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NULL CONTROLLABILITY OF DEGENERATE/SINGULAR PARABOLIC EQUATIONS

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ABSTRACT. The purpose of this paper is to provide a full analysis of the null controllability problem for the one dimensional degenerate/singular parabolic equation $u_t - (a(x)u_x)_x - \dfrac{\lambda}{x^\beta}u = 0$, $(t,x) \in (0,T) \times (0,1)$, where the diffusion coefficient $a(\cdot)$ is degenerate at $x = 0$. Also the boundary conditions are considered to be Dirichlet or Neumann type related to the degeneracy rate of $a(\cdot)$. Under some conditions on the function $a(\cdot)$ and parameters β, λ, we prove global Carleman estimates. The proof is based on an improved Hardy-type inequality.

1. Introduction

In the recent years, the study of the controllability for parabolic equations has become an active research area. After the pioneering works [9, 12, 13, 17, 18], there has been substantial progress in understanding the controllability properties of parabolic equations with variable coefficients. In particular, the null controllability for the following class of nondegenerate and nonsingular parabolic operators is well-known:

$$Pu = u_t - (a(x)u_x)_x, \quad x \in (0,1),$$

where the coefficient $a(x)$ is a positive continuous function on $[0,1]$.

On the contrary, when the coefficient $a(x)$ is zero at some points, the equation will be degenerate and few results are known in this case, even though many problems that are relevant for applications are described by parabolic equations degenerating at the boundary of the space domain. For instance, in [2,6,7,16], the reader will find a motivating example of a Crocco-type equation coming from the study of the velocity field of a laminar flow on a flat plate. In [5, 15], where the degenerate/nonsingular model

$$Pu = u_t - (a(x)u_x)_x, \quad x \in (0,1), \tag{1}$$

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is considered, observability results for the adjoint problem are obtained using Carleman estimates. Then the null controllability of (1) on $[0,1]$ is derived by some standard arguments. Also, in [4] the null controllability on $[0,1]$ of the semilinear degenerate parabolic equation

$$Pu = u_t - (a(x)u_x)_x + f(t,x,u), \qquad x \in (0,1),$$

is studied, where f is locally Lipschitz with respect to u.

Now, consider $N \geq 3$ and let $\Omega \subset \mathbb{R}^N$ be a bounded open set with the boundary Γ, of class C^2 such that $0 \in \Omega$. The heat equation perturbed by a singular inverse-square potential, namely

$$Pu = u_t - \Delta u + \frac{\lambda}{|x|^2}u, \qquad x \in \Omega,$$

arised in quantum mechanics and combustion theory. When $\lambda \leq (N-2)^2/4$, it is proved in [20] that the equation can be controlled to zero with a distributed control which surrounds the singularity. Next, it is shown in [10] that this geometric assumption is not necessary, and one can control the equation from any open subset as for the heat equation. But in the case $\lambda > (N-2)^2/4$, there exists a sequence of regularized potentials $\lambda/(|x|^2+\epsilon^2)$ such that one cannot stabilize the corresponding systems uniformly with respect to $\epsilon > 0$. Furthermore, Vancostenoble and Zuazua in [20] showed that in dimension $N = 1$, the following singular operator is null controllable for $\lambda \leq 1/4$, $m \in \mathbb{R}$ and $0 \leq \beta < 2$,

$$Pu = u_t - u_{xx} + \frac{\lambda}{|x|^2}u + \frac{m}{x^\beta}u, \qquad x \in (0,1).$$

Also, new Carleman estimates (and consequently null controllability property) were established in [19] for the one dimensional degenerate/singular operator

$$Pu = u_t - (x^\alpha u_x)_x - \frac{\lambda}{x^\beta}u, \qquad x \in (0,1),$$

with suitable boundary conditions and under the following assumptions

$$\begin{cases} \alpha \in [0,2), & 0 < \beta < 2-\alpha, \quad \lambda \in \mathbb{R}, \\ \alpha \in [0,2)\setminus\{1\}, & \beta = 2-\alpha, \qquad \lambda \leq \dfrac{(1-\alpha)^2}{4}. \end{cases}$$

Also, it has been proved that the null controllability is false when $\alpha \geq 2$, [7].

In this paper, we study the following one dimensional operator that couples a degenerate diffusion coefficient with a singular potential:

$$Pu = u_t - (a(x)u_x)_x - \frac{\lambda}{x^\beta}u, \qquad x \in (0,1). \tag{2}$$

Under suitable condition on a, β and λ, we prove the null controllability for the operator (2) which completes the results of [4] and [19].

The paper is organized as follows. Section 2 is devoted to discussing the assumptions on the coefficient degeneracy $a(\cdot)$ and the parameters β, λ. Then we state functional setting and well-posedness in Section 3. In Section 4, we state main results, especially Carleman estimates and its applications to observability and controllability. In Section 5, we study the improved Hardy inequality which is necessary in the proofs of well-posedness, Carleman estimates and observability results. Finally Section 6 is devoted to the proofs.

2. Assumptions and statement of the problem

2.1. **Hypothesis on the degeneracy coefficient.** In order to study the controllability of operator (2), we let some assumption on the degenerate diffusion coefficient $a(x)$ in the following way.

Hypothesis 1. We suppose that the degeneracy coefficient $a(\cdot)$ satisfies the following conditions:

(i) $a \in C([0,1]) \cap C^1((0,1])$, $a(x) > 0$ in $(0,1]$ and $a(0) = 0$.

(ii) $\exists\, \alpha \in (0,2)$ such that $xa'(x) \leq \alpha a(x)$ for every $x \in [0,1]$.

(iii) If $\alpha \in [1,2)$, there exist $m > 0$ and $\delta_0 > 0$ such that for every $x \in [0,\delta_0]$, we have

$$a(x) \geq m \sup_{0\leq y\leq x} a(y).$$

First of all, notice that every function which is nondecreasing near $x = 0$, satisfies in condition (iii) for $m = 1$. Thus Hypothesis 1 is weaker than the conditions of the coefficient $a(\cdot)$ stated in the paper [4],

$$\begin{cases} \text{If}\, \alpha \in (1,2), & \exists\, \theta \in (1,\alpha] \\ \text{such that}\, x \mapsto \dfrac{a(x)}{x^\theta} \text{ is nondecreasing near}\, x = 0, \\ \text{If}\, \alpha = 1, & \exists\, \theta \in (0,1) \\ \text{such that}\, x \mapsto \dfrac{a(x)}{x^\theta} \text{ is nondecreasing near}\, x = 0. \end{cases} \tag{3}$$

In fact, the condition (3) will imply that the coefficient $a(x)$ is nondecresing near $x = 0$. As stated in [4], in the case $\alpha \in (1,2)$, a sufficient condition to prove the controllability of the equation $u_t - (a(x)u_x)_x = f\chi_\omega$ is that the degeneracy coefficient $a(x)$ satisfies conditions (i) and (ii) of Hypothesis 1 as well as the Hardy type inequality holds, i.e. there exists $C > 0$ such that for any locally absolutely continuous function u on $(0,1]$ which

$$u(1) = 0, \qquad \int_0^1 a(x)|u'(x)|^2 dx < \infty,$$

we should have

$$\int_0^1 \frac{a(x)}{x^2} u^2(x)dx \le C \int_0^1 a(x)|u'(x)|^2 dx. \tag{4}$$

In fact, we replace the condition (iii) of Hypothesis 1 instead of Hardy inequality, however they are not equivalent. The following example shows us a degenerate coefficient $a(x)$ which satisfies Hypothesis 1, but the Hardy inequality does not hold for that. We use the following proposition in this example which was proved in [4].

Proposition 2. *Let $a, b : [0,1] \to \mathbb{R}$ be in $C([0,1]) \cap C^1((0,1])$, $a(0) = b(0) = 0$, $a > 0$ and $b > 0$ on $(0,1]$. Moreover, assume that there exist two positive constants c_1 and c_2 such that*

$$c_1 b \le a \le c_2 b,$$

in a neighborhood of zero. Then Hardy inequality (4) *holds for the function $a(\cdot)$ if and only if it holds for $b(\cdot)$.*

Example 3. Consider $a(x) = xe^{(\alpha-1)x}$ for some $\alpha \in (1,2)$, we will have

$$\frac{xa'(x)}{a(x)} = 1 + (\alpha-1)x \le 1 + (\alpha - 1) = \alpha,$$

for every $x \in [0,1]$. This type of function satisfies the conditions of Hypothesis 1, but the condition (3) or Hardy inequality does not hold. At first, we notice that there exists no $\theta \in (1, \alpha]$ such that the function $a_*(x) = \dfrac{a(x)}{x^\theta}$ is increasing near $x = 0$, because for every $\theta > 1$,

$$a_*'(x) = \frac{e^{(\alpha-1)x}[1 - \theta + (\alpha-1)x]}{x^\theta},$$

which is negative near $x = 0$. Furthermore, if Hardy inequality holds for $a(\cdot)$, according to Proposition 2 we conclude that Hardy inequality should hold for the function $b(x) = x$, because of equivalency $c_1 b(x) \le a(x) \le c_2 b(x)$ on the interval $x \in [0,1]$ for suitable positive constants c_1 and c_2. But by substituting the functions $u(x) = x^r(1-x)$ in the Hardy inequality (4) for $b(x) = x$ and let $r \to 0^+$, we imply that the inequality can't hold. So, by the method used in [4] one can not prove the controllability of the related equation to $a(x) = xe^{(\alpha-1)x}$. On the other hand $a(\cdot)$ satisfies the condition (iii) of Hypothesis 1 because of its increasing property.

Now we state a lemma which is useful along the paper. This shows $\dfrac{1}{a} \in L^1(0,1)$ in the case of $\alpha \in (0,1)$, and $\dfrac{1}{\sqrt{a}} \in L^1(0,1)$ for $\alpha \in [1,2)$. Note that if $\dfrac{1}{\sqrt{a}} \notin L^1(0,1)$, then the null controllability of (2) fails, see [7].

Lemma 4. *Let $a(\cdot)$ satisfies the Hypothesis 1, then the integral* $\int_0^1 \frac{dx}{a(x)}$ *is finite for* $\alpha \in (0,1)$ *and* $\int_0^1 \frac{dx}{\sqrt{a(x)}}$ *is finite for* $\alpha \in [1,2)$.

Remark 5. For $\alpha \in (1,2)$, the integral $\int_0^1 \frac{1}{a(x)}dx$ might be finite or infinite, for example consider the functions $a(x) = x^r e^{(\alpha-r)x}$ where $r \leq \alpha$.

