Travelling Waves

Morteza Fotouhi
Sharif Univ. of Technology
Mini Math NeuroScience
August 2008

Reaction Diffusion Equations

$$U_t = DU_{xx} + f(U)$$

$$x \in \mathbb{R} \qquad t > 0 \qquad U \in \mathbb{R}^n$$

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \qquad d_j > 0$$

Travelling wave is a solution of the form

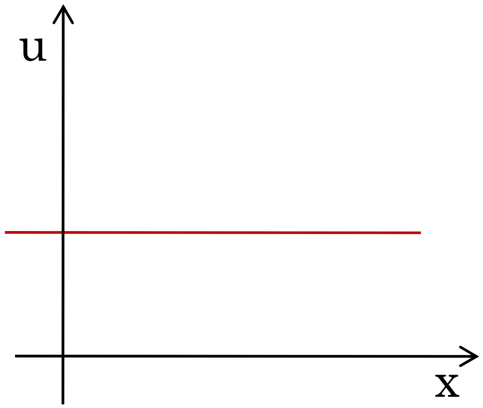
$$U(x,t) = V(x - ct)$$

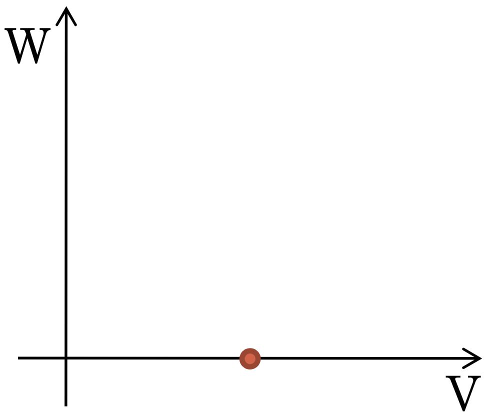
$$\xi = x - ct$$

$$-cV_\xi = DV_{\xi\xi} + f(V)$$

$$\begin{cases} V_\xi = W \\ W_\xi = -D^{-1}[cW + f(V)] \end{cases}$$

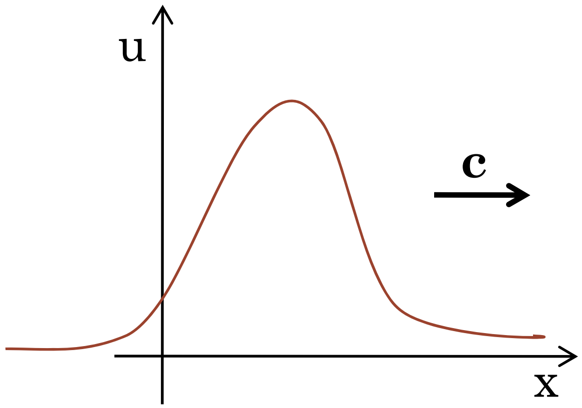
Rest State

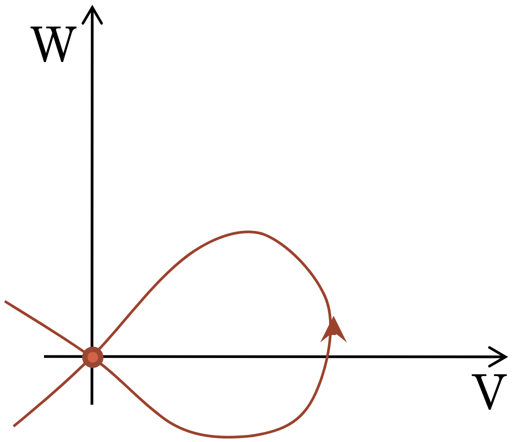




equilibrium

