i)^2+(\phi_x^i)^2]_x dxdt\\
&= -2\int_0^{\ell_i}\phi_t^i q\phi_x^i dx|_0^T + \int_0^T [(\phi_x^i)^2+(\phi_t^i)^2]q dt|_0^{\ell_i}\\
&\quad -\int_0^T\int_0^{\ell_i}[(\phi_t^i)^2+(\phi_x^i)^2]q' dxdt.
\end{aligned} \tag{6}$$

Summing over i, using the boundary conditions in Eq. 2 and the properties of q, one may obtain

$$\begin{aligned}
2\sum_1^M\int_0^T\int_0^{\ell_i} b_i q\phi^i\phi_x^i dxdt &= -2\sum_1^M\int_0^{\ell_i}\phi_t^i q\phi_x^i dx|_0^T + \int_0^T|\partial_n\phi^1(t,v_1)|^2 dt\\
&\quad -\sum_1^M\int_0^T\int_0^{\ell_i}[(\phi_t^i)^2+(\phi_x^i)^2]q' dxdt,
\end{aligned} \tag{7}$$

which results in

$$\begin{aligned}
\int_0^T|\partial_n\phi^1(t,v_1)|^2 dt &\le C\sum_1^M\int_0^T\int_0^{\ell_i}(\phi_t^i)^2+(\phi_x^i)^2 dxdt + 2\sum_1^M\int_0^T\int_0^{\ell_i} b_i q\phi^i\phi_x^i dxdt\\
&\quad + C\, ess\ sup_{0\le t\le T}(\|\phi(t)\|_V^2+\|\phi'(t)\|_H^2).
\end{aligned} \tag{8}$$

Now, we need to distinguish two cases.

Case 1. Since all the potentials b_i are bounded, one can use methods used for the proof of the regularity properties of solutions of the second order hyperbolic equations, i.e., Galerkin approximations to obtain

$$ess\ sup_{0\le t\le T}(\|\phi(t)\|_V^2+\|\phi'(t)\|_H^2) \le C(\|\phi_1\|_H^2+\|\phi_0\|_V^2), \tag{9}$$

therefore by Eq. 8, we get

$$\int_0^T|\partial_n\phi^1(t,v_1)|^2 dt \le C(\|\phi_1\|_H^2+\|\phi_0\|_V^2), \tag{10}$$

which gives the desired result.

Case 2. Here, we need to use the Hardy inequality (3) obtained before. Then

$$\begin{aligned}
\sum_{i\in I_{ext}}\int_0^T\int_0^{\ell_i}|b_i q\phi^i\phi_x^i|dxdt &\le C\sum_{i\in I_{ext}}\int_0^T\int_0^{\ell_i}\frac{(\phi^i)^2}{x^2}+(\phi_x^i)^2\\
&\le C\sum_{i\in I_{ext}}\int_0^T\int_0^{\ell_i}(\phi_x^i)^2.
\end{aligned}$$

Also by using the Hardy inequality (3), one can rewrite all the steps in Galerkin approximations to obtain

$$ess\ sup_{0\leq t\leq T}(\|\phi(t)\|_V^2+\|\phi'(t)\|_H^2)\leq C(\|\phi_1\|_H^2+\|\phi_0\|_V^2), \tag{11}$$

and therefore the proof will get completed. □

Therefore, the operator $A_t^*: L^2(0,T)\to H\times V'$, the adjoint of A_t, is also continuous (we have identified $L^2(0,T)$ and H with their duals). Furthermore, for every $h\in L^2(0,T)$, we define the solution of system (1) with initial state $(\bar{u}_0,\bar{u}_1)\in H\times V$ as

$$\bar{u}=A_t^*h+S_t(\bar{u}_0,\bar{u}_1), \tag{12}$$

where $S_t(\bar{u}_0,\bar{u}_1)$ is the solution of Eq. 5 with null control, i.e., $h\equiv 0$. To clarify the meaning of this formula, let us calculate the operator A_t^*. We will show that it coincides with the operator B defined for $h\in C^1([0,t])$ by

$$Bh=(\bar{u},\bar{u}_t),$$

where $\bar{u}$ is the solution in the classical sense of the problem (5) with null initial data $\bar{u}_0=\bar{u}_1=0$. Now multiply the first equation in Eq. 2 by u_i and integrate over $[0,t]\times[0,\ell_i]$. Note that in case 2, since Hardy inequality (3) is hold, one can rewrite all of the steps in proving the regularity of solutions of hyperbolic equations to obtain H^2 regularity of solutions ϕ^i and u^i in this case. Therefore, one can use integration by parts to get

$$\begin{aligned}0&=\int_0^t\int_0^{\ell_i}(\phi^i_{tt}-\phi^i_{xx}+b_i(x)\phi^i)u^i\,dxdt=\int_0^t\int_0^{\ell_i}(u^i_{tt}-u^i_{xx}+b_i(x)u^i)\phi^i\,dxdt\\&\quad+\int_0^{\ell_i}(u^i\phi^i_t-\phi^iu^i_t)\,dx|_0^t+\int_0^t(u^i_x\phi^i-u^i\phi^i_x)\,dt|_0^{\ell_i}.\end{aligned} \tag{13}$$

Now in view of the boundary conditions in Eqs. 2 and 5 and by adding equalities above, we get

$$\int_0^t h(\tau)\partial_n\phi^1(\tau,v_1)d\tau=\sum_1^M\int_0^{\ell_i}(u^i(t,x)\phi^i_t(t,x)-\phi^i(t,x)u^i_t(t,x))dx,$$

and this equality means that

$$<\partial_n\phi^1(t,v_1),h>_{L^2(0,t)}=<\bar{u}(t),\bar{\phi}_t(t)>_{H\times H}-<\bar{u}_t(t),\bar{\phi}(t)>_{V'\times V}. \tag{14}$$

Consequently, we have

$$<A_t\bar{\phi},h>_{L^2(0,t)}=<Bh,\bar{\phi}>_{(H\times V')\times(H\times V)}.$$

Thus,

$$<A_t^*h,\bar{\phi}>_{(H\times V')\times(H\times V)}=<Bh,\bar{\phi}>_{(H\times V')\times(H\times V)}.$$

That is, $Bh=A_t^*h$ for every $h\in C^1([0,t])$. Taking into account that the operator A_t^* is continuous and that $C^1([0,t])$ is dense in $L^2(0,t)$, we can ensure that A_t^* coincides with the extension of B to $L^2(0,t)$.