2.2. **Assumptions on the singular potential.** First note that when $\alpha \geq 2$, the null controllability might be false. For example consider $a(x) = x^\alpha$ for $\alpha \geq 2$. In this case the necessary condition for null controllability which was proved in [7] does not hold, i.e. $\frac{1}{\sqrt{a(x)}}$ does not belong to $L^1(0,1)$. On the other hand, since $\alpha = 1$ is a peculiar case in the following "Hardy type" inequality (which is proved in Lemma 18),

$$\int_0^1 a(x)u_x^2 dx \geq \lambda^*(a,\alpha) \int_0^1 \frac{u^2}{x^{2-\alpha}} dx, \tag{5}$$

this special case is considered separately. Also if $\alpha = 0$, then the nonnegative function $a(\cdot)$ is decreasing on $[0,1]$, therefore $a(x) = 0$ for every x, which is impossible since $a > 0$ in $(0,1]$. Now we assume that

$$\alpha \in (0,2) \setminus \{1\}.$$

For a given singular potential, Cabré and Martel in [3] proved that existence versus blow-up of positive solutions is connected to the existence of some Hardy inequality involving the considered potential, like as the following:

$$\int_0^1 a(x)u_x^2 dx \geq \lambda \int_0^1 \frac{u^2}{x^\beta} dx. \tag{6}$$

Therefore, the inequality (5) implies that the critical exponent could be $\beta = 2-\alpha$, so we assume that $\beta \leq 2-\alpha$. Here, we show well-posedness and controllability of (2) for

$$\begin{cases} \alpha \in (0,2), & 0 < \beta < 2-\alpha, \quad \lambda \in \mathbb{R}, \\ \alpha \in (0,2) \setminus \{1\}, & \beta = 2-\alpha, \qquad \lambda < \lambda^*(a,\alpha), \end{cases} \tag{7}$$

where $\lambda^*(a,\alpha)$ is the optimal constant in "Hardy type" inequality (5).

When $\alpha = 1$, the fact that $\sqrt{a(x)}u_x \in L^2(0,1)$ does not imply that $u/\sqrt{x}$ belongs to $L^2(0,1)$, because λ^* may be equal to zero. So the case $\beta = 2-\alpha = 1$ is now forbidden. However, we have the following improved version of the preceding "Hardy type" inequality which is valid for all values $n > 0$, $\gamma < 2-\alpha$ and some positive constant $C_0 = C_0(a,\alpha,\gamma,n) > 0$. (See Theorem 21)

$$\int_0^1 a(x)u_x^2 dx + C_0 \int_0^1 u^2 dx \geq n \int_0^1 \frac{u^2}{x^\gamma} dx.$$

Applying the above inequality for $\gamma = \beta$, from the fact $\sqrt{a(x)}u_x \in L^2(0,1)$ one can imply $\dfrac{u}{x^{\beta/2}} \in L^2(0,1)$ for all $\beta < 2-\alpha$. Thus, in this case we assume

$$0 < \beta < 2 - \alpha = 1,$$

and no condition on λ is necessary.

2.3. **Statement of the problem.** Consider the operator

$$Au := (a(x)u_x)_x + \frac{\lambda}{x^\beta}u \tag{8}$$

where the coefficient $a(\cdot)$ satisfies Hypothesis 1. As mentioned in the last subsection, we will study the operator A under one of the two assumptions (7). Now let ω be a nonempty subinterval of $(0,1)$ and consider the following initial-boundary value problem in the domain $Q_T = (0,T) \times (0,1)$.

$$\begin{cases} u_t - Au = h\chi_\omega, & & (t,x) \in Q_T, \\ u(t,0) = u(t,1) = 0, & \text{in the case } \alpha \in (0,1), & t \in (0,T), \\ (a(x)u_x)(t,0) = u(t,1) = 0, & \text{in the case } \alpha \in [1,2), & t \in (0,T), \\ u(0,x) = u_0(x), & & x \in (0,1), \end{cases} \tag{9}$$

where the initial condition u_0 is given in $L^2(0,1)$ and $h \in L^2(Q_T)$. Depending on the value of α, we choose some natural boundary conditions. The problem is considered in appropriate weighted spaces which will be described in the next section. We will see that in these spaces, the Dirichlet boundary condition makes sense when $\alpha \in (0,1)$. (See Lemma 7). But for $\alpha \geq 1$, the trace at $x = 0$ does not make sense anymore, so we choose some suitable Neumann boundary condition in this case. (Lemma 10).

We are interested in the null controllability of (9) in time $T > 0$ with a distributed control supported in ω, i.e. for all $u_0 \in L^2(0,1)$, does there exist $h \in L^2(Q_T)$ such that $u(T,x) = 0$ for every $x \in [0,1]$? In this purpose, we drive Carleman estimates for operator A in Section 4. But before going any further, we describe in the following section the functional setting in which problem (9) is well-posed.

3. Functional setting and well-posedness

In order to investigate the well-posedness of equation (9), the unbounded operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ has to be studied in appropriate weighted spaces whose definitions depend on the values of α. Indeed a natural functional setting involves the space

$$H^1_a(0,1) := \{u \in L^2(0,1) \cap H^1_{loc}((0,1]) : \int_0^1 a(x)u_x^2 dx < \infty\},$$

which is a Hilbert space for the scalar product

$$\forall u, v \in H^1_a(0,1), \qquad (u,v)_{H^1_a} = \int_0^1 uv + a(x)u_x v_x dx.$$

The elements of $H^1_a(0,1)$ are characterized by the following property.

Lemma 6. *For $\alpha \in (0,2)$ and every $u \in H^1_a(0,1)$, we have $\lim_{x\to 0} xu^2 = 0$ and $\lim_{x\to 0} xu = 0$.*

Let us mention that for the well-posedness of boundary conditions in problem (9), we need to define the trace of u at boundary points $x = 0$ and $x = 1$ for any $u \in H^1_a(0,1)$. The trace at $x = 1$ obviously makes sense which allows to consider the Dirichlet boundary condition at this point. (Note that the function $u \in H^1_a(0,1)$ belongs to the Sobolev space $W^{1,2}$ in a neighborhood of $x = 1$). On the other hand, if $\alpha < 1$, the trace of u at $x = 0$ is meaningful because of the following lemma.

Lemma 7. *If $a(\cdot)$ satisfies Hypothesis 1 and $\alpha \in (0,1)$, then for every $u \in H^1_a(0,1)$ we have $u \in W^{1,1}(0,1) = \{u \in L^1(0,1) : u_x \in L^1(0,1)\}$ and so $u(0)$ is meaningful.*

Thus we could introduce the following space $H^1_{a,0}(0,1)$ depending on the values of α:

Definition 8. (i) For $\alpha \in (0,1)$, we define

$$H^1_{a,0}(0,1) := \{u \in H^1_a(0,1) : u(0) = u(1) = 0\},$$

(ii) For $\alpha \in [1,2)$, we let

$$H^1_{a,0}(0,1) := \{u \in H^1_a(0,1) : u(1) = 0\}.$$

There exists an important density result for the elements of $H^1_{a,0}$.

Proposition 9. *(i) For $\alpha \in (0,1)$, the space $C^\infty_c(0,1)$ is dense in $H^1_{a,0}(0,1)$.*
(ii) In the case $\alpha \in [1,2)$, the subset of $C^\infty([0,1])$ which vanishes at $x = 1$ is dense in $H^1_{a,0}(0,1)$.

Before going to define $D(A)$ we state the following lemma which results that the boundary condition makes sense in the case $\alpha \in [1,2)$.

Lemma 10. *Assume that $\alpha \in [1,2)$ and the condition (7) holds. Then for all $u \in H^1_a(0,1)$ such that $(au_x)_x + \dfrac{\lambda}{x^\beta}u \in L^2(0,1)$, we have $au_x \in W^{1,1}(0,1)$.*

Definition 11. (i) For $\alpha \in (0,1)$, we define

$$D(A) := \{u \in H^1_{a,0}(0,1) \cap H^2_{loc}((0,1]) : (au_x)_x + \frac{\lambda}{x^\beta}u \in L^2(0,1)\},$$

(ii) For $\alpha \in [1,2)$, we change the definition of $D(A)$ in the following way

$$D(A) := \{u \in H^1_{a,0}(0,1) \cap H^2_{loc}((0,1]) : (au_x)_x + \frac{\lambda}{x^\beta}u \in L^2(0,1), (au_x)(0) = 0\}.$$

Thus, in the case $\alpha \in (0,1)$, if $u \in D(A)$, then u satisfies the Dirichlet boundary conditions $u(0) = u(1) = 0$ and in the case $\alpha \in [1,2)$, every $u \in D(A)$ satisfies the Neumann boundary condition $(au_x)(0) = 0$ and the Dirichlet boundary condition $u(1) = 0$.

For the well-posedness of (9), it suffices to show that for suitable $k \geq 0$ the operator $-(A - kI)$ is self-adjoint and positive.

Proposition 12. *Assume that the condition* (7) *holds, then there exists a constant $k \geq 0$ such that the operator $(-(A - kI), D(A))$ is a self-adjoint and positive operator.*

Remark 13. As one can see in the proof of Proposition 12, the bilinear form associated to $-(A - kI)$ is coercive in $H^1_{a,0}(0,1)$.

Consequently, we have the following well-posedness result (see e.g. [8]).

Theorem 14. *Assume that the condition* (7) *holds and consider the problem* (9) *with $h \equiv 0$. Then, for all initial condition $u_0 \in L^2(0,1)$, the problem* (9) *has a unique solution*

$$u \in C^0([0,T], L^2(0,1)) \cap C^0((0,T], D(A)) \cap C^1((0,T], L^2(0,1)). \tag{10}$$

Moreover, if $u_0 \in D(A)$, then

$$u \in C^0([0,T], D(A)) \cap C^1([0,T], L^2(0,1)). \tag{11}$$

In addition, the inhomogeneous problem (9) *with $h \in L^2(Q_T)$, has a unique solution $u \in C^0([0,T], L^2(0,1))$ for all initial condition $u_0 \in L^2(0,1)$.*

4. Carleman estimates and Applications to controllability

As it is well-known, in order to get controllability results, we need to derive some observability inequalities for the adjoint problem

$$\begin{cases} v_t + (a(x)v_x)_x + \dfrac{\lambda}{x^\beta} v = 0, & & (t,x) \in Q_T, \\ v(t,1) = 0, & & t \in (0,T), \\ v(t,0) = 0, & \text{in the case } \alpha \in (0,1), & t \in (0,T), \\ (av_x)(t,0) = 0, & \text{in the case } \alpha \in [1,2), & t \in (0,T), \\ v(T,x) = v_T(x), & & x \in (0,1). \end{cases} \tag{12}$$

More precisely, we need to prove the following inequality.

Proposition 15. *Assume that the coefficient $a(\cdot)$ satisfies Hypothesis* 1*, let $T > 0$ be given and ω be a nonempty subinterval of $(0,1)$. Then there exists a positive constant $C = C(T, a, \alpha, \lambda)$ such that the following observability inequality is valid for every solution v of* (12)*,*

$$\int_0^1 v^2(0,x)dx \leq C \int_0^T \int_\omega v^2(t,x)dxdt. \tag{13}$$

Now, by standard arguments, a null controllability result follows.

Theorem 16. *Assume that* (7) *holds. Let* $T > 0$ *be given, and let* ω *be a nonempty subinterval of* $(0, 1)$. *Then, for all* $u_0 \in L^2(0, 1)$, *there exists* $h \in L^2((0, T) \times \omega)$ *such that the solution of* (9) *satisfies* $u(T) \equiv 0$ *in* $(0, 1)$. *Furthermore, we have the estimate*

$$\|h\|_{L^2((0,T)\times\omega)} \le C'\|u_0\|_{L^2(0,1)},$$

for some $C' = C'(T, a, \alpha, \lambda) > 0$.

Here, we summarisely explain how one can deduce null controllability from observability. For the complete proof, one can see [11] and [14].

