PATTERN FORMATION

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SOME FEATURES

- Spotted body and striped tail or legs
- Cheetah (upper), Okapi (lower)
- Tiger (upper), Leopard (lower)

Why do animals' coats have patterns like spots, or stripes?

Problem we consider

- A population whose density \( u(x,t) \) depends on the time \( t \) and the location \( x \)

Question: to persist, or extinct?
determine which of these modes are stabilized by the nonlinearities of the system we use techniques such as Liapunov-Schmidt reduction and Poincare-Lindstedt perturbation theory to reduce the dynamics to a set of nonlinear equations for the amplitudes appearing in equation (33) (Walgraef 1997). These amplitude equations, which effectively describe the dynamics on a finite-dimensional centre manifold, then determine the selection and stability of patterns (at least sufficiently close to the bifurcation point). The symmetries of the system severely restrict the allowed forms (Golubitsky et al. 1988); however, the coefficients in this form are inherently model-dependent and have to be calculated explicitly. In this section we determine the amplitude equation for our cortical model up to cubic order and use this to...
PATTERNS IN NEURAL FIELDS

\[ u_t = -u + \int w(x, y) f(u(y, t)) \, dy \]
Example 4: the 1D 'orientational' representation

Example 1: trivial representation

Examples of representations

If the only invariant subspaces of \( \rho \) are the origin and the whole space, \( \rho \) is said to be reducible. Otherwise, \( \rho \) is irreducible.

A representation \( \rho \) is called orthogonal if the representation is orthogonal. Since the representation is orthogonal, so the only invariant subspaces of \( \rho \) are the origin and the whole space. Hence, \( \rho \) is irreducible.

The representation is said to be reducible, otherwise.

The representation of \( G \) is called a symmetric representation if every element \( g \) is in the natural way, as the symmetries of a square:

Suppose that \( (123) \), \( (12)(3) \), and \( (13) \) are the only elements of \( G \) that preserve orientation. Now we can return to the initial observation of 'system symmetry'.

The representation is composed of an even number of transpositions.

Let \( (123) = \rho(1) \rho(2) \rho(3) \) and \( (12)(3) = \rho(1) \rho(2) \rho(3) \rho(4) \) and \( (13) = \rho(1) \rho(3) \).

From differentiating the equivariance condition we find

\[ \rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

\[ m = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ D_4 = \{ I, m_x, m_y, m_d, m_{d'}, \rho, \rho^2, \rho^3 \} \]
SYMMETRIC EQUATIONS

- **Vector Field** \( \frac{dx}{dt} = f(x, \mu) \)

- **Group Action** \( \Gamma \times \mathbb{R}^n \rightarrow \mathbb{R}^n \)

- Vector field \( f(x, \mu) \) has the symmetry \( \Gamma \) if for every solution \( x(t) \), the trajectory \( \gamma \cdot x(t) \) is also a solution for every \( \gamma \in \Gamma \)

\[ \gamma \cdot f(x, \mu) = f(\gamma \cdot x, \mu) \]
An Example of $D_4$ Symmetry

\[
\begin{align*}
\frac{dx_1}{dt} &= f_1(x_1, x_2, \mu) \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, \mu)
\end{align*}
\]

\[m \cdot f(x) = f(m \cdot x) \implies \begin{cases} 
-f_1(x_1, x_2) = f_1(-x_1, x_2) \\
f_2(x_1, x_2) = f_2(-x_1, x_2)
\end{cases}\]

\[f(x) = \mu \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + a_1 \begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} + a_2 \begin{pmatrix} x_1 x_2^2 \\ x_1^2 x_2 \end{pmatrix} + \cdots\]
SYMMETRIC BIFURCATION

Definition 1: (Isotropy Subgroup)
\[ \Sigma_x = \{ \sigma \in \Gamma : \sigma.x = x \} \]

Isotropic subgroups are constant along solution curves.