This fact gives sense to the equality (12). In the classical case, $h\in C^1([0,t])$, $(\bar{u}_0,\bar{u}_1)\in(H\times V')$ and $u_0^i,u_1^i\in C^1([0,\ell_i])$, then formula (12) simply expresses the fact that the solution of the inhomogeneous problem with initial state $(\bar{u}_0,\bar{u}_1)$ can be represented as the sum of the solution of the homogeneous problem with initial state $(\bar{u}_0,\bar{u}_1)$ and the solution of the inhomogeneous problem with null initial state $(\bar{0},\bar{0})$. This fact is an immediate consequence of the linearity of the system (1). Finally, note that in view of Eq. 14 and the estimate (10), it follows that for every $h \in L^2(0, T)$ the solution $\bar{u}$ of Eq. 1 defined by Eq. 12 has the property

$$\bar{u} \in L^\infty(0, T; H) \cap W^{1,\infty}(0, T; V'), \tag{15}$$

together with the estimate

$$\|\bar{u}\|_{L^\infty(0,T;H)} + \|\bar{u}\|_{W^{1,\infty}(0,T;V')} \leq C[\|(\bar{u}_0, \bar{u}_1)\|_{H\times V'} + \|h\|_{L^2(0,T)}]. \tag{16}$$

Remark 3.5 If all the potentials b_i are smooth, then as a consequence of a density argument, (15), (16) and the fact that, for smooth data, the solution $\bar{u}$ is smooth as well, one may get

$$\bar{u} \in C([0, T]; H) \cap C^1(0, T; V').$$

Summarizing the previous results, we can formulate

Theorem 3.6 *Consider the inhomogeneous problem* (5) *with* $h \in L^2(0, T)$. *Then, for all* $(\bar{u}_0, \bar{u}_1) \in H \times V'$, *it has a unique solution*

$$\bar{u} \in L^\infty(0, T; H) \cap W^{1,\infty}(0, T; V').$$

Moreover, if all the potentials b_i *are smooth, then*

$$\bar{u} \in C([0, T]; H) \cap C^1(0, T; V').$$

4 The Control Problem

4.1 Basic Definitions

Definition 1 Let $T > 0$. We say that the initial state $(\bar{u}_0, \bar{u}_1) \in H \times V'$ is controllable in time T, if there exists function $h \in L^2(0, T)$ such that the solution of Eq. 5 with initial state $(\bar{u}_0, \bar{u}_1)$ satisfies

$$\bar{u}|_T = \bar{u}_t|_T = \bar{0}.$$

When for every $\epsilon > 0$ there exists control h^ϵ such that the corresponding solutions $\bar{u}^\epsilon$ verify

$$\|(\bar{u}^\epsilon|_T, \bar{u}_t^\epsilon|_T)\|_{H\times V'} < \epsilon,$$

it is said that $(\bar{u}_0, \bar{u}_1)$ is approximately controllable in time T.

Definition 2 Let $T > 0$. We say that the set $K \subset H \times V'$ is controllable in time T, if all the initial states $(\bar{u}_0, \bar{u}_1) \in K$ are controllable in time T. Then, we shall say that the system (5) is

1) **approximately** controllable in time T if there exists a dense set $K \subset H \times V'$, which is approximately controllable in time T. (And consequently, all the initial states $(\bar{u}_0, \bar{u}_1) \in H \times V'$ are approximately controllable).
2) **spectrally** controllable in time T if the subspace $Z \times Z \subset H \times V'$ is controllable in time T, where Z is the set of all the finite linear combinations of the eigenfunctions of the operator A.

Remark 4.1 Obviously, spectral controllability in time T results the approximate controllability in T. We will show later that under some conditions, these two notions are equivalent for system (5).

We start the study of the control problem for a network of strings as follows: Consider the simplest case, i.e., all the potentials b_i in the system (1) are equal to zero:

$$\begin{cases} u^i_{tt} - u^i_{xx} = 0, & (t,x) \in \mathbb{R} \times [0,\ell_i], i = 1,\dots,M \\ u^{i(j)}(t,v_j) = h_j(t), & t \in \mathbb{R}, v_j \in \mathcal{C} \\ u^{i(j)}(t,v_j) = 0, & t \in \mathbb{R}, v_j \in \mathcal{V_S} \setminus \mathcal{C}, \\ u^i(t,v) = u^j(t,v), & t \in \mathbb{R}, v \in \mathcal{V_M}, i,j \in I_v, \\ \sum_{i\in I_v} \partial_n u^i(t,v) = 0, & t \in \mathbb{R}, v \in \mathcal{V_M}, \\ u^i(0,x) = u^i_0(x), u^i_t(0,x) = u^i_1(x), & x \in [0,\ell_i]. \end{cases} \tag{17}$$

It has been proved in [6] that even in the case that the underlying graph G has a simple topological configuration, for example G is a tree, then for the exact controllability of the system (17), high number of controls on the exterior vertices are needed. In other words

Theorem 4.2 *If G is a tree and there are at least two uncontrolled nodes, then system (17) is not exactly controllable whatever $T > 0$ is, i.e., there exists initial states in $H \times V'$ which are not controllable in any finite time T.*

Consequently, in our case, where we have only one control, i.e., $r = 1$, one only expect the controllability of the system (1) to hold in strict subspaces of $H \times V'$. In fact, we show that for T sufficiently large (i.e., T be greater than twice the sum of the length of the edges) system (1) is approximately controllable if and only if all the eigenfunctions of the network are observable. (see Theorem 4.7).

So far, we do not know of any necessary and sufficient condition guaranteeing that all the eigenfunctions are observable in the general graphs. However, this condition, in the particular case of stars and trees, turns out to be sharp: the lengths of the strings are mutually irrational in the case of stars or the spectra of all pairs of subtrees with a common end-point are mutually disjoint in the more general case of trees. (see [6]).

Now, let us give an equivalent formulation of the control problem in term of operators in a superficial level. Let $\mathbf{P}_T : U \to H \times V'$ be the operator defined by

$$\mathbf{P}_T \bar{h} := (\bar{u}(T), \bar{u}_t(T)),$$

where $\bar{u}$ is the solution of system (5) with initial state $(\bar{0}, \bar{0})$. Also denote by $\mathcal{W}_T$ the rank of $\mathbf{P}_T$. Due to the linearity and time reversibility of system (1), one can see that the control problem in time T is reduced to study the rank $\mathcal{W}_T$ of the operator $\mathbf{P}_T$. On the other hand, with the aid of a general result in functional analysis, one can describe the space $\mathcal{W}_T$ in terms of the adjoint operator of $\mathbf{P}_T$. This is essentially the Hilbert Uniqueness Method (HUM). Now, as shown in Section 3, the operator $\mathbf{P}^*_T$ coincides with A_T. (A_T has been defined in Section 3). Therefore, the following theorems hold (see [6] for the proofs).

Theorem 4.3 *The initial state $(\bar{u}_0, \bar{u}_1) \in H \times V'$ is controllable in time T if and only if, there exists a constant $C > 0$ such that*

$$\int_0^T |\phi^1_x(t,v_1)|^2 dt \geq C| < \bar{u}_0, \bar{\phi}_1 >_H - < \bar{u}_1, \bar{\phi}_0 >_{V'\times V} |^2, \tag{18}$$

for every solution $\bar{\phi}$ of system (2) with initial state $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$.