Let us fix $T > 0$ and $u_0 \in L^2(0, 1)$. We consider for any $\epsilon > 0$, the functional $v_T \mapsto J_\epsilon(v_T)$ on $L^2(0, 1)$:

$$J_\epsilon(v_T) = \frac{1}{2}\left[\int_0^T \int_\omega v^2(t, x)dxdt\right]^2 + \epsilon\|v_T\|_{L^2(0,1)} + \int_0^1 v(0, x)u_0(x)dx,$$

where v is the corresponding solution of (12). One can see that $v_T \mapsto J_\epsilon(v_T)$ is a continuous and strictly convex function on $L^2(0, 1)$. Moreover, J_ϵ is coercive and achieves its minimum at a unique function $\hat{v}_\epsilon$. From $\hat{v}_\epsilon$, one can construct a control h_ϵ such that

$$\|u_\epsilon(T, \cdot)\|_{L^2(0,1)} \le \epsilon,$$

for u_ϵ be the corresponding solution of (9). Now, by observaility inequality (13), we obtain

$$\|h_\epsilon\|_{L^2((0,T)\times\omega)} \le C\|u_0\|_{L^2(0,1)}.$$

Therefore, by extracting an appropriate subsequence, we get

$$h_\epsilon \to h \qquad \text{weakly-* in} \quad L^2((0, T) \times \omega).$$

Since it is for all $\epsilon > 0$, we deduce that h is such that the correspondin solution u of (9) satisfies $u(T) \equiv 0$.

For the proof of the observability inequality (13), we need Carleman estimates for the degenerate and singular problems

$$\begin{cases} v_t + (a(x)v_x)_x + \dfrac{\lambda}{x^\beta}v - rv = h, & & (t, x) \in Q_T, \\ v(t, 1) = 0, & & t \in (0, T), \\ v(t, 0) = 0, & \text{in the case } \alpha \in (0, 1), & t \in (0, T), \\ (av_x)(t, 0) = 0, & \text{in the case } \alpha \in [1, 2), & t \in (0, T), \\ v(T, x) = v_T(x), & & x \in (0, 1), \end{cases} \tag{14}$$

where r is a nonnegative fixed constant. To define this estimates, consider $0 < \gamma < 2 - \alpha$ and $\sigma(t, x) := \theta(t)p(x)$ where

$$\theta(t) := \frac{1}{[t(T - t)]^k}, \qquad k := 1 + \frac{2}{\gamma} > 1, \tag{15}$$

$$p(x) := \frac{c_1}{2-\alpha}\left(\int_0^x \frac{y}{a(y)}dy - c_2\right). \tag{16}$$

Observe that there exists some $c > 0$ such that for all $t \in (0, T)$

$$|\theta_t(t)| \le c\theta^{1+\frac{1}{k}}(t), \qquad |\theta_{tt}(t)| \le c\theta^{1+\frac{2}{k}}(t). \tag{17}$$

For the moment we assume $c_1 > 0$ and $c_2 > \dfrac{1}{a(1)(2-\alpha)}$, so that $p(x) < 0$ for all $x \in [0, 1]$. As we shall see, a nice choice is $c_1 = 3$, and the following theorem is valid.

Theorem 17. *Assume that the function $a(\cdot)$ satisfies Hypothesis 1 and let $T > 0$. In the case $\alpha \in (0, 2)$, $\beta < 2 - \alpha$ and $\lambda \in \mathbb{R}$, for every $\gamma < 2 - \alpha$ there exists $R_0 = R_0(a, \alpha, \gamma, \lambda) > 0$ such that for all $R \ge R_0$ and all solutions v of (14), we have*

$$\begin{aligned}\frac{R^3}{(2-\alpha)^2}\int_0^T\int_0^1 \theta^3 + \frac{x^2}{a(x)}v^2 e^{2R\sigma(t,x)}dxdt + R\int_0^T\int_0^1 \theta a(x) v_x^2 e^{2R\sigma(t,x)}dxdt \\ + \frac{a(1)(1-\alpha)^2}{4}R\int_0^T\int_0^1 \theta\frac{v^2}{x^{2-\alpha}}e^{2R\sigma(t,x)}dxdt \\ + R\int_0^T\int_0^1 \theta\frac{v^2}{x^\gamma}e^{2R\sigma(t,x)}dxdt \\ \le \frac{1}{2}\int_0^T\int_0^1 |h|^2 e^{2R\sigma(t,x)}dxdt \\ + \frac{3Ra(1)}{2-\alpha}\int_0^T \theta(t)v_x^2(t,1)e^{2R\sigma(t,1)}dt. \end{aligned} \tag{18}$$

Also, in the case $\alpha \in (0, 2) \setminus \{1\}$, $\beta = 2 - \alpha$ and $\lambda < \lambda^*(a, \alpha)$, for every $\gamma < 2 - \alpha$ there exists a constant $R_0 = R_0(a, \alpha, \gamma, \lambda) > 0$ such that, for all $R \ge R_0$ and all solutions v of (14), we have

$$\begin{aligned}\frac{R^3}{(2-\alpha)^2}\int_0^T\int_0^1 \theta^3\frac{x^2}{a(x)}v^2 e^{2R\sigma(t,x)}dxdt + \frac{RC}{2}\int_0^T\int_0^1 \theta(a(x)v_x^2 \\ - \lambda\frac{v^2}{x^{2-\alpha}})e^{2R\sigma(t,x)}dxdt + R\int_0^T\int_0^1 \theta\frac{v^2}{x^\gamma}e^{2R\sigma(t,x)}dxdt \\ \le \frac{1}{2}\int_0^T\int_0^1 |h|^2 e^{2R\sigma(t,x)}dxdt \\ + \frac{3Ra(1)}{2-\alpha}\int_0^T \theta(t)v_x^2(t,1)e^{2R\sigma(t,1)}dt, \end{aligned} \tag{19}$$

where $C = \min\left\{1, \dfrac{\lambda^*-\lambda}{|\lambda|}\right\}$.

5. Hardy-Type inequalities

For the proof of Proposition 12, we need to use some "Hardy-type" inequalities. One of them is the following.

Lemma 18. *Let $\alpha \in (0,2)$. There exists an optimal constant $\lambda^*(a,\alpha)$ such that for every $u \in H^1_{a,0}(0,1)$, we have*

$$\int_0^1 a(x)u_x^2 dx \geq \lambda^*(a,\alpha)\int_0^1 \frac{u^2}{x^{2-\alpha}}dx. \tag{20}$$

In fact, one has $\lambda^*(a,\alpha) \geq a(1)(1-\alpha)^2/4$. Therefore $\lambda^*(a,\alpha) > 0$ for every $\alpha \in (0,2)\setminus\{1\}$, but $\lambda^*(a,1)$ might be equal to zero.

Remark 19. Observe that in the case $\alpha = 1$, the inequality (20) doesn't take any estimate for $\displaystyle\int_0^1 \frac{u^2}{x}dx$. However, in a similar way one can prove a "weaker" Hardy inequality in this case:

$$\int_0^1 a(x)u_x^2 dx \geq \frac{a(1)}{4}\int_0^1 \frac{u^2}{x(\ln x)^2}dx, \tag{21}$$

for every $u \in H^1_{a,0}(0,1)$.

Remark 20. For $\alpha \in (0,2)$, by (20) and (21) we have

$$\int_0^1 a(x)u_x^2 dx \geq C\int_0^1 u^2 dx, \qquad \forall u \in H^1_{a,0}(0,1).$$

Then we can consider the new equivalent norm $\|u\|_{H^1_{a,0}(0,1)} := \left(\int_0^1 a(x)u_x^2 dx\right)^{\frac{1}{2}}$ on $H^1_{a,0}(0,1)$.

Another useful inequality to achieve the desired result is the following improved Hardy inequality.

Theorem 21. *Suppose that the function $a(\cdot)$ satisfies Hypothesis 1 and let $\lambda < \lambda^*(a,\alpha)$ be given. Then for all $n > 0$ and $\gamma < 2-\alpha$, there exists some positive constant $C_0 = C_0(a,\alpha,\lambda,\gamma,n) > 0$ such that, for all $u \in H^1_{a,0}(0,1)$, the following inequality holds:*

$$\int_0^1 a(x)u_x^2 dx + C_0\int_0^1 u^2 dx \geq \lambda\int_0^1 \frac{u^2}{x^{2-\alpha}}dx + n\int_0^1 \frac{u^2}{x^\gamma}dx. \tag{22}$$

Proof. For the simplest example of the function $a(\cdot)$, namely x^α, Vancostenoble in [19] proved that if $\alpha \in [0,2)$, then for all $n > 0$ and $\gamma < 2-\alpha$ there exists some positive constant $C_0' = C_0'(\alpha,\gamma,n) > 0$ such that, for all $u \in C_c^\infty(0,1)$, the following inequality holds:

$$\int_0^1 x^\alpha u_x^2 dx + C_0'\int_0^1 u^2 dx \geq \frac{(1-\alpha)^2}{4}\int_0^1 \frac{u^2}{x^{2-\alpha}}dx + n\int_0^1 \frac{u^2}{x^\gamma}dx. \tag{23}$$

Also $C_0'(\alpha,\gamma,n)$ is explicitly given by

$$C_0'(\alpha,\gamma,n) = (n+1)^{\frac{2-\alpha+\gamma}{2-\alpha-\gamma}} \frac{2-\alpha-\gamma}{2-\alpha+\gamma}\left(\frac{4\gamma}{(2-\alpha)^2-\gamma^2}\right)^{\frac{2\gamma}{2-\alpha-\gamma}}.$$

Now, considering Proposition 9, we imply that the inequality (23) is true for every $u \in H^1_{a,0}(0,1)$ in the case $\alpha \in (0,1)$. For $\alpha \in [1,2)$, we can prove the inequality (23) for every $u \in C^\infty([0,1])$ such that $u(1) = 0$, by the similar method used in [19]. Thus the inequality will be true for every $u \in H^1_{a,0}(0,1)$ and $\alpha \in (0,2)$. On the other hand for every $x \in [0,1]$, we have $a(x) \geq a(1)x^\alpha$, so for all $n > 0$ and $\gamma < 2-\alpha$, there exists some positive constant $C_0 = C_0(a,\alpha,\gamma,n) > 0$ such that for all $u \in H^1_{a,0}(0,1)$, the following inequality holds.

$$\int_0^1 a(x)u_x^2dx + C_0\int_0^1 u^2dx \geq \frac{a(1)(1-\alpha)^2}{4}\int_0^1 \frac{u^2}{x^{2-\alpha}}dx + n\int_0^1 \frac{u^2}{x^\gamma}dx. \quad (24)$$

In fact, $C_0 = a(1)C_0'(\alpha,\gamma,\frac{n}{a(1)})$. Now, suppose that improved Hardy (22) is true for some $\mu \leq \lambda^* = \lambda^*(a,\alpha)$, i.e. for all $n > 0$ and $\gamma < 2-\alpha$ there exists some positive constant $C_0 = C_0(a,\alpha,\gamma,n)$ such that, for all $u \in H^1_{a,0}(0,1)$,

$$\int_0^1 a(x)u_x^2dx + C_0\int_0^1 u^2dx \geq \mu\int_0^1 \frac{u^2}{x^{2-\alpha}}dx + n\int_0^1 \frac{u^2}{x^\gamma}dx.$$