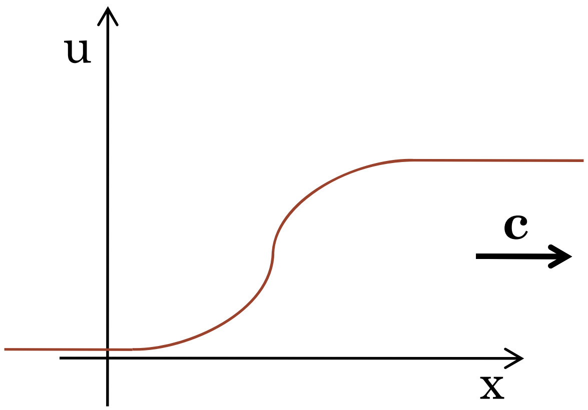
PULSE WAVE

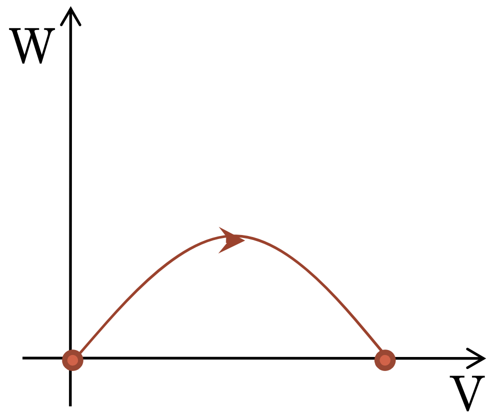




Homoclinic Orbit

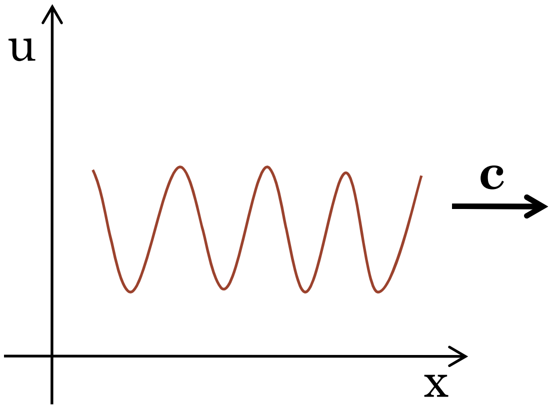
FRONT WAVE

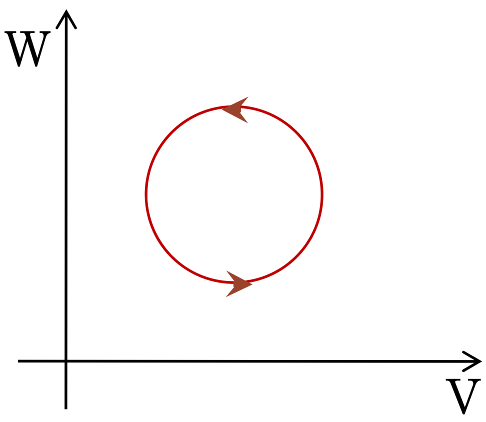




Hetroclinic Orbit

Periodic Wave





Periodic Orbit

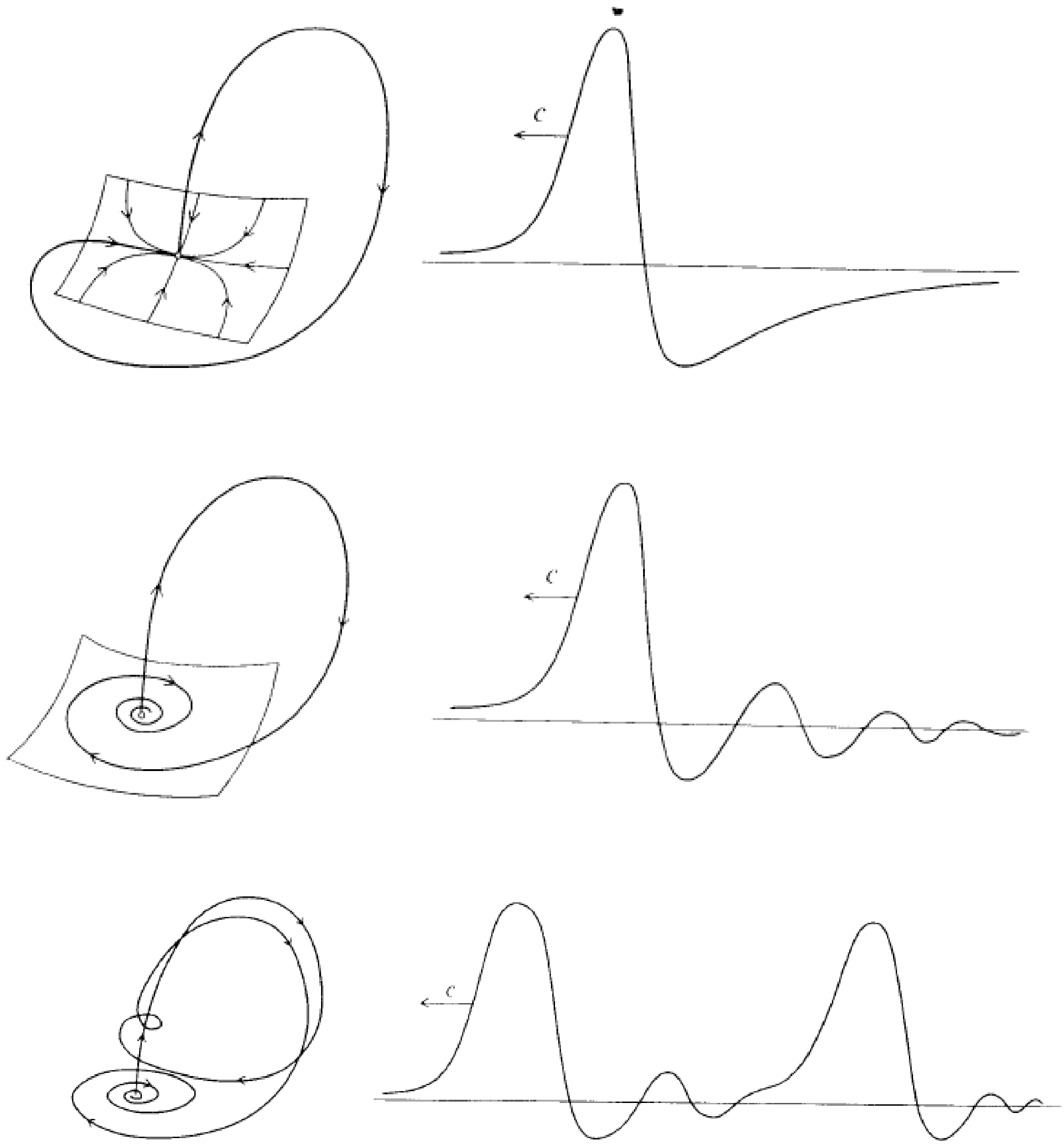
EXAMPLE

- FitzHugh-Nagumo

$$
\begin{aligned}
v_t &= v_{xx} + f(v) - u \\
u_t &= \beta v
\end{aligned}
$$

$$f(v) = v(\alpha - v)(v - 1), 0 < \alpha < 1, 0 < \beta$$

$$
\begin{aligned}
V_\xi &= W \\
W_\xi &= cW - f(V) + U \\
U_\xi &= \frac{\beta}{c} V
\end{aligned}
$$


c
c
c

STABILITY

If U(x) is a stationary solution of Reaction-Diffusion equation, $U_t = DU_{xx} + f(U)$
we call it stable when for every initial value u_0
close to U in some norm, i.e. $\left\| U - u_0 \right\|_X < \varepsilon$
the solution satisfies

$$\left\| u(.,t) - U(\cdot + h) \right\|_X < \delta$$

Furthermore, it is called asymptotically stable, if it is stable and tends towards $U(x+h)$, for a constant h.

$$\left\| u(.,t) - U(\cdot + h) \right\|_X \to 0$$

STABLITY OF TRAVELLING WAVES

$$U(x,t) = \tilde{V}(x-ct,t)$$

$$\tilde{V}_t = D\tilde{V}_{\xi\xi} + c\tilde{V}_{\xi} + f(\tilde{V})$$

Travelling wave $V(x-ct)$ is a stationary solution of this PDE. We mean its (asymptotically) stability as a stable solution of this PDE.