Theorem 4.4 *System (1) is approximately controllable in time T if and only if the following unique continuation property for the homogeneous system (2) is verified*

$$\phi^1_x(t,v_1) = 0, \quad a.e. \quad t \in [0,T] \Rightarrow (\bar{\phi}_0, \bar{\phi}_1) = (\bar{0}, \bar{0}).$$

Let $\bar{\theta}_n$ be the corresponding eigenfunction of μ_n such that $\{\bar{\theta}_n\}_{n\in I^+\cup I^-}$ form an orthonormal basis of H. Then, the solution of the homogeneous system (2) with initial data $(\bar{\phi}_0, \bar{\phi}_1) \in V_1 \times H_1$ with the expansions

$$\bar{\phi}_0 = \sum_{n\in I^+\cup I^-} \phi_{0,n}\bar{\theta}_n, \qquad \bar{\phi}_1 = \sum_{n\in I^+\cup I^-} \phi_{1,n}\bar{\theta}_n, \tag{19}$$

is defined by the formula

$$\bar{\phi}(t,x) = \sum_{n\in I^+} (\phi_{0,n}\cos\lambda_n t + \frac{\phi_{1,n}}{\lambda_n}\sin\lambda_n t)\bar{\theta}_n + \sum_{n\in I^-} (d_{1,n}e^{\lambda_n t} + d_{2,n}e^{-\lambda_n t})\bar{\theta}_n, \tag{20}$$

in which $\lambda_n = \sqrt{\xi_n}$ for $n \in I^+$ and $\lambda_n = \sqrt{-\xi_n}$ for $n \in I^-$ (Remember that $\{\xi_n\}$ is the sequence of eigenvalues of A). Furthermore,

$$d_{1,n} = \frac{1}{2}(\phi_{0,n} + \frac{\phi_{1,n}}{\lambda_n}),\ d_{2,n} = \frac{1}{2}(\phi_{0,n} - \frac{\phi_{1,n}}{\lambda_n}).$$

By this notations and similar methods used in [6], one can prove

Theorem 4.5 *Suppose that there exists a constant $C > 0$ and non-vanishing coefficients c_n such that for every initial data $(\bar{\phi}_0, \bar{\phi}_1)$ with Fourier coefficients $\{\phi_{0,n}\}$ and $\{\phi_{1,n}\}$ as defined in* Eq. 19*, the following observability inequality holds*

$$\int_0^T |\phi_x^1(t,v_1)|^2 dt \geq C \sum_{n\in\mathbb{N}} c_n^2(\mu_n\phi_{0,n}^2 + \phi_{1,n}^2),$$

then the space of initial states $(\bar{u}_0, \bar{u}_1) \in H \times V'$ defined by

$$\sum_{n\in\mathbb{N}} \frac{1}{c_n^2}u_{0,n}^2 < \infty, \qquad \sum_{n\in\mathbb{N}} \frac{1}{c_n^2\mu_n}u_{1,n}^2 < \infty,$$

is controllable in time T. In particular, any initial state $(\bar{u}_0, \bar{u}_1) \in Z \times Z$ is controllable in time T.

In particular, the system is spectrally controllable (and then approximately controllable) in time T.

Proposition 4.6 (i) *Suppose $T > 2L$. Consider two sequences $\{\lambda_n\}_{n\in I^-}$ and $\{\lambda_n\}_{n\in I^+}$. If each of them has distinct elements, then for every $n \in \mathbb{N}$, there exists a constant C_n such that for every function f of the form*

$$f(t) = \sum_{j_1\in J^+} f_{j_1}e^{i\lambda_{j_1}t} + \sum_{j_2\in J^+} f_{j_2}e^{-i\lambda_{j_2}t} + \sum_{k_1\in J^-} f_{k_1}e^{\lambda_{k_1}t} + \sum_{k_2\in J^-} f_{k_2}e^{-\lambda_{k_2}t}, \tag{21}$$

in which J^+, J^- are finite subsets of I^+ and I^-, respectively, and $n \in J^+ \cup J^-$, the following inequality satisfies

$$|f_n| \leq C_n[\int_0^T |f(t)|^2 dt]^{1/2}. \tag{22}$$

(ii) *On the other hand, for every $T < 2L$ and each finite sequence $\{\alpha_n\}$ of complex numbers having a non-zero term, there exists no $C > 0$ such that*

$$|\sum_{n\in F} \alpha_n f_n| \leq C\{\int_0^T |f(t)|^2 dt\}^{1/2}. \tag{23}$$

for every function f of the form Eq. 21 *with* $F \subset J^+ \cup J^-$.

The proof of this proposition needs some preliminaries of the spectrum of the operator A. So, we prove it in the next section.

Now, as a consequence of proposition 4.6, we state the main result of this paper.

Theorem 4.7 (a) *For every $T > 2L$, the following properties of the system (5) are equivalent*

(1) *The system is approximately controllable in time T.*
(2) *The system is spectrally controllable in time T.*
(3) *The spectral unique continuation property holds, i.e., $\omega^1_x(v_1) \neq 0$ is verified by any non-zero eigenfunction $\bar{\omega}$.*

(b) *When $T < 2L$, system (1) is not spectrally controllable; no element of $Z \times Z$ is controllable in time T.*

Proof Clearly, (2) $\Rightarrow$ (1), so it suffices to show (1) $\Rightarrow$ (3) $\Rightarrow$ (2).

(1) $\Rightarrow$ (3): First denote by $\bar{\theta}_n$ the eigenfunction corresponding to the eigenvalue μ_n and set $\chi_n := \theta^1_{n,x}(v_1)$. Then observe that if $\chi_n = 0$ for some $n = n_0$, then the function

$$\bar{\phi}(t, x) = \cos \lambda_{n_0} t \, \bar{\theta}_{n_0}(x),$$

is a solution of Eq. 2 for which $\bar{\phi}^1_x(t, v_1) = 0$ for every $t \in \mathbb{R}$ which means that the unique continuation property in not valid for any $T > 0$. Therefore, by Theorem 4.4, system (5) is not approximately controllable in time T.

(3) $\Rightarrow$ (2): Suppose that ϕ is a solution of the homogeneous system (2) with initial conditions $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$. For the proof of spectral controllability, it suffices to show that there exist nonzero coefficients c_n such that the following observability inequality is satisfied

$$\int_0^T |\phi^1_x(t, v_1)|^2 dt \geq \sum_{n \in I^+ \cup I^-} c_n^2((\mu_n + K)|\phi_{0,n}|^2 + |\phi_{1,n}|^2), \tag{24}$$

where the positive constant K has been defined in Proposition 3.2. Indeed, if Eq. 24 holds, then by an argument similar to one used in the proof of assertion (ii) of Theorem 4.4 (see [6]), one can deduce that all the initial data $(\bar{u}_0, \bar{u}_1) \in H \times V'$ satisfiyng

$$\sum_{n \in I^+ \cup I^-} \frac{1}{c_n^2} |u_{0,n}|^2 + \sum_{n \in I^+ \cup I^-} \frac{1}{c_n^2(\mu_n + K)} |u_{1,n}|^2 < \infty, \tag{25}$$

are controllable in time T. In particular, since all c_n's s are nonzero, then every element of $Z \times Z$ is controllable in time T. Now for initial data

$$\bar{\phi}_0 = \sum_{n \in I^+ \cup I^-} \phi_{0,n} \bar{\theta}_n, \qquad \bar{\phi}_1 = \sum_{n \in I^+ \cup I^-} \phi_{1,n} \bar{\theta}_n,$$

the solution of the system (2) is defined by

$$\bar{\phi}(t, x) = \sum_{n \in I^+} (\phi_{0,n} \cos \lambda_n t + \frac{\phi_{1,n}}{\lambda_n} \sin \lambda_n t) \bar{\theta}_n + \sum_{n \in I^-} (d_{1,n} e^{\lambda_n t} + d_{2,n} e^{-\lambda_n t}) \bar{\theta}_n,$$

in which $\lambda_n = \sqrt{\mu_n}$ for $n \in I^+$ and $\lambda_n = \sqrt{-\mu_n}$ for $n \in I^-$. Also

$$d_{1,n} = \frac{1}{2}(\phi_{0,n} + \frac{\phi_{1,n}}{\lambda_n}), \, d_{2,n} = \frac{1}{2}(\phi_{0,n} - \frac{\phi_{1,n}}{\lambda_n}).$$