By summation with (20), we obtain

$$\int_0^1 a(x)u_x^2dx + \frac{C_0}{2}\int_0^1 u^2dx \geq \frac{\lambda^*+\mu}{2}\int_0^1 \frac{u^2}{x^{2-\alpha}}dx + \frac{n}{2}\int_0^1 \frac{u^2}{x^\gamma}dx.$$

Therefore, if improved Hardy (22) is true for μ, then it is also true for $\frac{\lambda^*+\mu}{2}$. Now, if we define the sequence $\{\mu_k\}$ with

$$\mu_0 = \frac{a(1)(1-\alpha)^2}{4}, \qquad \mu_{k+1} = \frac{\lambda^*+\mu_k}{2},$$

then the improved Hardy inequality is true for every $\lambda < \lambda^* = \lim_{k\to\infty}\mu_k$. $\square$

Remark 22. Clearly, if $\lambda^* = a(1)(1-\alpha)^2/4$, Improved Hardy (22) is true for λ^*. But if $\lambda^* > a(1)(1-\alpha)^2/4$, the above proof doesn't work for λ^* in the general case. In fact the constant coefficient C_0 in the inequality tends to infinity in the limit case $\mu_k \to \lambda^*$. If we let $C_0^k = C_0^k(a,\alpha,\gamma,n)$ the constant value in the Improved Hardy (22) for μ_k, we have

$$C_0^0(a,\alpha,\gamma,n) = a(1)C_0'\left(\alpha,\gamma,\frac{n}{a(1)}\right),$$

$$C_0^{k+1}(a,\alpha,\gamma,n) = \frac{1}{2}C_0^k\left(a,\alpha,\gamma,2n\right).$$

So, for $\gamma > 0$, we obtain

$$C_0^k(a, \alpha, \gamma, n) = \frac{1}{2^k} C_0^0 \left(a, \alpha, \gamma, 2^k n\right) \to \infty.$$

Therefore one can not deduce Improved Hardy for λ^* in such cases with the same method used for $\lambda < \lambda^*$.

6. Proofs

6.1. **Proof of Lemmas 4, 6, 7 and 10.**

Proof of Lemma 4. Since $xa'(x) \le \alpha a(x)$ for every x, the function $x \mapsto \dfrac{a(x)}{x^\alpha}$ is decreasing on $(0,1)$, so $a(x) \ge a(1)x^\alpha$ and for $\alpha < 1$

$$\int_0^1 \frac{dx}{a(x)} \le \int_0^1 \frac{dx}{a(1)x^\alpha} < +\infty.$$

□

Proof of Lemma 6. At first, we show that $xu^2 \in W^{1,1}$. It is obvious that $xu^2 \in L^1(0,1)$ for every $u \in H_a^1(0,1)$. On the other hand

$$(xu^2)_x = u^2 + 2xuu_x,$$
$$xuu_x = \left(\frac{x}{\sqrt{a(x)}} u\right)\left(\sqrt{a(x)} u_x\right),$$

and by Hypothesis 1 one can easily see that the function $x \mapsto \dfrac{x^2}{a(x)}$ is increasing, so

$$\frac{x}{\sqrt{a(x)}} \le \frac{1}{\sqrt{a(1)}} \Rightarrow \frac{x}{\sqrt{a(x)}} u \in L^2(0,1) \Rightarrow xuu_x \in L^1(0,1).$$

Hence $xu^2 \in W^{1,1}(0,1)$ and it follows that $xu^2 \to L \ge 0$ as $x \to 0$. If $L > 0$, then one could have

$$u \sim_{x \to 0^+} \sqrt{\frac{L}{x}} \notin L^2(0,1),$$

thus $L = 0$. Similarly one can see that $\lim_{x\to 0} xu = 0$. □

Proof of Lemma 7. For any $u \in H_a^1(0,1)$ we have $u \in L^2(0,1)$, so $u \in L^1(0,1)$. We show that $u_x \in L^1(0,1)$.

$$\int_0^1 |u_x| dx = \int_0^1 |\sqrt{a(x)} u_x \frac{1}{\sqrt{a(x)}}| dx \le \left(\int_0^1 \frac{1}{a(x)} dx \cdot \int_0^1 a(x) u_x^2 dx\right)^{\frac{1}{2}},$$

But by Lemma 4 the integral $\displaystyle\int_0^1 \frac{1}{a(x)} dx$ is finite, so $u_x \in L^1(0,1)$.

Now, consider a sequence $\{u_n\}$ of smooth functions which converge to u in $W^{1,1}(0,1)$ and let χ be a smooth cut-off function such that $\chi|_{[0,\frac{1}{2}]} \equiv 1$

and χ vanishes in some neighbourhood of 1. Then one has $\chi u_n \to \chi u$ in $W^{1,1}$. On the other hand for every x, we have

$$\chi u_n(x) = -\int_x^1 (\chi u_n)_x(t)dt,$$

which means that the $\lim_{x\to 0} \chi u_n(x)$ exists, thus $\lim_{x\to 0} u_n(x)$ exists and we define $u(0)$ to be equal to this value. □

Proof of Lemma 10. Let us denote $w = a(x)u_x$ and choose $M > 0$, such that $a(x) \le M$ for every $x \in [0,1]$. One has

$$\int_0^1 |w|dx = \int_0^1 |a(x)u_x|dx \le \sqrt{\int_0^1 a(x)^2 u_x^2 dx} \le \sqrt{M\int_0^1 a(x)u_x^2 dx} < +\infty,$$

since $u \in H_a^1(0,1)$. Next we write

$$\begin{aligned}\int_0^1 |w_x|dx = \int_0^1 |(a(x)u_x)_x|dx &\le \int_0^1 |(a(x)u_x)_x + \frac{\lambda}{x^\beta}u|dx + \int_0^1 |\frac{\lambda}{x^\beta}u|dx\\ &\le \sqrt{\int_0^1 |(a(x)u_x)_x + \frac{\lambda}{x^\beta}u|^2 dx} + |\lambda|\int_0^1 \frac{|u|}{x^\beta}dx.\end{aligned}$$

The first integral in the right hand side of the latter inequality is finite since $(a(x)u_x)_x + \lambda u/x^\beta \in L^2(0,1)$. Furthermore we have

$$\int_0^1 \frac{|u|}{x^\beta}dx = \int_0^1 \frac{1}{x^{\beta/2}}\frac{|u|}{x^{\beta/2}}dx \le \sqrt{\int_0^1 \frac{1}{x^\beta}\int_0^1 \frac{u^2}{x^\beta}dx}.$$

Note that $\beta < 1$ since $1 \le \alpha < 2$, thus the first integral in the right hand side is finite. Also the second one is finite by (22) using the fact that $\beta \le 2-\alpha$ for $\alpha \in (1,2)$ and $\beta < 2-\alpha$ for $\alpha = 1$. □

6.2. Proof of Proposition 9 and Lemma 18.

We first state the following lemma which will be useful for the proof of Proposition 9. For the proof see [4].

Lemma 23. *Suppose that $\alpha \in (0,1)$, then for every $u \in H_a^1(0,1)$ such that $u(0) = 0$, we have*

$$\int_0^1 \frac{a(x)}{x^2}u^2(x)dx \le \frac{4}{(1-\alpha)^2}\int_0^1 a(x)|u'(x)|^2 dx. \tag{25}$$

Remark 24. As one can see in [4], the inequality (25) is true for $\alpha \in (0,2)\setminus\{1\}$ by the hypothesis on the function $a(\cdot)$ defined in [4]. But our hypothesis is weaker than one defined in [4] for $\alpha \in (1,2)$ and the inequality fails in this case (See Example 3). We will prove a similar inequality for $\alpha \in (0,2)$ in Lemma 18.

Proof of Proposition 9. (i)**:** Since $C_c^\infty(0,1)$ is dense in $H_0^1(0,1)$ and the embedding of $H_0^1(0,1)$ into $H_{a,0}^1(0,1)$ is continuous, it suffices to prove that $H_0^1(0,1)$ is dense in $H_{a,0}^1(0,1)$. Let $v \in H_{a,0}^1(0,1)$ be given and define the family $\{v_\delta\}_{\delta\geq 0}$, with $\delta \in (0,1)$ in the following way

$$v_\delta(x) := \begin{cases} \dfrac{x}{\delta} v(x), & 0 \leq x \leq \delta, \\ v(x), & \delta < x \leq 1. \end{cases}$$

We want to show that:

(1) $v_\delta \in H_0^1(0,1)$ for all $\delta \in (0,1)$.

(2) $v_\delta \to v$ in $H_{a,0}^1(0,1)$ as $\delta \to 0$.

One has

$$\begin{aligned} \int_0^1 |v_\delta'(x)|^2 dx &= \int_0^\delta |\frac{1}{\delta} v(x) + \frac{x}{\delta} v'(x)|^2 dx + \int_\delta^1 |v'(x)|^2 dx \\ &\leq 2\int_0^\delta \frac{1}{\delta^2}|v(x)|^2 + \frac{x^2}{\delta^2}|v'(x)|^2 dx + \int_\delta^1 |v'(x)|^2 dx. \qquad (26) \end{aligned}$$

By Hypothesis 1 the function $x \mapsto \dfrac{a(x)}{x^2}$ is decreasing on $(0,1]$, so $a(x) \geq a(1)x^2$ for every $x \in [0,1]$. Therefore for every $\delta > 0$ we have

$$\int_0^\delta \frac{x^2}{\delta^2}|v'(x)|^2 dx \leq \frac{1}{a(1)\delta^2}\int_0^\delta a(x)|v'(x)|^2 dx. \qquad (27)$$

On the other hand, since $a(x) > 0$ in $(0,1]$, there exists $M_\delta > 0$ such that $a(x) \geq M_\delta$ in $[\delta, 1]$, So

$$\int_\delta^1 |v'(x)|^2 dx \leq M_\delta^{-1}\int_\delta^1 a(x)|v'(x)|^2 dx. \qquad (28)$$

Combining (26), (27) and (28) we obtain $C_\delta > 0$ such that

$$\int_0^1 |v_\delta'(x)|^2 dx \leq C_\delta \int_0^1 |v(x)|^2 + a(x)|v'(x)|^2 dx$$

The right part of the last inequality is finite since $v \in H_a^1(0,1)$. Then $v_\delta \in H_0^1(0,1)$. Also

$$\begin{aligned} \|v - v_\delta\|^2_{H_{a,0}^1} &= \int_0^1 |v - v_\delta|^2 + a(x)|v' - v_\delta'|^2 dx \\ &= \int_0^\delta \left|v - \frac{x}{\delta}v\right|^2 dx + \int_0^\delta a(x)\left|v' - \frac{v}{\delta} - \frac{x}{\delta}v'\right|^2 dx \\ &\leq \int_0^\delta v^2 dx + 2\int_0^\delta a(x)|v'|^2 dx + 2\int_0^\delta a(x)\frac{v^2}{\delta^2} dx. \end{aligned}$$

Now $\int_0^\delta v^2dx + 2\int_0^\delta a(x)|v'|^2dx \to 0$ as $\delta \to 0$, since $v \in H^1_{a,0}(0,1)$. On the other hand by Lemma 23 we obtain

$$\int_0^\delta a(x)\frac{v^2}{\delta^2}dx \le \int_0^\delta a(x)\frac{v^2}{x^2}dx \le \frac{4}{(1-\alpha)^2}\int_0^\delta a(x)|v'(x)|^2dx.$$

Indeed, we can rewrite the proof of Lemma 23 in the interval $[0,\delta]$ instead of $[0,1]$ and derive a new inequality with the same constant $\frac{4}{(1-\alpha)^2}$. Now the right part of the last inequality tends to zero as $\delta \to 0$, because $v \in H^1_{a,0}(0,1)$ and the proof of (i) is complete.