Linear Stability

Linearize the equation

$$\tilde{V}_t = D\tilde{V}_{\xi\xi} + c\tilde{V}_\xi + f(\tilde{V})$$

around the stationary solution $V(\xi)$

$$\tilde{V}_t = D\tilde{V}_{\xi\xi} + c\tilde{V}_\xi + f_U(V)\tilde{V}$$

$$L = D\partial_{\xi\xi} + c\partial_\xi + f_U(V)$$

Spectrum

$$L : D(L) \subset X \to X$$

Resolvent set

$$\rho(L) = \{\lambda \mid L - \lambda I \text{ has a bounded inverse}\}$$

$$\exists K > 0 : \forall h \in X \exists! U \in X, (L - \lambda I)U = h$$

$$\|U\|_X \leq K\|h\|_X$$

$$Spec(L) = \mathbb{C} \setminus \rho(L) = \Sigma_{pt} \cup \Sigma_{ess}$$

Σ_{pt} : point spectrum or eigenvalue defined as the kernel of $L - \lambda I$ is nontrivial.

$\Sigma_{ess} = Spec(L) \setminus \Sigma_{pt}$ is essential spectrum

Example:

$$L : \ell^\infty \to \ell^\infty \qquad (a_0, a_1, \cdots) \mapsto (0, a_0, a_1, \cdots)$$

$\lambda = 0$ is not an eigenvalue but $0 \in Spec(L)$ because $Lu = (1, 0, 0, \cdots)$ doesn't have solution in ℓ^∞

Spectrum of Linear Equation

Proposition:

If $V(\xi)$ *is a travelling wave solution and* $V_\xi \neq 0$,
then $0 \in Spec(L)$.

$$L = D\partial_{\xi\xi} + c\partial_\xi + f_U(V)$$

Proof.

Differentiate $DV_{\xi\xi} + cV_\xi + f(V) = 0$

We find

$$LV_\xi = 0$$

Nonlinear Stability

If $V(\xi)$ *is a travelling wave solution of* $U_t = DU_{xx} + f(U)$ *with* $\lambda = 0$*, is a simple eigenvalue of L and the other spectrum are located in* $\{\mathrm{Re}(\lambda) \le -\alpha < 0\}$*, then V is asymptotically stable.*

CASE1: REST STATE, $V(\xi) = V_0$

Substitute $u(\xi) = e^{\nu\xi} u_0$ for some $\nu \in \mathbb{C}, u_0 \in \mathbb{C}^n \setminus \{0\}$ in the equation $Lu = \lambda u$ to find eigenvalues.

We find

$$\lambda u_0 = [\nu^2 D + c\nu I + f_U(V_0)] u_0$$

$$d(\lambda, \nu) := \det[\nu^2 D + (c\nu - \lambda) I + f_U(V_0)]$$

CASE1: REST STATE, $V(\xi) = V_0$

Theorem:

$$Spec(L) = \{\lambda \in \mathbb{C} \mid d(\lambda, ik) = 0 \ \textit{for some} \ k \in \mathbb{R}\}$$

Proof.

Assume $d(\lambda, ik) \neq 0, \forall k \in \mathbb{R}$, we will show that

$$Du_{\xi\xi} + cu_{\xi} + (f_U(V_0) - \lambda I)u = h(\xi)$$

for every $h \in X$, has solution and $\|u\|_X \leq K\|h\|_X$

$$\begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} = A(\lambda)\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ h \end{pmatrix}$$

$$A(\lambda) = \begin{bmatrix} 0 & I \\ -D^{-1}[\lambda I - f_U(V_0)] & -cD^{-1} \end{bmatrix}$$

$$\det[A(\lambda) - \nu I] = d(\lambda, \nu)\frac{1}{\det D}$$

$A(\lambda)$ is hyperbolic since $d(\lambda, ik) \neq 0, \forall k \in \mathbb{R}$

$E^s(\lambda) =$ stable subspace, $E^u(\lambda) =$ unstable subspace

$$E^s(\lambda) \oplus E^u(\lambda) = \mathbb{C}^n$$

STRUCTURE OF *SPEC(L)*

$$Spec(L) = \{\lambda \in \mathbb{C} \mid d(\lambda, ik) = 0 \textit{ for some } k \in \mathbb{R}\}$$
$$= \{\lambda \mid SpecA(\lambda) \cap i\mathbb{R} \neq \varnothing\}$$