Now set $I_* = I^+ \cup -I^+$ and for $n \in I^+$ define $\lambda_n = -\lambda_{-n}$ and

$$a_n := \frac{1}{2}(\phi_{0,|n|} + \frac{\phi_{1,|n|}}{i\lambda_n}).$$

Therefore, by setting $\chi_n = \theta^1_{n,x}(v_1)$, one can obtain

$$\sum_{n \in I^+} (\phi_{0,n} \cos \lambda_n t + \frac{\phi_{1,n}}{\lambda_n} \sin \lambda_n t)\chi_n = \sum_{n \in I_*} a_n \chi_{|n|} e^{i\lambda_n t},$$

and inequality (24) becomes

$$\int_0^T |\sum_{n \in I_*} a_n \chi_{|n|} e^{i\lambda_n t} + \sum_{n \in I^-} (d_{1,n} e^{\lambda_n t} + d_{2,n} e^{-\lambda_n t})\chi_n|^2 dt \geq 4 \sum_{n \in I^+} c_n^2 \mu_n |a_n|^2$$
$$+ \sum_{n \in I^-} c_n^2 (2K + 4\mu_n) d_{1,n} d_{2,n} + K \sum_{n \in I^+} c_n^2 |\phi_{0,n}|^2 + K \sum_{n \in I^-} c_n^2 (d_{1,n}^2 + d_{2,n}^2). \quad (26)$$

Let us observe that the eigenvalues μ_n are all simple. Indeed, if $\bar{\varphi}$ and $\bar{\psi}$ are two linearly independent eigenfunctions corresponding to μ_n, then the function

$$\bar{w} = \psi_x(v_1)\bar{\varphi} - \varphi_x(v_1)\bar{\psi}$$

is also a non-trivial eigenfunction. Besides

$$w^1_x(v_1) = 0,$$

which contradicts our hypothesis on the network. Thus, all the eigenvalues are simple and the sequence $\{\lambda_n\}$, $n \in I^- \cup I^+$ is strictly increasing. Now from proposition 4.6, there exists a sequence $\{C_n\}$ of positive numbers such that

$$\int_0^T |\phi^1_x(t, v_1)|^2 dt \ \geq \ C_n \chi_n^2 |a_n|^2, \qquad \forall n \in I^+,$$
$$\int_0^T |\phi^1_x(t, v_1)|^2 dt \ \geq \ C_n \chi_n^2 (d_{1,n}^2 + d_{2,n}^2), \qquad \forall n \in I^-,$$

Thus for suitable sequence $\{\gamma_n\}$ of positive numbers (for example choose $\gamma_n^2 = 4C_n \chi_n^2/(n^2+1)$), one obtain

$$\int_0^T |\phi^1_x(t, v_1)|^2 dt \geq 2 \sum_{n \in I^+} \gamma_n^2 |a_n|^2 + 2 \sum_{n \in I^-} \gamma_n^2 (d_{1,n}^2 + d_{2,n}^2). \quad (27)$$

Now remark that for every $n \in I^-$, we have

$$d_{1,n}^2 + (K + 2\mu_n)^2 d_{2,n}^2 \geq 2(K + 2\mu_n) d_{1,n} d_{2,n}. \quad (28)$$

Hence for $C = max\{1, K, (K + 2\mu_n)^2\}_{n \in I^-}$, we get (observe that $C < \infty$ since $|I^-|$ is finit)

$$C\gamma_n^2 (d_{1,n}^2 + d_{2,n}^2) \geq \gamma_n^2 (d_{1,n}^2 + (K + 2\mu_n)^2 d_{2,n}^2). \quad (29)$$

Thus from Eqs. 28 and 29, one can obtain

$$\gamma_n^2 (d_{1,n}^2 + d_{2,n}^2) \geq \frac{\gamma_n^2}{C}(2K + 4\mu_n) d_{1,n} d_{2,n}. \quad (30)$$

Now for any $n \in I^+$ choose positive number c_n such that

$$c_n^2 \leq \min\{\frac{\gamma_n^2}{4\mu_n}, \frac{\gamma_n^2}{K}\}. \quad (31)$$

Also for $n \in I^-$, set $c_n^2 = \frac{\gamma_n^2}{C}$. Note that $\gamma_n^2 \geq K c_n^2$, because $C \geq K$. Now by Eqs. 27, 29 and the choice of c_n, we can write

$$\int_0^T |\phi_x^1(t, v_1)|^2 dt \geq \sum_{n \in I^+} \gamma_n^2 |a_n|^2 + \sum_{n \in I^+} \gamma_n^2 |a_n|^2 + \sum_{n \in I^-} \gamma_n^2 (d_{1,n}^2 + d_{2,n}^2)$$
$$+ \sum_{n \in I^-} \gamma_n^2 (d_{1,n}^2 + d_{2,n}^2) \geq 4 \sum_{n \in I^+} c_n^2 \mu_n |a_n|^2 + K \sum_{n \in I^+} c_n^2 |a_n|^2$$
$$\sum_{n \in I^-} c_n^2 (2K + 4\mu_n) d_{1,n} d_{2,n} + K \sum_{n \in I^-} c_n^2 (d_{1,n}^2 + d_{2,n}^2).$$

On the other hand, it is obvious that $|a_n|^2 \geq |\phi_{0,n}|^2$. Therefore, the above inequality results in

$$\int_0^T |\phi_x^1(t, v_1)|^2 dt \geq 4 \sum_{n \in I^+} c_n^2 \mu_n |a_n|^2 + \sum_{n \in I^-} c_n^2 (2K + 4\mu_n) d_{1,n} d_{2,n}$$
$$+ K \sum_{n \in I^+} c_n^2 |\phi_{0,n}|^2 + K \sum_{n \in I^-} c_n^2 (d_{1,n}^2 + d_{2,n}^2),$$

which is exactly (26).