(ii)**:** Similarly in this case, for $v \in H^1_{a,0}(0,1)$ it suffices to construct functions $\{v_\delta\}_{\delta\ge 0}$ such that

(1) $v_\delta \in H^1(0,1)$ and $v_\delta(1) = 0$.

(2) $v_\delta \to v$ in $H^1_{a,0}(0,1)$ as $\delta \to 0$.

Define

$$v_\delta(x) := \begin{cases} v(2\delta - x), & 0 \le x \le \delta, \\ v(x), & \delta < x \le 1. \end{cases}$$

We have

$$v'_\delta(x) := \begin{cases} -v'(2\delta - x), & 0 < x < \delta, \\ v'(x), & \delta < x < 1. \end{cases}$$

Since a is strictly positive on $(0,1]$ and in the computing of $\int_0^1 |v'_\delta|^2dx$, we are far from the boundary, it is easy to see that $v_\delta \in H^1(0,1)$. Also

$$\|v_\delta - v\|^2_{H^1_a(0,1)} = \|v_\delta - v\|^2_{H^1_a(0,\delta)} \le 2[\|v\|^2_{H^1_a(0,\delta)} + \|v_\delta\|^2_{H^1_a(0,\delta)}].$$

Since $v \in H^1_a(0,1)$, the term $\|v\|^2_{H^1_a(0,\delta)}$ tends to zero as $\delta \to 0$. Also if $\delta \le \delta_0/2$ where δ_0 is the constant introduced in property (iii) of Hypothesis 1, then

$$\begin{aligned}\int_0^\delta v_\delta^2(x) + a(x)|v'_\delta(x)|^2dx &= \int_0^\delta v^2(2\delta - x) + a(x)|v'(2\delta - x)|^2dx \\ &\le \int_\delta^{2\delta} v^2(x) + \frac{1}{m}\int_\delta^{2\delta} a(x)|v'(x)|^2dx,\end{aligned}$$

which tends to zero as $\delta \to 0$. Observe that $v_\delta(1) = 0$, so the subset of $C^\infty([0,1])$ which vanishes at $x = 1$ is dense in $H^1_{a,0}(0,1)$. □

Before proving Lemma 18, we remark that for $\alpha \in (0,1)$ this lemma is a simple consequence of Lemma 23 and the fact that $a(x) \ge a(1)x^\alpha$ for $x \in [0,1]$. But our proof works for $\alpha \in (0,2)$.

Proof of Lemma 18. Note that by Proposition 9 it is enough to prove (20) for $u \in C_c^\infty(0,1)$ in the case $\alpha \in (0,1)$ and for $u \in C^\infty([0,1])$ such that

$u(1) = 0$ in the case $\alpha \in [1, 2)$. In these two cases we write

$$\int_0^1 \left(x^{\frac{\alpha}{2}} u_x - \frac{1-\alpha}{2} \frac{u}{x^{\frac{2-\alpha}{2}}} \right)^2 dx \geq 0.$$

Thus we get

$$\int_0^1 x^\alpha u_x^2 dx + \frac{(1-\alpha)^2}{4} \int_0^1 \frac{u^2}{x^{2-\alpha}} dx - \frac{(1-\alpha)}{2} \int_0^1 \frac{1}{x^{1-\alpha}} (u^2)_x dx \geq 0.$$

Now, if we use integration by parts, in the case $\alpha \in (0, 1]$ there exists no boundary term, also in the case $\alpha \in (1, 2)$, since $u \in C^\infty([0, 1])$ the term $\dfrac{u^2}{x^{1-\alpha}}$ tends to zero as $x \to 0$. Therefore, we obtain

$$\int_0^1 x^\alpha u_x^2 dx - \frac{(1-\alpha)^2}{4} \int_0^1 \frac{u^2}{x^{2-\alpha}} dx \geq 0.$$

Note that the function $x \mapsto \dfrac{a(x)}{x^\alpha}$ is decreasing on $(0, 1]$, so $a(x) \geq a(1)x^\alpha$ for every $x \in [0, 1]$ and deducely

$$\int_0^1 a(x) u_x^2 dx \geq a(1) \frac{(1-\alpha)^2}{4} \int_0^1 \frac{u^2}{x^{2-\alpha}} dx.$$

□

Proof of Proposition 12. **(i)** $-(A - kI)$ **is positive:** Let $k \geq 0$ arbitrarily for the moment, then for all $u \in D(A)$ we have

$$\langle -(A - kI)u, u \rangle = \int_0^1 a(x) u_x^2 - \frac{\lambda}{x^\beta} u^2 + k u^2.$$

Now, we distinguish three different cases:

$$\begin{array}{llll} \text{case 1:} & \alpha \neq 1, & \beta \leq 2 - \alpha, & \lambda < \lambda^*(a, \alpha), \\ \text{case 2:} & \alpha \neq 1, & \beta < 2 - \alpha, & \lambda \geq \lambda^*(a, \alpha), \\ \text{case 3:} & \alpha = 1, & \beta < 2 - \alpha, & \lambda \in \mathbb{R}. \end{array}$$

For the first case, we can write

$$\int_0^1 a(x) u_x^2 - \lambda \frac{u^2}{x^\beta} \geq \left(1 - \frac{\lambda}{\lambda^*(a, \alpha)} \right) \int_0^1 a(x) u_x^2 + \frac{\lambda}{\lambda^*(a, \alpha)} \int_0^1 \left(a(x) u_x^2 - \lambda^* \left(a, \alpha \right) \frac{u^2}{x^\beta} \right).$$

Therefore, by Lemma 18, we obtain

$$\int_0^1 a(x) u_x^2 - \lambda \frac{u^2}{x^\beta} \geq \left(1 - \frac{\lambda}{\lambda^*(a, \alpha)} \right) \|u\|^2_{H^1_{a,0}}.$$

Hence the result holds with $k = 0$. In the second and third case, since $\beta < 2 - \alpha$, one can use (22) with $\gamma = \beta$ and $n = 2\lambda$, so there exists $C_0 > 0$ such that

$$\int_0^1 \left(a(x)u_x^2 - 2\lambda \frac{u^2}{x^\beta} \right) + C_0 \int_0^1 u^2 \geq 0,$$

then

$$\int_0^1 a(x)u_x^2 - \lambda \frac{u^2}{x^\beta} + \frac{C_0}{2}u^2 \geq \frac{1}{2}\int_0^1 a(x)u_x^2 = \frac{1}{2}\|u\|^2_{H^1_{a,0}}.$$

Therefore the result holds with $k = \frac{C_0}{2}$.

(ii) $-(A - kI)$ **is self-adjoint:** Clearly, it is sufficient to show that A is self-adjoint, i.e. that $(A^*, D(A^*)) = (A, D(A))$. We have

$$D(A^*) = \left\{ v \in L^2(0,1) : \exists C > 0, \forall u \in D(A), |\langle v, Au\rangle| \leq C\|u\|_{L^2(0,1)} \right\}.$$

We first show $D(A) \subseteq D(A^*)$ and $A^*|_{D(A)} = A$. Suppose that $v \in D(A)$, for every $u \in D(A)$, we have

$$\langle v, Au\rangle = \langle v, (a(x)u_x)_x + \frac{\lambda}{x^\beta}u\rangle = \int_0^1 -a(x)u_x v_x + \frac{\lambda}{x^\beta}uv dx = \langle Av, u\rangle,$$

so $v \in D(A^*)$ and $A^*v = Av$. Now we show that $D(A^*) \subseteq D(A)$. If we define

$$\langle u, \phi\rangle_1 := \int_0^1 ku\phi + a(x)u_x\phi_x - \frac{\lambda}{x^\beta}u\phi dx,$$

then by the part (i):

$$\langle u, u\rangle_1 = \int_0^1 ku^2 + a(x)u_x^2 - \frac{\lambda}{x^\beta}u^2 dx \geq C^*\|u\|^2_{H^1_{a,0}},$$

and by Lemma 18 there exists $C_* > 0$ such that $\langle u, u\rangle_1 \leq C_*\|u\|^2_{H^1_{a,0}}$. Thus $\langle\ ,\ \rangle_1$ defines a scalar product on $H^1_{a,0}$ which corresponding norm is equivalent with $\|\cdot\|_{H^1_{a,0}}$, therefore $H^1_{a,0}$ with the inner product $\langle\ ,\ \rangle_1$ is a Hilbert space. So for every $f \in L^2(0,1)$ there exists a unique $u \in H^1_{a,0}$ such that

$$\langle u, \phi\rangle_1 = \int_0^1 f\phi dx, \qquad \forall\phi \in H^1_{a,0}.$$

Thus

$$ku - (a(x)u_x)_x - \frac{\lambda}{x^\beta}u = f,$$

then $u \in D(A)$, and satisfies in the equation $(kI - A)u = f$. Now suppose that $v \in D(A^*)$, we want to show $v \in D(A)$. Let $w := kv - A^*v \in L^2$ and consider the solution $(kI - A)u = w$. So for every $\phi \in D(A)$, we will have

$$\langle v, (kI - A)\phi\rangle = \langle (kI - A^*)v, \phi\rangle = \langle w, \phi\rangle = \langle (kI - A)u, \phi\rangle = \langle u, (kI - A)\phi\rangle,$$

since $u, \phi \in D(A)$ and $A^*|_{D(A)} = A$. Now let ϕ be the solution of $(kI - A)\phi = v - u$, we can imply that $v = u$. Therefore $v \in D(A)$ and $Av = A^*v$. □

6.3. **Proof of the Carleman estimates.** In order to prove Theorem 17 we define the function $w(t,x) := e^{R\sigma(t,x)}v(t,x)$, where v is the solution of (14) and $R > 0$ is a positive constant. Then w satisfies

$$\begin{cases} (e^{-R\sigma}w)_t + (a(x)(e^{-R\sigma}w)_x)_x + \frac{\lambda}{x^\beta}(e^{-R\sigma}w) - re^{-R\sigma}w = h, & (t,x) \in Q_T, \\ w(t,1) = 0, & t \in (0,T), \\ w(t,0) = 0, \qquad \text{for } \alpha \in (0,1), & t \in (0,T), \\ (aw_x)(t,0) = R(\sigma_x aw)(t,0), \qquad \text{for } \alpha \in [1,2), & t \in (0,T), \\ w(T,x) = w(0,x) = 0, & x \in (0,1). \end{cases} \tag{29}$$

Thanks to the definitions of p and σ, we have $(\sigma_x aw)(t,x) = \frac{c_1}{2-\alpha}x\theta(t)w(t,x)$. Since, for $t \in [0,T]$, the function $w(t,\cdot)$ is in $H^1_a(0,1)$, we deduce that $xw(t,x)|_{x=0} = 0$ for $t \in [0,T]$ using Lemma 6. Thus $(\sigma_x aw)(t,x)|_{x=0} = 0$ and the previous problem can be recast as follows. Set

$$Lv := v_t + (a(x)v_x)_x + \frac{\lambda}{x^\beta}v - rv, \qquad L_R w := e^{R\sigma}L(e^{-R\sigma}w).$$

Then (29) becomes

$$\begin{cases} L_R w = he^{R\sigma}, & (t,x) \in Q_T, \\ w(t,1) = 0, & t \in (0,T), \\ w(t,0) = 0, \qquad \text{in the case } \alpha \in (0,1), & t \in (0,T), \\ (aw_x)(t,0) = 0, \qquad \text{in the case } \alpha \in [1,2), & t \in (0,T), \\ w(T,x) = w(0,x) = 0, & x \in (0,1). \end{cases} \tag{30}$$

We have the following proposition which implies the Carleman estimates.