Definition 2: (Fixed Point Invariant Subspace)
\[ \text{Fix}(\Sigma) = \{ x \in \mathbb{R}^n : \sigma.x = x, \forall \sigma \in \Sigma \} \]

One possible line of finding the $\Sigma$ symmetry solution is to restrict the dynamics to $\text{Fix}(\Sigma)$. 
EQUIVARIANT BRANCHING LEMMA

Let $\Gamma$ be a finite group acting on $\mathbb{R}^n$ with $\text{Fix}(\Gamma) = \{0\}$.

Let $\frac{dx}{dt} = f(x, \mu)$ be a $\Gamma$ symmetry with $f(0, \mu) = 0$,

$D_x f\big|_{(0,0)} = 0$, $D_x f_\mu\big|_{(0,0)} v \neq 0$ for a $0 \neq v \in \text{Fix}(\Sigma)$

$\Sigma$ is an isotropy subgroup of $\Gamma$ where $\dim \text{Fix}(\Sigma) = 1$.

Then there is a curve $x = sv$, $\mu = \mu(s)$ of critical points.

$$f(sv, \mu(s)) = 0$$
\[ \Sigma = \{e, m\} \implies \text{Fix}(\Sigma) = \{(0, y) : y \in \mathbb{R}\} \]

\[ f(0, y, \mu(y)) = 0, \quad \mu \approx -a_1 y^2 \]

**Pitchfork Bifurcation**
LATTICE PATTERNS

\[ u_t = F(u, \mu) \]

\[ u(x + \vec{l}, t) = u(x, t) \]

\[ \mathcal{L} = \{ n_1 \vec{l}_1 + n_2 \vec{l}_2 : n_1, n_2 \in \mathbb{Z} \} \]

\[ u(x, t) = \sum_{k \in \mathcal{L}^*} z_k(t) \exp(ik.x) + c.c. \]

\[ \frac{dz}{dt} = g(z, \mu) \]
LATTICE SYMMETRIES GROUP

Rotation:

\[ u(x_1, x_2, t) = z_1 \exp(ix_1) + z_2 \exp(ix_2) + c.c. \]
\[ u(-x_2, x_1, t) = z_1 \exp(-ix_2) + z_2 \exp(ix_1) + c.c. \]

\[ \implies \rho.(z_1, z_2) = (z_2, \overline{z_1}) \]

Reflection:

\[ m.(z_1, z_2) = (\overline{z_1}, z_2) \]

Translation:

\[ p.(z_1, z_2) = (e^{-ip_1} z_1, e^{-ip_2} z_2) \]
Example: (Square Lattice) \[ \dot{z} = g(z, \mu) \]

symmetry group: \( \Gamma = D_4 \times T^2 \)

\[ \gamma . g(z) = g(\gamma . z) \quad \forall \gamma \in \Gamma \]

\[
\begin{align*}
\frac{dz_1}{dt} & = \mu z_1 - \alpha |z_1|^2 z_1 - \beta |z_2|^2 z_1 + \ldots \\
\frac{dz_2}{dt} & = \mu z_2 - \beta |z_1|^2 z_2 - \alpha |z_2|^2 z_2 + \ldots 
\end{align*}
\]
\[ \Sigma = D_2 \times S^1 = \{ e, m \} \times (p_1, 0) \]

\[ \text{Fix}(\Sigma) = \{ (z_1, 0) : z_1 \in \mathbb{R} \} \]

\[ u(x, t, \mu) = \sqrt{\frac{\mu}{\alpha}} \exp(ix_1) + \text{c.c.} = 2\sqrt{\frac{\mu}{\alpha}} \cos(x_1) \]
\[ \Sigma = D_4 \quad \text{Fix}(\Sigma) = \{(z_1, z_1) : z_1 \in \mathbb{R}\} \]

\[
u(x, t, \mu) \approx 2\sqrt{\frac{\mu}{\alpha + \beta}}(\cos(x_1) + \cos(x_2))
\]
Natural Patterns of \( \cos(kx) \) and \( \cos(x) \):

- Valais goat (single color: \( f(x)=1 \), a lot of examples)
REFERENCE

Marty Golubitsky

Rebecca Hoyle