(b) Let $\mathcal{I} \subset \mathbb{N}$ be a finite set. By Theorem 4.3, the initial state

$$(\bar{u}_0, \bar{u}_1) = (\sum_{n \in \mathcal{I}} \alpha_n \bar{\theta}_n, \sum_{n \in \mathcal{I}} \beta_n \bar{\theta}_n) \in Z \times Z \tag{32}$$

is controllable in time T if, and only if, there exists a constant $C > 0$ such that

$$\int_0^T |\phi_x^1(t, v_1)|^2 dt \geq C |\sum_{n \in \mathcal{I}} \alpha_n \phi_{1,n} - \sum_{n \in \mathcal{I}} \beta_n \phi_{0,n}|^2,$$

for any solution $\bar{\phi}$ of system (2) with initial state $(\bar{\phi}_0, \bar{\phi}_1) \in Z \times Z$. Therefore, if the initial state defined by Eq. 32 is controllable in time T, then there exists a constant $C > 0$ such that

$$\int_0^T |\sum_{n \in I_*} a_n \chi_{|n|} e^{i\lambda_n t} + \sum_{n \in I^-} (d_{1,n} e^{\lambda_n t} + d_{2,n} e^{-\lambda_n t}) \chi_n|^2 dt \geq C \left| \sum_{n \in \mathcal{I} \cap I^+} \alpha_n (a_n - a_{-n}) i\lambda_n \right.$$
$$\left. - \beta_n (a_n + a_{-n}) + \sum_{n \in \mathcal{I} \cap I^-} \alpha_n \lambda_n (d_{1,n} - d_{2,n}) - \beta_n (d_{1,n} + d_{2,n}) \right|^2$$
$$= C \left| \sum_{n \in \mathcal{I} \cap I^+} (i\alpha_n \lambda_n - \beta_n) a_n + (-i\alpha_n \lambda_n - \beta_n) a_{-n} \right.$$
$$\left. + \sum_{n \in \mathcal{I} \cap I^-} (\alpha_n \lambda_n - \beta_n) d_{1,n} + (-\alpha_n \lambda_n - \beta_n) d_{2,n} \right|^2$$
$$= C \left| \sum_{n \in (\mathcal{I} \cap I^+) \cup -(\mathcal{I} \cap I^+)} \rho_n a_n + \sum_{n \in \mathcal{I} \cap I^-} \rho_{1,n} d_{1,n} + \rho_{2,n} d_{2,n} \right|^2, \tag{33}$$

for every finite sequences (a_n), $(d_{1,n})$, and $(d_{2,n})$, where ρ_n, $\rho_{1,n}$, and $\rho_{2,n}$ can be chosen appropriately. On the other hand, since $T < 2L$, in account of proposition 4.6, we can ensure that there are no sequences ρ_n, $\rho_{1,n}$, and $\rho_{2,n}$ satisfying (33). Therefore, the initial state $(\bar{u}_0, \bar{u}_1)$ defined by Eq. 32 is not controllable in time T, if $T < 2L$. □

Remark 4.8 The proof of assertion (a) shows why we assume all of the eigenvalues of A are nonzero. In fact for the zero eigenvalue, the expression $(\phi_{1,0}t + \phi_{0,0})\bar{\theta}_0$ would appear in the expansion of solution $\bar{\phi}$, then because of the term t one can not rewrite the proofs of proposition 4.6 and theorem 4.5. On the other hand in [9], a model of one-dimensional wave equation with potential has been considered for which by the use of Carleman estimates an observability inequality like (18) has been proved. Thus, one can guess that even in the case of zero eigenvalue, by using Carleman estimates, Theorem 4.5 holds. But the subject of Carleman estimates for a general network is far from being complete. We refer to [2] for the analysis of the wave equation with potential on a star-shaped network by using Carleman estimates.

5 Spectrum of A

Suppose that $\lambda = \{\lambda_n\}$ be a sequence of complex numbers none of which is zero. Also denote by Θ the set of all the finite linear combinations of the functions $\{e^{i\lambda_n t}\}$.

Definition The completeness radius of the sequence $\lambda = \{\lambda_n\}_{n\in I}$ is defined as

$$I(\lambda) := \sup\{r \in \mathbb{R} : \Theta \text{ is dense in } C([-r, r])\}.$$

The following has been proved in [14].

Theorem 5.1 *Suppose that $\lambda = \{\lambda_n\}$ and $\eta = \{\eta_n\}$ are sequences of complex numbers none of which is zero. Then under condition*

$$\sum_n \left|\frac{1}{\lambda_n} - \frac{1}{\eta_n}\right| < \infty, \tag{34}$$

one can deduce $I(\lambda) = I(\eta)$.

Also, the celebrated Buerling- Malliavin's Theorem gives a relation between the completeness radius and density of real sequences, [12]:

Theorem 5.2 *Let $\lambda = \{\lambda_n\}$ be a sequence of real numbers. Assume that there exists constants $d^+, d^- \geq 0$ and $0 \leq \alpha < 1$, such that*

$$|\{\lambda \in \Lambda : 0 \leq \lambda \leq t\}| = d^+ t + O(t^\alpha),$$

$$|\{\lambda \in \Lambda : -t \leq \lambda \leq 0\}| = d^- t + O(t^\alpha).$$

Then

$$I(\lambda) = \pi d, \qquad d = \max\{d^+, d^-\}.$$

In the previous section, we see that the eigenvalues of the operator $A : V \to V'$ can be represented in the form $\{\xi_n\}_{n\in I^+\cup I^-}$ where I^+ and I^- are indices with respect to the positive and negative eigenvalues respectively, also $|I^-| < \infty$.

Now, set $\lambda_n = \sqrt{\xi_n}$ for $n \in I^+$ and $\lambda_n = \sqrt{-\xi_n}$ for $n \in I^-$. Also sort the eigenvalues of A such that the sequence $\{\mu_n\}$ be an increasing sequence and $\lambda^+ = (\lambda_n)$. As we will show later, the number $I(\lambda^+)$ gives the optimal time for spectral controllability. On the other hand, by Theorem 5.2, it equals to the density of the sequence λ^+.

Now, it is not difficult to obtain certain information on the asymptotic behavior of the sequence of eigenvalues of the network. In fact, using the min-max principle of Courant [13], the eigenvalues of the network may be compared with the eigenvalues of the strings with Dirichlet and Neumann boundary conditions. For this purpose, some preliminaries are needed.

Let V, H be Hilbert spaces, V densely and continuously embedded in H. Identify H and H', so that

$$V \hookrightarrow H = H' \hookrightarrow V'.$$

Proposition 5.3 *Let $a : V \times V \to H$ be a continuous, symmetric, and coercive (i.e., $\exists \alpha > 0 : |a(u,u)| \geq \alpha \|u\|_V^2, \quad \forall u \in V$) bilinear form. Define $\tilde{A} : V \to V'$ by*

$$(\tilde{A}u, v) = a(u, v), \qquad \forall u, v \in V.$$

and consider

$$A : D(A) \subset V \to H, \text{ where } D(A) := \{u \in V | \tilde{A}u \in H\} = \tilde{A}^{-1}H \text{ and } A := \tilde{A}|_{D(A)}.$$

Also put the following norm on $D(A)$:

$$\|u\|_{D(A)} := \|Au\|_H.$$

Then, $(A, D(A))$ is self adjoint and A is an isomorphism.

Proof In fact, this is one of the forms of the Friedrichs extenstion for semibounded symmetric operators. See section XI.7 in [17]. □

Now, we have the following important result which compares the eigenvalues of operators. In fact, it is a consequence of the Minimax principle of Courant. See [11], theorem 2.2, p. 31.

Theorem 5.4 *Consider Hilbert spaces H, V and bilinear forms a, a_1, a_2 which satisfy the hypothesis of proposition 5.3. Also assume that the embedding $V \hookrightarrow H$ be compact and*

$$a(u,u) \geq 0, a_j(u,u) \geq 0, \forall u \in V, \ j = 1, 2.$$

(i) *Let A_1 and A_2 with corresponding eigenvalues λ_n^1 and λ_n^2 be constructed from the triples (V, H, a_1) and (V, H, a_2) as in Proposition 5.3 with*

$$a_1(u,u) \leq a_2(u,u), \ \forall u \in V.$$

Then

$$\lambda_n^1 \leq \lambda_n^2, \qquad \forall n \in \mathbb{N}.$$

(ii) *Consider subspace $W \subset V$ which is densely and continuously embedded in H. Let A^V and A^W with corresponding eigenvalues λ_n^V and λ_n^W be constructed from the triples (V, H, a) and (W, H, a) as previous. Then*

$$\lambda_n^V \leq \lambda_n^W, \qquad \forall n \in \mathbb{N}.$$

Now, let us go back to the main problem and investigate $I(\lambda^+)$ defined after Theorem 5.2.