Proposition 25. *Let $T > 0$ be given.*
(i) In the case $\alpha \in (0,2)$, $\beta < 2-\alpha$, $\lambda \in \mathbb{R}$, for every $\gamma < 2-\alpha$ there exists a constant $R_0 = R_0(a,\alpha,\gamma,\lambda) > 0$ such that, for all $R \geq R_0$ and all solutions w of (30), we have

$$\begin{aligned} &\frac{19R^3}{(2-\alpha)^2}\int_0^T\int_0^1 \theta^3\frac{x^2}{a(x)}w^2dxdt + 2R\int_0^T\int_0^1 \theta a(x)w_x^2dxdt \\ &\quad + \frac{a(1)(1-\alpha)^2}{4}R\int_0^T\int_0^1 \theta\frac{w^2}{x^{2-\alpha}}dxdt + R\int_0^T\int_0^1 \theta\frac{w^2}{x^\gamma}dxdt \\ &\qquad \leq \frac{1}{2}\int_0^T\int_0^1 |h|^2e^{2R\sigma(t,x)}dxdt + \frac{3Ra(1)}{2-\alpha}\int_0^T \theta(t)w_x^2(t,1)dt. \end{aligned} \tag{31}$$

(ii) In the case $\alpha \in (0,2)\setminus\{1\}$, $\beta = 2-\alpha$, $\lambda < \lambda^*(a,\alpha)$, for every $\gamma < 2-\alpha$ there exists a constant $R_0 = R_0(a,\alpha,\gamma,\lambda) > 0$ such that, for all $R \geq R_0$ and all solutions w of (30)*, we have*

$$\frac{10R^3}{(2-\alpha)^2}\int_0^T\int_0^1 \theta^3\frac{x^2}{a(x)}w^2dxdt + \frac{3R}{2}\int_0^1 \theta(a(x)w_x^2 - \lambda\frac{w^2}{x^{2-\alpha}})dxdt$$

$$+ R\int_0^T\int_0^1 \theta\frac{w^2}{x^\gamma}dxdt \le \frac{1}{2}\int_0^T\int_0^1 |h|^2e^{2R\sigma(t,x)}dxdt + \frac{3Ra(1)}{2-\alpha}\int_0^T \theta(t)w_x^2(t,1)dt. \quad (32)$$

For the proof of this proposition, separate self-adjoint and skew-adjoint terms. Then one has

$$L_Rw = L_R^+w + L_R^-w,$$

where

$$L_R^+w := (a(x)w_x)_x - R\sigma_t w + R^2a(x)\sigma_x^2w + \frac{\lambda}{x^\beta}w - rw,$$
$$L_R^-w := w_t - 2Ra(x)\sigma_xw_x - R(a(x)\sigma_x)_xw.$$

Therefore, we have

$$\|he^{R\sigma}\|^2 = \|L_R^+w\|^2 + \|L_R^-w\|^2 + 2\langle L_R^+w, L_R^-w\rangle \ge 2\langle L_R^+w, L_R^-w\rangle, \quad (33)$$

where $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the usual norm and scalar product in $L^2(Q_T)$, respectively. The proof of Proposition 25, is based on the computation of the scalar product $\langle L_R^+w, L_R^-w\rangle$, which comes in the following lemma. The proof is similar to one stated in [4, 19].

Lemma 26. *The scalar product $\langle L_R^+w, L_R^-w\rangle$ may be written as a sum of a distributed term A and a boundary term B, $\langle L_R^+w, L_R^-w\rangle = A + B$, where the distributed term A is given by*

$$A = -2R^2\int_0^T\int_0^1 \theta\theta_tp_x^2a(x)w^2dxdt + \frac{R}{2}\int_0^T\int_0^1 \theta_{tt}pw^2dxdt$$
$$+ R\int_0^T\int_0^1 \theta(2a^2(x)p_{xx} + a(x)a'(x)p_x)w_x^2dxdt$$
$$+ R^3\int_0^T\int_0^1 \theta^3(2a^2(x)p_{xx} + a(x)a'(x)p_x)p_x^2w^2dxdt$$
$$- \beta\lambda R\int_0^T\int_0^1 \theta p_x\frac{a(x)}{x^{\beta+1}}w^2,$$

whereas the boundary term B is given by

$$B = \frac{-Ra(1)c_1}{2-\alpha}\int_0^T \theta(t)w_x^2(t,1)dt,$$

where $\theta(t)$ and $p(x)$ defined by (15) *and* (16)*.*

Now we estimate the distributed term A in the following lemma.

Lemma 27. (*i*) *In the case $\alpha \in (0,2), \beta < 2-\alpha$, $\lambda \in \mathbb{R}$ for every $\gamma < 2-\alpha$, there exists a constant $R_0 = R_0(a,\alpha,\gamma,\lambda) > 0$ such that, for all*

$R \geq R_0$ we have

$$A \geq \frac{19R^3}{(2-\alpha)^2}\int_0^T\int_0^1 \theta^3 \frac{x^2}{a(x)} w^2 dxdt + 2R\int_0^T\int_0^1 \theta a(x) w_x^2 dxdt$$
$$+ \frac{a(1)(1-\alpha)^2}{4} R\int_0^T\int_0^1 \theta \frac{w^2}{x^{2-\alpha}} dxdt + R\int_0^T\int_0^1 \theta \frac{w^2}{x^\gamma} dxdt.$$

(ii) In the case $\alpha \in (0,2)\setminus\{1\}$, $\beta = 2-\alpha$, $\lambda < \lambda^*(a,\alpha)$, for every $\gamma < 2-\alpha$, there exists a constant $R_0 = R_0(a,\alpha,\gamma,\lambda) > 0$ such that, for all $R \geq R_0$ we have

$$A \geq \frac{10R^3}{(2-\alpha)^2}\int_0^T\int_0^1 \theta^3 \frac{x^2}{a(x)} w^2 dxdt + \frac{3R}{2}\int_0^T\int_0^1 \theta (a(x) w_x^2 - \lambda \frac{w^2}{x^{2-\alpha}}) dxdt$$
$$+ R\int_0^T\int_0^1 \theta \frac{w^2}{x^\gamma} dxdt.$$

Proof . By Lemma 26, assumption $xa'(x) \leq \alpha a(x)$ and the fact that $p_x = \dfrac{c_1 x}{(2-\alpha)a(x)}$, one can estimate A in the following way:

$$A \geq -\frac{2R^2c_1^2}{(2-\alpha)^2}\int_0^T\int_0^1 \theta\theta_t \frac{x^2}{a(x)} w^2 dxdt + \frac{R}{2}\int_0^T\int_0^1 \theta_{tt} p w^2 dxdt$$
$$+ Rc_1\int_0^T\int_0^1 \theta a(x) w_x^2 dxdt + \frac{R^3c_1^3}{(2-\alpha)^2}\int_0^T\int_0^1 \theta^3 \frac{x^2}{a(x)} w^2 dxdt$$
$$- \frac{\beta\lambda R c_1}{2-\alpha}\int_0^T\int_0^1 \theta \frac{w^2}{x^\beta} dxdt.$$

Let

$$A_1 := -\frac{2R^2c_1^2}{(2-\alpha)^2}\int_0^T\int_0^1 \theta\theta_t \frac{x^2}{a(x)} w^2 dxdt,$$
$$A_2 := \frac{R}{2}\int_0^T\int_0^1 \theta_{tt} p w^2 dxdt,$$
$$A_3 := Rc_1\int_0^T\int_0^1 \theta a(x) w_x^2 dxdt + \frac{R^3c_1^3}{(2-\alpha)^2}\int_0^T\int_0^1 \theta^3 \frac{x^2}{a(x)} w^2 dxdt,$$
$$A_4 := -\frac{\beta\lambda R c_1}{2-\alpha}\int_0^T\int_0^1 \theta \frac{w^2}{x^\beta} dxdt.$$

First, we estimate the term A_1. According to the relation (17), we know that $|\theta\theta_t| \leq c\theta^{2+\frac{1}{k}} \leq \tilde{c}\theta^3$, and obtain

$$|A_1| \leq \frac{2R^2c_1^2\tilde{c}}{(2-\alpha)^2}\int_0^T\int_0^1 \theta^3 \frac{x^2}{a(x)} w^2 dxdt.$$

Therefore

$$A \geq \left(\frac{R^3 c_1^3}{(2-\alpha)^2} - \frac{2R^2 c_1^2 \tilde{c}}{(2-\alpha)^2} \right) \int_0^T \int_0^1 \theta^3 \frac{x^2}{a(x)} w^2 dxdt + Rc_1 \int_0^T \int_0^1 \theta a(x) w_x^2 dxdt + A_2 + A_4. \quad (34)$$

In the following, we produce estimates of the last two terms A_2 and A_4 to complete the proof. We distinguish two cases: the case of a sub-critical exponent $\beta < 2-\alpha$ and the case of a critical exponent $\beta = 2-\alpha$.

First case: $\alpha \in (0,2)$, $0 < \beta < 2-\alpha$ and $\lambda \in \mathbb{R}$.
We want to prove the result for all γ satisfying $0 < \gamma < 2-\alpha$. However, if the result holds for any γ such that $\beta \leq \gamma < 2-\alpha$, then it obviously also holds for all γ such that $0 < \gamma < 2-\alpha$. Therefore, we consider here γ, such that $\beta \leq \gamma < 2-\alpha$ and we study the term A_4. In the case $\lambda > 0$, we apply the improved Hardy inequality (24) with $n = \lambda c_1 + 3 - \alpha \geq \dfrac{\lambda\beta c_1}{2-\alpha} + 3 - \alpha$, which gives:

$$\int_0^1 a(x) w_x^2 dx + C_0 \int_0^1 w^2 dx \geq \frac{a(1)(1-\alpha)^2}{4} \int_0^1 \frac{w^2}{x^{2-\alpha}} dx + \left(\frac{\beta\lambda c_1}{2-\alpha} + 3 - \alpha \right) \int_0^1 \frac{w^2}{x^\gamma} dx,$$

for a suitable $C_0 = C_0(a, \alpha, n, \gamma) = C_0(a, \alpha, \lambda, \gamma, c_1)$. Therefore we can write

$$\begin{aligned} A_4 &= -\frac{\beta\lambda R c_1}{2-\alpha} \int_0^T \int_0^1 \theta \frac{w^2}{x^\beta} dxdt \geq -\frac{\beta\lambda R c_1}{2-\alpha} \int_0^T \int_0^1 \theta \frac{w^2}{x^\gamma} dxdt \\ &\geq \frac{a(1)(1-\alpha)^2}{4} R \int_0^T \int_0^1 \theta \frac{w^2}{x^{2-\alpha}} dxdt - R \int_0^T \int_0^1 \theta a(x) w_x^2 dxdt \\ &\quad + R(3-\alpha) \int_0^T \int_0^1 \theta \frac{w^2}{x^\gamma} dxdt - C_0 R \int_0^T \int_0^1 \theta w^2 dxdt. \end{aligned}$$

For $\lambda \leq 0$, we have

$$A_4 = -\frac{\beta\lambda R c_1}{2-\alpha} \int_0^T \int_0^1 \theta \frac{w^2}{x^\beta} dxdt \geq 0.$$

Applying (24) with $n = 3 - \alpha$, that is

$$\int_0^1 a(x) w_x^2 + C_0 \int_0^1 w^2 \geq \frac{a(1)(1-\alpha)^2}{4} \int_0^1 \frac{w^2}{x^{2-\alpha}} + (3-\alpha) \int_0^1 \frac{w^2}{x^\gamma},$$

we obtain the same estimate as in the case $\lambda > 0$. It follows that

$$\begin{aligned} A \geq &\left(\frac{R^3 c_1^3}{(2-\alpha)^2} - \frac{2R^2 c_1^2 \tilde{c}}{(2-\alpha)^2} \right) \int_0^T \int_0^1 \theta^3 \frac{x^2}{a(x)} w^2 dxdt \\ &+ R(c_1 - 1) \int_0^T \int_0^1 \theta a(x) w_x^2 dxdt \end{aligned}$$

$$+\frac{a(1)(1-\alpha)^2}{4}R\int_0^T\int_0^1\theta\frac{w^2}{x^{2-\alpha}}dxdt$$
$$+R(3-\alpha)\int_0^T\int_0^1\theta\frac{w^2}{x^\gamma}dxdt$$
$$+A_2-C_0R\int_0^T\int_0^1\theta w^2dxdt. \tag{35}$$

Finally, we need to estimate the two last terms in the above inequality. By (17), we have

$$|\theta_{tt}|\|p\|_\infty \le K\theta^{1+\frac{2}{k}},$$

for some $K>0$. It follows that

$$\left|A_2-C_0R\int_0^T\int_0^1\theta w^2dxdt\right| \le RK'\int_0^T\int_0^1\theta^{1+\frac{2}{k}}w^2dxdt, \tag{36}$$

for some $K'=K'(a,\alpha,\lambda,\gamma,c_1)>0$. At this stage, we use the special choice of k, that is

$$k=1+\frac{2}{\gamma},$$

and consider $q=\frac{k}{k-1}$ and $q'=k$, so that $\frac{1}{q}+\frac{1}{q'}=1$. Then, for all $\epsilon>0$, we have

$$\int_0^T\int_0^1\theta^{1+\frac{2}{k}}w^2dxdt=\int_0^T\int_0^1\left(\theta^{1+\frac{2}{k}-\frac{3}{q'}}a^{\frac{1}{q'}}x^{\frac{-2}{q'}}w^{\frac{2}{q}}\right)\left(\theta^{\frac{3}{q'}}x^{\frac{2}{q'}}a^{\frac{-1}{q'}}w^{\frac{2}{q'}}\right)dxdt$$
$$\le\epsilon\int_0^T\int_0^1\theta^{(1+\frac{2}{k}-\frac{3}{q'})q}a^{\frac{q}{q'}}x^{\frac{-2q}{q'}}w^2dxdt+C(\epsilon)\int_0^T\int_0^1\theta^3\frac{x^2}{a(x)}w^2dxdt, \tag{37}$$

where $C(\epsilon)=(\epsilon q)^{\frac{-q'}{q}}q'^{-1}$. Note that

$$q\left(1+\frac{2}{k}-\frac{3}{q'}\right)=1,\qquad \frac{2q}{q'}=\gamma.$$

Now if $K''>0$ be such that $a(x)^{\frac{q}{q'}}\le K''$ for every $x\in[0,1]$, then we obtain

$$R\int_0^T\int_0^1\theta^{1+\frac{2}{k}}w^2dxdt\le\epsilon K''R\int_0^T\int_0^1\theta\frac{w^2}{x^\gamma}dxdt$$
$$+C(\epsilon)R\int_0^T\int_0^1\theta^3\frac{x^2}{a(x)}w^2dxdt. \tag{38}$$

Putting the estimate (38) in (36) and using (35), we obtain

$$A\ge\left(\frac{R^3c_1^3}{(2-\alpha)^2}-\frac{2R^2c_1^2\tilde{c}}{(2-\alpha)^2}-C(\epsilon)K'R\right)\int_0^T\int_0^1\theta^3\frac{x^2}{a(x)}w^2dxdt$$

$$+R(c_1-1)\int_0^T\int_0^1 \theta a(x)w_x^2dxdt+\frac{a(1)(1-\alpha)^2}{4}R\int_0^T\int_0^1\theta\frac{w^2}{x^{2-\alpha}}dxdt$$
$$+R(3-\alpha-\epsilon K''K')\int_0^T\int_0^1\theta\frac{w^2}{x^\gamma}dxdt.$$

Now take $\epsilon=\epsilon(a,\alpha,\lambda,\gamma,c_1)>0$ such that $3-\alpha-\epsilon K''K'=1$, also let $c_1=3$. Thus there exists $R_0=R_0(a,\alpha,\lambda,\gamma)>0$ such that for all $R\geq R_0$

$$\begin{aligned}A\geq&\frac{19R^3}{(2-\alpha)^2}\int_0^T\int_0^1\theta^3\frac{x^2}{a(x)}w^2dxdt+2R\int_0^T\int_0^1\theta a(x)w_x^2dxdt\\&+\frac{a(1)(1-\alpha)^2}{4}R\int_0^T\int_0^1\theta\frac{w^2}{x^{2-\alpha}}dxdt+R\int_0^T\int_0^1\theta\frac{w^2}{x^\gamma}dxdt.\end{aligned}$$

Second case: $\alpha\in(0,2)\setminus\{1\}$, $\beta=2-\alpha$ and $\lambda<\lambda^*$.
Let us fix γ such that $0<\gamma<2-\alpha$. In the present case, we observe that

$$A_4=-\lambda Rc_1\int_0^T\int_0^1\theta\frac{w^2}{x^{2-\alpha}}.$$

Therefore by (34) one has

$$\begin{aligned}A\geq&\left(\frac{R^3c_1^3}{(2-\alpha)^2}-\frac{2R^2c_1^2\tilde{c}}{(2-\alpha)^2}\right)\int_0^T\int_0^1\theta^3\frac{x^2}{a(x)}w^2\\&+Rc_1\int_0^T\int_0^1\theta\left(a(x)w_x^2-\lambda\frac{w^2}{x^{2-\alpha}}\right)+A_2.\end{aligned}$$

Now, if we apply (22) with $n=2$, we obtain

$$\int_0^1 a(x)w_x^2dx-\lambda\int_0^1\frac{w^2}{x^{2-\alpha}}dx\geq 2\int_0^1\frac{w^2}{x^\gamma}dx-C_0\int_0^1 w^2dx,$$

for a constant $C_0=C_0(a,\alpha,\gamma)$. Therefore

$$\begin{aligned}A\geq&\left(\frac{R^3c_1^3}{(2-\alpha)^2}-\frac{2R^2c_1^2\tilde{c}}{(2-\alpha)^2}\right)\int_0^T\int_0^1\theta^3\frac{x^2}{a(x)}w^2dxdt+\frac{Rc_1}{2}\int_0^T\int_0^1\theta(a(x)w_x^2\\&-\lambda\frac{w^2}{x^{2-\alpha}})dxdt+Rc_1\int_0^T\int_0^1\theta\frac{w^2}{x^\gamma}dx-\frac{Rc_1C_0}{2}\int_0^T\int_0^1\theta w^2dx+A_2.\end{aligned}$$

Next we need to estimate the term A_2 and the last term in the above inequality. Proceeding as in the previous case, similar to the relations (36) and (38), we get

$$\begin{aligned}\left|A_2-\frac{Rc_1C_0}{2}\int_0^T\int_0^1\theta w^2dxdt\right|\leq&\ \epsilon K'K''R\int_0^T\int_0^1\theta\frac{w^2}{x^\gamma}dxdt\\&+C(\epsilon)K'R\int_0^T\int_0^1\theta^3\frac{x^2}{a(x)}w^2dxdt,\end{aligned}$$

for suitable positive K' and K''. Then as in the previous case, we can set $c_1 = 3$ and there exists $R_0 = R_0(a, \alpha, \gamma, \lambda)$ such that for every $R \geq R_0$, one has

$$
\begin{aligned}
A \geq & \frac{10R^3}{(2-\alpha)^2} \int_0^T \int_0^1 \theta^3 \frac{x^2}{a(x)} w^2 dxdt \\
& + \frac{3R}{2} \int_0^T \int_0^1 \theta \left(a(x) w_x^2 - \lambda \frac{w^2}{x^{2-\alpha}} \right) dxdt + R \int_0^T \int_0^1 \theta \frac{w^2}{x^\gamma} dxdt.
\end{aligned}
$$

□

Now from Lemma 26, Lemma 27, the inequality (33) and the fact that $c_1 = 3$, we can easily imply Proposition 25. On the other hand, since $v = e^{-R\sigma} w$ and $c_1 = 3$, one has

$$
v_x^2 \leq 2e^{-2R\sigma} \left(w_x^2 + \frac{9R^2}{(2-\alpha)^2} \theta^2 \frac{x^2}{a(x)^2} w^2 \right).
$$

So, the left hand of (18) is smaller than

$$
\begin{aligned}
& \left(\frac{R^3}{(2-\alpha)^2} + \frac{18R^3}{(2-\alpha)^2} \right) \int_0^T \int_0^1 \theta^3 \frac{x^2}{a(x)} w^2 dxdt + 2R \int_0^T \int_0^1 \theta a(x) w_x^2 dxdt \\
& \qquad + \frac{a(1)(1-\alpha)^2}{4} R \int_0^T \int_0^1 \theta \frac{w^2}{x^{2-\alpha}} dxdt + R \int_0^T \int_0^1 \theta \frac{w^2}{x^\gamma} dxdt.
\end{aligned}
$$

Also, note that $w_x(t,1) = v_x(t,1) e^{R\sigma(t,1)}$, since $v(t,1) = 0$. Now, by Proposition 25 part (i) of Theorem 17 follows immediately.
For the proof of part (ii), note that the left hand of (19) is smaller than

$$
\begin{aligned}
& \frac{10R^3}{(2-\alpha)^2} \int_0^T \int_0^1 \theta^3 \frac{x^2}{a(x)} w^2 dxdt + RC \int_0^T \int_0^1 \theta (a(x) w_x^2 - \lambda \frac{w^2)}{x^{2-\alpha}}) dxdt \\
& \qquad + \frac{\lambda RC}{2} \int_0^T \int_0^1 \theta \frac{w^2}{x^{2-\alpha}} dxdt + R \int_0^T \int_0^1 \theta \frac{w^2}{x^\gamma} dxdt.
\end{aligned}
$$

Now, if $\lambda \leq 0$, the above expression is smaller than the left hand of (32) and similar to (i) we can prove the desired result. But for $\lambda > 0$, we use Lemma 18 to get

$$
\begin{aligned}
\frac{\lambda RC}{2} \int_0^T \int_0^1 \theta \frac{w^2}{x^{2-\alpha}} dxdt & \leq \frac{\lambda RC}{2(\lambda^* - \lambda)} \int_0^T \int_0^1 \theta \left(a(x) w_x^2 - \lambda \frac{w^2}{x^{2-\alpha}} \right) dxdt \\
& \leq \frac{R}{2} \int_0^T \int_0^1 \theta \left(a(x) w_x^2 - \lambda \frac{a(x)}{x^{2-\alpha}} w^2 \right) dxdt. \qquad (39)
\end{aligned}
$$

Therefore we can easily complete the proof of part (ii).