Proposition 5.5 *Let $\lambda = \{\lambda_n\}_{n \in I^+}$. Then $I(\lambda) = L = \sum_1^M \ell_i$.*

Proof **Case 1.** Let K be as defined in Lemma 3.2 and choose $M_1 > 0$ large enough so that

$$\|b_i\|_\infty + K < M_1, \qquad i = 1, \dots, M.$$

Also define the spaces

$$W_1 := \{\bar{u} \in \prod_{i=1}^{M} H_0^1(0, \ell_i) : \bar{u}(v) = 0 \quad \text{for every } v \in \mathcal{V}\},$$

$$W_2 := \prod_{i=1}^{M} H^1(0, \ell_i),$$

and consider the operators

$$A_1 : W_1 \to W_1', \qquad < A_1\bar{u}, \bar{v} > := \sum_{i=1}^{M} \int_0^{\ell_i} u_x^i v_x^i + M_1 u^i v^i \, dx,$$

$$A_2 : W_2 \to W_2', \qquad < A_2\bar{u}, \bar{v} > := \sum_{i=1}^{M} \int_0^{\ell_i} u_x^i v_x^i \, dx.$$

So one has

$$W_1 \hookrightarrow V \hookrightarrow W_2,$$

and according to the properties of K, M_1, we have

$$0 \le < A_2\bar{u}, \bar{u} > \le < (A + KI)\bar{u}, \bar{u} > \le < A_1\bar{u}, \bar{u} > .$$

Let us denote by $\{\mu_n^D\}$ and $\{\mu_n^N\}$ the strictly increasing sequences of eigenvalues of the operators A_1 and A_2, respectively. Then by Theorem 5.4, one may obtain

$$\mu_n^N \le \xi_n + K \le \mu_n^D, \qquad \forall n \in \mathbb{N}. \tag{35}$$

But note that $\{\mu_n^D\}$ (resp. $\{\mu_n^N\}$) is equal to $\bigcup_{i=1}^M \{\mu_n^{i,D}\}$ (resp. $\bigcup_{i=1}^M \{\mu_n^{i,N}\}$) in which $\{\mu_n^{i,D}\}$ (resp. $\{\mu_n^{i,N}\}$) is the sequence of eigenvalues of the corresponding operator on the string e_i of length ℓ_i with Dirichlet (resp. Neumann) boundary conditions. We know that the eigenvalues $\{\mu_n^{i,D}\}$ and $\{\mu_n^{i,N}\}$ may be computed explicitly:

$$\mu_n^{i,D} = (\frac{n\pi}{\ell_i})^2 + M_1, \qquad \mu_n^{i,N} = (\frac{(n-1)\pi}{\ell_i})^2.$$

Therefore,

$$\mu_n^{i,D} = \mu_{n+1}^{i,N} + M_1,$$

and consequently

$$\mu_n^D = \mu_{n+1}^N + M_1,$$

According to (35), we will have

$$\mu_n^N \le \xi_n + K \le \mu_{n+1}^N + M_1, \tag{36}$$

This inequality leads us to obtain some asymptotic information of the sequence $\{\lambda_n\}$. In fact, define $\lambda_n^N = \sqrt{\mu_n^N}$, then for $r > 0$ from the left-hand side of Eq. 36 one gets

$$n(r, (\lambda_n)) \le n(f(r), (\lambda_n^N)),$$

in which $f(r) := \sqrt{r^2 + K}$.

On the other hand, if $\lambda_{n+1}^N \in (0, r)$ from the right-hand side of Eq. 36, we deduce that

$$\xi_n < r^2 + M_1 - K,$$

so

$$n(g(r), (\lambda_n)) + |I^-| \geq n(r, (\lambda_n^N)) - 1,$$

in which $g(r) := \sqrt{r^2 + M_1 - K}$ and I^- is the finite index set of the negative eigenvalues of A. Therefore,

$$n(g^{-1}(r), (\lambda_n^N)) - 1 - |I^-| \leq n(r, \lambda_n) \leq n(f(r), (\lambda_n^N)).$$

Now

$$n(r, (\lambda_n^N)) = \sum_{i=1}^{M} n(r, (\lambda_n^{i,N})) = \sum_{i=1}^{M} 1 + [\frac{\ell_i}{\pi} r].$$

Thus

$$\sum_{i=1}^{M} \{1 + [\frac{\ell_i}{\pi} g^{-1}(r)]\} - 1 - |I^-| \leq n(r, (\lambda_n)) \leq \sum_{i=1}^{M} 1 + [\frac{\ell_i}{\pi} f(r)].$$

Now from dividing the above inequality by $r > 0$ and letting $r \to \infty$, the density of the sequence (λ_n) is obtained:

$$D(\lambda_n) := \lim_{r\to\infty} \frac{n(r, (\lambda_n))}{r} = \frac{L}{\pi},$$

where

$$L = \sum_{i=1}^{M} \ell_i.$$

Thus from the theorem 5.2, one can deduce that

$$I(\lambda) = L.$$

Case 2. Let K be as in case 1 and choose $M_1 > 0$ such that $\|b_i\|_\infty + K < M_1$ for every $i \in I_{int}$ and $b_i(x) + K \leq M_1/x^2$ for every $i \in I_{ext}$ and all $x \in (0, \ell_i)$. Also define the spaces W_1 and W_2 as in case 1 and the operators $A_i : W_i \to W_i'$ $(i = 1, 2)$ such that

$$< A_1\bar{u}, \bar{v} > := \sum_{i\in I_{int}}^{M} \int_0^{\ell_i} u_x^i v_x^i + M_1 u^i v^i \, dx + \sum_{i\in I_{ext}}^{M} \int_0^{\ell_i} u_x^i v_x^i + \frac{M_1}{x^2} u^i v^i \, dx,$$

$$< A_2\bar{u}, \bar{v} > := \sum_{i=1}^{M} \int_0^{\ell_i} u_x^i v_x^i \, dx.$$

for every $\bar{u} \in W_1$ we have

$$\begin{aligned} < (A + KI)\bar{u}, \bar{u} > = &\sum_1^M \int_0^{\ell_i} (u_x^i)^2 dx + \sum_{i\in I_{int}} \int_0^{\ell_i} (b_i(x) + K) u^2 dx \\ &+ \sum_{i\in I_{ext}} \int_0^{\ell_i} (b_i(x) + K) u^2 dx \leq \sum_1^M \int_0^{\ell_i} (u_x^i)^2 dx \\ &+ \sum_{i\in I_{int}} \int_0^{\ell_i} M_1 u^2 dx + \sum_{i\in I_{ext}} \int_0^{\ell_i} \frac{M_1}{x^2} u^2 dx = < A_1\bar{u}, \bar{u} > . \end{aligned}$$

Also for $K > 0$ sufficiently large, we have

$$< (A + KI)\bar{u}, \bar{u} >\geq \sum_{1}^{M} \int_{0}^{\ell_i} (u_x^i)^2 =< A_2\bar{u}, \bar{u} >,$$

and in summerize

$$0 \leq< A_2\bar{u}, \bar{u} >\leq< (A + KI)\bar{u}, \bar{u} >\leq< A_1\bar{u}, \bar{u} > .$$

Now define the sets $\{\mu_n^{i,D}\}, \{\mu_n^{i,N}\}, \{\mu_n^D\}, \{\mu_n^N\}, \{\hat{\mu}_n\}$, and $\{\lambda_n^N\}$ as in case 1. Then, by Theorem 5.4, the following holds:

$$\mu_n^N \leq \xi_n + K \leq \mu_n^D. \tag{37}$$

But $\mu_n^{i,N} = (\frac{(n-1)\pi}{\ell_i})^2$. Therefore, as in case 1

$$n(r, (\lambda_n)) \leq n(f(r), (\lambda_n^N)) = \sum_{i=1}^{M} 1 + [\frac{\ell_i}{\pi} f(r)], \tag{38}$$

where $f(r) = \sqrt{r^2 + K}$. On the other hand by Eq. 37, we get $\xi_n \leq \mu_n^D - K$, and so

$$n(g(r), (\lambda_n)) + |I^-| \geq n(r, (\lambda_n^D)) = n_1(r) + n_2(r), \tag{39}$$

in which $\lambda_n^D = \sqrt{\mu_n^D}$, $g(r) = \sqrt{r^2 - K}$. Also $n_1(r)$ and $n_2(r)$ are the number of $\{\lambda_n^{i,D}\}$ for $i \in I_{int}$ and $i \in I_{ext}$, respectively. Now observe that if $i \in I_{int}$ then

$$\mu^{i,D} = (\frac{n\pi}{\ell_i})^2 + M_1. \tag{40}$$

Thus for $r \geq \sqrt{M_1}$, we have

$$n_1(r) = \sum_{i\in I_{int}} [\frac{\ell_i}{\pi}\sqrt{r^2 - M_1}]. \tag{41}$$

But if $i \in I_{ext}$, then one should compute the eigenvalues of the following Sturm-Liouvill problem with Dirichlet boundary conditions

$$- u_{xx}^i + \frac{M_1}{x^2} u^i = \lambda u^i. \tag{42}$$

Let $M_1 = \nu^2 - \frac{1}{4}$ for $\nu > 0$. Then one can see that solutions of Eq. 42 are in the form

$$u^i(x) = x^{\nu+\frac{1}{2}}[1 + \sum_{1}^{\infty} \frac{(-1)^m \lambda^m}{m!(1+\nu)(2+\nu)\dots(m+\nu)} (\frac{x}{2})^{2m}].$$

The condition $u^i(\ell_i) = 0$ implies that $\frac{\ell_i\sqrt{\lambda}}{2}$ is the zero of Bessel function of order ν:

$$J_\nu(x) = 1 + \sum_{1}^{\infty} \frac{(-1)^m x^m}{m!(1+\nu)(2+\nu)\dots(m+\nu)}.$$

Now let us denote by $J_{\nu,n}$ the nth zero of J_ν. It is proved in [16] that for sufficiently large n

$$J_{\nu,n} = (n + \frac{\nu}{2} - \frac{1}{4})\frac{\pi}{\ell_i} + O(n^{-1}), \qquad n \to \infty \tag{43}$$

Thus if n_0^i be the first index for which Eq. 43 is satisfied, then for $r > n_0^i$, one can write

$$\sum_{i\in I_{ext}} N_0^i+[\frac{r\ell_i}{\pi}-(\frac{\nu}{2}-\frac{1}{4})\frac{\pi}{\ell_i}]-1 \le n_2(r) \le \sum_{i\in I_{ext}} N_0^i+[\frac{r\ell_i}{\pi}-(\frac{\nu}{2}-\frac{1}{4})\frac{\pi}{\ell_i}], \quad (44)$$

where N_0^i is the number of roots of the equation $J_\nu = 0$ on the interval $(0, n_0^i)$. Now from Eqs. 39–44, we get

$$\sum_{i\in I_{int}} [\frac{\ell_i}{\pi}\sqrt{r^2 - M_1}] + \sum_{i\in I_{ext}} \{N_0^i+[\frac{r\ell_i}{\pi}-(\frac{\nu}{2}-\frac{1}{4})\frac{\pi}{\ell_i}]-1\} \le n(g(r), (\lambda_n))+|I^-|.$$

Thus according to Eq. 38, we obtain

$$\sum_{i\in I_{int}} [\frac{\ell_i}{\pi}\sqrt{g^{-1}(r)^2 - M_1}] + \sum_{i\in I_{ext}} \{N_0 + [\frac{g^{-1}(r)\ell_i}{\pi} - (\frac{\nu}{2} - \frac{1}{4})\frac{\pi}{\ell_i}] - 1\} - |I^-|$$
$$\le n(r, (\lambda_n)) \le \sum_{i=1}^{M} 1 + [\frac{\ell_i f(r)}{\pi}],$$

and consequently $D(\lambda_n) = \dfrac{L}{\pi}$,so again we get $I(\lambda) = L$.

□

Remark 5.6 Note that from theorem 5.1, the value of $I(\lambda)$ does not change if a finite number of points are removed or adjoined to $\lambda = (\lambda_n)_{n\in I^+}$. Therefore, from the above theorem, one has

$$I\left((\lambda_n)_{n\in I^+} \cup (\lambda_n)_{n\in I^-}\right) = L.$$

Now we prove Proposition 4.6 which played a key role in the proof of the controllability result in the previous section.

Proof of Proposition 4.6 (i) Let $\mathcal{U} = (0, T)$, if Eq. 22 is not satisfied for some n, then one can find a sequence of the functions $\{f^p\}$ in the form of Eq. 21 such that $|f_n^p| = 1$ and $\int_{\mathcal{U}} |f^p(t)|^2 dt \to 0$ as $p \to \infty$. Now there are four cases:

If $n \in J^+$, then we have two cases:

(1) The constant function 1 is the limit in $L^2(\mathcal{U})$ of the functions g^+ of the form

$$g^+(t) = \sum_{j_1\in J^+} g_{j_1} e^{i\mu_{j_1} t} + \sum_{j_2\in J^+} g_{j_2} e^{-i\mu_{j_2} t} + \sum_{k_1\in J^-} g_{k_1} e^{\lambda_{k_1} t - i\lambda_n t} + \sum_{k_2\in J^-} g_{k_2} e^{-\lambda_{k_2} t - i\lambda_n t}, \quad (45)$$

in which $\mu_{j_1} = \lambda_{j_1} + \lambda_n$ and $\mu_{j_2} = \lambda_{j_2} - \lambda_n$.

(2) The constant function 1 is the limit in $L^2(\mathcal{U})$ of the functions g^- of the form

$$g^-(t) = \sum_{j_1\in J^+} g_{j_1} e^{i\mu_{j_1} t} + \sum_{j_2\in J^+} g_{j_2} e^{-i\mu_{j_2} t} + \sum_{k_1\in J^-} g_{k_1} e^{\lambda_{k_1} t - i\lambda_n t} + \sum_{k_2\in J^-} g_{k_2} e^{-\lambda_{k_2} t - i\lambda_n t},$$

in which $\mu_{j_1} = \lambda_{j_1} - \lambda_n$ and $\mu_{j_2} = \lambda_{j_2} + \lambda_n$.