6.4. **Proof of the observability inequality.** In this subsection we will prove Proposition 15 as a consequence of the Carleman estimates stated in Section 4. First, define $\tilde{v}(t,x) := e^{rt}v(t,x)$, where v is the solution of (14) and r will be defined later. Then $\tilde{v}$ satisfies

$$\begin{cases} \tilde{v}_t + (a(x)\tilde{v}_x)_x + \frac{\lambda}{x^\beta}\tilde{v} - r\tilde{v} = 0, & & (t,x) \in Q_T, \\ \tilde{v}(t,1) = 0, & & t \in (0,T), \\ \tilde{v}(t,0) = 0, & \text{in the case } \alpha \in (0,1), & t \in (0,T) \\ (a\tilde{v}_x)(t,0) = 0 & \text{in the case } \alpha \in [1,2), & t \in (0,T), \\ \tilde{v}(T,x) = e^{rT}v_T(x), & & x \in (0,1). \end{cases} \tag{40}$$

Furthermore, it is obvious that if the observability inequality (13) is true for $\tilde{v}$, then it is also true for v. First, let us state the following standard inequality to be proved later on.

Lemma 28. *(Caccioppoli's inequality) Let $\omega' \subset\subset \omega$, then there exist a positive constant $\tilde{C}$ such that for every solution v of* (14)*,*

$$\int_0^T \int_{\omega'} \tilde{v}_x^2 e^{2R\sigma} dxdt \le \tilde{C} \int_0^T \int_\omega \tilde{v}^2 dxdt.$$

Proof of Proposition 15. We consider $\omega' = (x_0', x_1') \subset\subset \omega = (x_0, x_1)$ and a smooth cut-off function $0 \le \psi \le 1$ such that $\psi = 1$ in $(0, x_0')$ and $\psi = 0$ in $(x_1', 1)$. Also define $w := \psi\tilde{v}$ where $\tilde{v}$ is the solution of (40). Then w satisfies (14) with some right-hand side h explicitly given in term of $\tilde{v}$ and $\tilde{v}_x$ and supported in ω'. Applying Theorem 17 with $\gamma = \frac{2-\alpha}{2}$ and $R_0 = R_0(a, \alpha, \lambda)$, we get (note that $w_x(t,1) = 0$):

$$\begin{aligned} R_0 \int_0^T \int_0^1 \theta w^2 e^{2R_0\sigma} dxdt &\le R_0 \int_0^T \int_0^1 \theta \frac{w^2}{x^\gamma} e^{2R_0\sigma} dxdt \\ &\le \hat{C} \int_0^T \int_{\omega'} (\tilde{v}_x^2 + \tilde{v}^2) e^{2R_0\sigma} dxdt. \end{aligned}$$

According to Lemma 28, we obtain

$$R_0 \int_0^T \int_0^1 \theta w^2 e^{2R_0\sigma} dxdt \le \tilde{C} \int_0^T \int_\omega \tilde{v}^2 dxdt,$$

next we use the definition of ψ to obtain a bound for v on $(0, x_0)$

$$\int_0^T \int_0^{x_0} \theta \tilde{v}^2 e^{2R_0\sigma} dxdt \le \frac{\tilde{C}}{R_0} \int_0^T \int_\omega \tilde{v}^2 dxdt.$$

Using the properties of $\theta(t)$ and $p(x)$, for a positive constant $c(T)$ we have

$$e^{-c(T)R_0} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^{x_0} \tilde{v}^2 dxdt \le \frac{\tilde{C}}{R_0} \int_0^T \int_\omega \tilde{v}^2 dxdt.$$

Clearly, using a similar cut-off argument, $\tilde{v}$ can be estimated on $(x_1, 1)$ in the same way. In this case, we use the classical Carleman estimates, since the operator A is neither degenerate nor singular on $(x_1, 1)$. Therefore we get

$$\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 \tilde{v}^2(t,x)dxdt \leq \frac{\tilde{C}e^{c(T)R_0}}{R_0}\int_0^T\int_\omega \tilde{v}^2 \leq \tilde{C}e^{c(T)R_0}\int_0^T\int_\omega \tilde{v}^2 dxdt.$$

Now, consider the function $t \mapsto \int_0^1 \tilde{v}^2(t,x)dx$, its derivative is

$$\begin{aligned}\int_0^1 2\tilde{v}\tilde{v}_t dx &= 2\int_0^1 \tilde{v}\left[-(a(x)\tilde{v}_x)_x - \frac{\lambda}{x^\beta}\tilde{v} + r\tilde{v}\right]dx\\ &= 2\int_0^1 a(x)\tilde{v}_x^2 - \frac{\lambda}{x^\beta}\tilde{v}^2 + r\tilde{v}^2,\end{aligned}$$

and if let $r = C_0$ where C_0 be as defined in (24) when we set $n = \lambda$, then the last term is positive. (Note that this is true in the case $\beta < 2-\alpha$. For $\beta = 2-\alpha$, we can set $r = 0$ and use Lemma 18). Therefore the function $t \mapsto \int_0^1 \tilde{v}^2(t,x)dx$ is nondecreasing in $[0,T]$ and we obtain

$$\int_0^1 \tilde{v}^2(0,x)dx \leq \frac{2}{T}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 \tilde{v}^2(t,x)dxdt \leq \frac{2\tilde{C}e^{c(T)R_0}}{T}\int_0^T\int_\omega \tilde{v}^2 dxdt.$$

□

Proof of Lemma 28. Let us consider a smooth function $\xi : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} 0 \leq \xi(x) \leq 1, & \forall x \in \mathbb{R},\\ \xi(x) = 1, & x \in \omega',\\ \xi(x) = 0, & x < x_0 \quad \textit{and} \quad x > x_1,\end{cases}$$

and $\xi > 0$ for $x > x_0$ and $x < x_1$. Then

$$\begin{aligned}0 =& \int_0^T \frac{d}{dt}\int_0^1 \xi^2 e^{2R\sigma}\tilde{v}^2 dxdt = \int_0^T\int_0^1 (2R\xi^2\sigma_t e^{2R\sigma}\tilde{v}^2 + 2\xi^2 e^{2R\sigma}\tilde{v}\tilde{v}_t)dxdt\\ =&2\int_0^T\int_0^1 R\xi^2\sigma_t e^{2R\sigma}\tilde{v}^2 dxdt \qquad (41)\\ &+ 2\int_0^T\int_0^1 \xi^2 e^{2R\sigma}\tilde{v}\left(-\frac{\lambda}{x^\beta}\tilde{v} - (a\tilde{v}_x)_x + r\tilde{v}\right)dxdt\\ =&2\int_0^T\int_0^1 R\xi^2\sigma_t e^{2R\sigma}\tilde{v}^2 dxdt - 2\int_0^T\int_0^1 \frac{\lambda}{x^\beta}\xi^2 e^{2R\sigma}\tilde{v}^2 dxdt\\ &+ 2\int_0^T\int_0^1 (\xi^2 e^{2R\sigma}\tilde{v})_x a\tilde{v}_x dxdt \qquad (42)\\ &+ 2r\int_0^T\int_0^1 \xi^2 e^{2R\sigma}\tilde{v}^2\\ =&2\int_0^T\int_0^1 R\xi^2\sigma_t e^{2R\sigma}\tilde{v}^2 dxdt - 2\int_0^T\int_0^1 \frac{\lambda}{x^\beta}\xi^2 e^{2R\sigma}\tilde{v}^2 dxdt\end{aligned}$$

$$+ 2\int_0^T\int_0^1 a(\xi^2 e^{2R\sigma})_x \tilde{v}\tilde{v}_x dxdt + 2\int_0^T\int_0^1 a\xi^2 e^{2R\sigma}\tilde{v}_x^2 dxdt \qquad (43)$$
$$+ 2r\int_0^T\int_0^1 \xi^2 e^{2R\sigma}\tilde{v}^2.$$

Hence,

$$\begin{aligned}2\int_0^T\int_0^1 \xi^2 e^{2R\sigma} a\tilde{v}_x^2 dxdt &= -2\int_0^T\int_0^1 R\xi^2\sigma_t e^{2R\sigma}\tilde{v}^2 dxdt\\ &- 2\int_0^T\int_0^1 a(\xi^2 e^{2R\sigma})_x\tilde{v}\tilde{v}_x dxdt + 2\int_0^T\int_0^1 \frac{\lambda}{x^\beta}\xi^2 e^{2R\sigma}\tilde{v}^2\\ &- 2r\int_0^T\int_0^1 \xi^2 e^{2R\sigma}\tilde{v}^2 dxdt.\end{aligned}$$

Case1: For $\lambda \le 0$, we obtain

$$\begin{aligned}2\int_0^T\int_0^1 \xi^2 e^{2R\sigma} a\tilde{v}_x^2 dxdt &\le -2\int_0^T\int_0^1 R\xi^2\sigma_t e^{2R\sigma}\tilde{v}^2 dxdt\\ &- 2\int_0^T\int_0^1 a(\xi^2 e^{2R\sigma})_x\tilde{v}\tilde{v}_x dxdt \le -2\int_0^T\int_0^1 R\xi^2\sigma_t e^{2R\sigma}\tilde{v}^2 dxdt\\ &+ \int_0^T\int_0^1 (\sqrt{a}\xi e^{R\sigma}\tilde{v}_x)^2 dxdt + \int_0^T\int_0^1 (\sqrt{a}\frac{(\xi^2 e^{2R\sigma})_x}{\xi e^{R\sigma}}\tilde{v})^2 dxdt.\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^T\int_0^1 \xi^2 e^{2R\sigma} a\tilde{v}_x^2 dxdt &\le -2\int_0^T\int_0^1 R\xi^2\sigma_t e^{2R\sigma}\tilde{v}^2 dxdt\\ &+ \int_0^T\int_0^1 \left(\sqrt{a}\frac{(\xi^2 e^{2R\sigma})_x}{\xi e^{R\sigma}}\tilde{v}\right)^2 dxdt \le \tilde{C}\int_0^T\int_\omega \tilde{v}^2 dxdt.\end{aligned}$$

Therefore

$$\min_{\omega'} a(x)\int_0^T\int_{\omega'} e^{2R\sigma}\tilde{v}_x^2 dxdt \le \tilde{C}\int_0^T\int_\omega \tilde{v}^2 dxdt.$$

Case 2: For $\lambda > 0$, since the support of ξ is located in ω, then

$$\int_0^T\int_0^1 \xi^2 e^{2R\sigma}\frac{\lambda}{x^\beta}\tilde{v}^2 dxdt \le \hat{C}\int_0^T\int_\omega \tilde{v}^2 dxdt,$$

for some suitable $\hat{C} > 0$. So, rewriting the relations of the case 1, we get

$$\min_{\omega'} a(x)\int_0^T\int_{\omega'} e^{2R\sigma}\tilde{v}_x^2 dxdt \le (\tilde{C}+\hat{C})\int_0^T\int_\omega \tilde{v}^2 dxdt.$$

□

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