If $n \in J^-$, similarly we have

(3) The constant function 1 is the limit in $L^2(\mathcal{U})$ of the functions g^+ of the form

$$g^+(t) = \sum_{j_1 \in J^+} g_{j_1} e^{i\lambda_{j_1} t - \lambda_n t} + \sum_{j_2 \in J^+} g_{j_2} e^{-i\lambda_{j_2} t - \lambda_n t} + \sum_{k_1 \in J^-} g_{k_1} e^{\beta_{k_1} t} + \sum_{k_2 \in J^-} g_{k_2} e^{\beta_{k_2} t}, \tag{46}$$

in which $\beta_{k_i} = \lambda_{k_i} - \lambda_n$ for $i = 1, 2$,

(4) The constant function 1 is the limit in $L^2(\mathcal{U})$ of the functions g^- in the form

$$g^-(t) = \sum_{j_1 \in J^+} g_{j_1} e^{i\lambda_{j_1} t - \lambda_n t} + \sum_{j_2 \in J^+} g_{j_2} e^{-i\lambda_{j_2} t - \lambda_n t} + \sum_{k_1 \in J^-} g_{k_1} e^{\beta_{k_1} t} + \sum_{k_2 \in J^-} g_{k_2} e^{\beta_{k_2} t}, \tag{47}$$

where $\beta_{k_i} = \lambda_{k_i} + \lambda_n$ for $i = 1, 2$.

Now, consider the first case. In this case by repeated integration in t, we can deduce that all polynomials of t with complex coefficients are also limits in $L^2(\mathcal{U})$ of some functions g of the form Eq. 45. Indeed, let $p \in \mathbb{N} \cup \{0\}$ and suppose for $\epsilon > 0$ we have

$$\|t^p - \sum_{j_1 \in J^+} g_{j_1} e^{i\mu_{j_1} t} - \sum_{j_2 \in J^+} g_{j_2} e^{-i\mu_{j_2} t} - \sum_{k_1 \in J^-} g_{k_1} e^{\lambda_{k_1} t - i\lambda_n t} - \sum_{k_2 \in J^-} g_{k_2} e^{-\lambda_{k_2} t - i\lambda_n t}\|_2 \le \epsilon,$$

then by integrating in t one can obtain

$$\begin{aligned}
\| \frac{t^{p+1}}{p+1} &- \sum_{j_1 \in J^+} \frac{g_{j_1}}{i\mu_{j_1}} e^{i\mu_{j_1} t} + \sum_{j_1 \in J^+} \frac{g_{j_1}}{i\mu_{j_1}} + \sum_{j_2 \in J^+} \frac{g_{j_2}}{i\mu_{j_2}} e^{-i\mu_{j_1} t} - \sum_{j_2 \in J^+} \frac{g_{j_2}}{i\mu_{j_2}} \\
&- \sum_{k_1 \in J^-} \frac{g_{k_1}}{\lambda_{k_1} - i\lambda_n} e^{\lambda_{k_1} t - i\lambda_n t} + \sum_{k_1 \in J^-} \frac{g_{k_1}(p+1)}{\lambda_{k_1} - i\lambda_n} \\
&- \sum_{k_2 \in J^-} \frac{g_{k_2}}{-\lambda_{k_2} - i\lambda_n} e^{-\lambda_{k_2} t - i\lambda_n t} + \sum_{k_2 \in J^-} \frac{g_{k_2}}{-\lambda_{k_2} - i\lambda_n} \|_\infty \le \epsilon T^{1/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|t^{p+1} - (p+1)\{ &\sum_{j_1 \in J^+} \frac{g_{j_1}}{i\mu_{j_1}} e^{i\mu_{j_1} t} - \sum_{j_2 \in J^+} \frac{g_{j_2}}{i\mu_{j_2}} e^{-i\mu_{j_2} t} + \sum_{k_1 \in J^-} \frac{g_{k_1}}{\lambda_{k_1} - i\lambda_n} e^{(\lambda_{k_1} - i\lambda_n)t} \\
&+ \sum_{k_2 \in J^-} \frac{g_{k_2}}{-\lambda_{k_2} - i\lambda_n} e^{(-\lambda_{k_2} - i\lambda_n)t}\} + C\|_\infty \le \epsilon(p+1)T,
\end{aligned}$$

where the constant C is equal to

$$C = (p+1)\{ \sum_{j_1 \in J^+} \frac{g_{j_1}}{i\mu_{j_1}} - \sum_{j_2 \in J^+} \frac{g_{j_2}}{i\mu_{j_2}} + \sum_{k_1 \in J^-} \frac{g_{k_1}}{\lambda_{k_1} - i\lambda_n} + \sum_{k_2 \in J^-} \frac{g_{k_2}}{-\lambda_{k_2} - i\lambda_n}\}.$$

Then by approximating C in $L^2(\mathcal{U})$ by functions of the form Eq. 45, one can find sequences of coefficients $\{h_n\}$ such that

$$\begin{aligned}
\|t^{p+1} &- \sum_{j_1 \in J^+} h_{j_1} e^{i\mu_{j_1} t} - \sum_{j_2 \in J^+} h_{j_2} e^{-i\mu_{j_2} t} - \sum_{k_1 \in J^-} h_{k_1} e^{(\lambda_{k_1} - i\lambda_n)t} \\
&- \sum_{k_2 \in J^-} h_{k_2} e^{(-\lambda_{k_2} - i\lambda_n)t} \|_\infty \le 2\epsilon(p+1)T.
\end{aligned}$$

This proves the claim by induction in p since it has been proved already for $p = 0$. Finally, by Stone-Weierstrass density theorem, one can obtain that the functions g

of the form Eq. 45 are dense in $L^2(\mathcal{U})$, and the same property follows at once for functions f of the form Eq. 21 which contradict the assumption $T > 2L$. In other three cases by exactly similar arguments, one can get the above contradiction.

(ii) For any finite subset $F \subset J^+ \cup J^-$, consider the new sequence $\bar{\lambda} := (\lambda)_{(J^+ \cup J^-)\setminus F}$. Then as stated in remark 5.6, one has $I(\bar{\lambda}) = I(\lambda)$. Thus for any $T < 2L$ the set of functions f of the form Eq. 21 with $\{J^+ \cup J^-\} \cap F = \emptyset$ are dense in $L^2(\mathcal{U})$ and as a consequence for each non trivial sequence $\{\alpha_n\}_F$ of complex numbers, the function

$$a(t) = \sum_{n \in F \cap J^+} \alpha_n e^{i\lambda_n t} + \sum_{n \in F \cap J^-} \alpha_n e^{\lambda_n t}$$

can be approached by functions f of the form Eq. 21 with $\{J^+ \cup J^-\} \cap F = \emptyset$. By taking the difference, we find a sequence of functions of the form Eq. 21 tending to zero in $L^2(\mathcal{U})$ and for which the left-hand side in Eq. 23 is positive and constant. This clearly proves assertion (ii).

□

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