- ✓ All eigenvalues of $f_U(V_0)$ lie in $Spec(L)$
- ✓ If $d(\lambda_0, ik_0) = 0, d_\lambda(\lambda_0, ik_0) \neq 0$, then there is a curve $\lambda(ik)$ defined for $k \approx k_0$ such that

$$\lambda(ik_0) = \lambda_0, \qquad \lambda(ik) \in SpecA(\lambda)$$

- ✓ As $|\, k \,| \to \infty$, we have $\operatorname{Re} \lambda \to -\infty$

STRUCTURE OF *SPEC(L)*

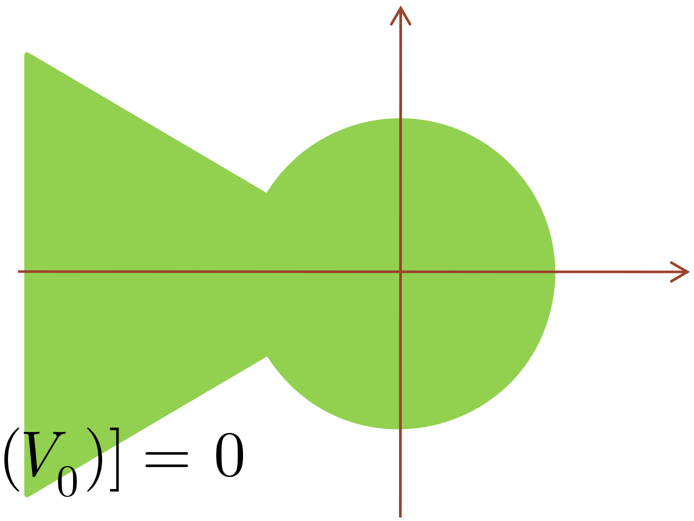
- ✓ Spectrum lies in sector $\{|\lambda| < R\} \cup \{|\arg\lambda| > \pi - \delta\}$

Assume that $\lambda = \dfrac{e^{i\varphi}}{\varepsilon^2} \in SpecL$

where $0 < \varepsilon \ll 1, |\varphi| \le \pi - \delta$

then we show that roots of

$$d(\lambda,\nu) = \det[\nu^2 D + (c\nu - \frac{e^{i\varphi}}{\varepsilon^2})I + f_U(V_0)] = 0$$



are far from $i\mathbb{R}$. Let $\nu = \dfrac{\tilde{\nu}}{\varepsilon}$

$$\Rightarrow \det[-\tilde{\nu}^2 D + (c\tilde{\nu}\varepsilon - e^{i\varphi})I + \varepsilon^2 f_U(V_0)] = 0$$

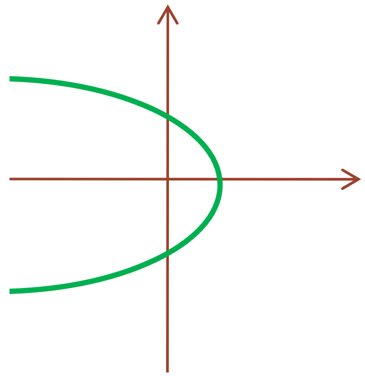
For $\varepsilon = 0$, we have $\tilde{\nu}_0 = \pm \dfrac{e^{i\varphi/2}}{\sqrt{d_j}}$ is far from imaginary axis. And for $0 < \varepsilon \ll 1$, $\tilde{\nu} = \tilde{\nu}_0 + O(\varepsilon)$ is far too.

EXAMPLE

$$u_t = u_{xx} + au$$

$$v_{\xi\xi} + cv_{\xi} + av = 0$$

$$d(\lambda, ik) = -k^2 + ick + a - \lambda = 0$$



For negative parameter a<0

the rest wave u(x,t)=0 is stable.

CASE 2: PERIODIC WAVE, $V(\xi + q) = V(\xi)$

$$0 = (L - \lambda I)u = Du_{\xi\xi} + cu_{\xi} + (f_U(V(\xi)) - \lambda I)u$$

$$\begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} = A(\xi,\lambda)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 0 & I \\ -D^{-1}[\lambda I - f_U(V(\xi))] & -cD^{-1} \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$A(\xi + q, \lambda) = A(\xi,\lambda)$$

Floquet representation $\Rightarrow \begin{pmatrix} u \\ v \end{pmatrix}(\xi) = R(\xi,\lambda)e^{B(\lambda)\xi}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$

- ✓ *The point spectrum is empty.*
- ✓ $Spec(L) = \{\lambda \mid \det(B(\lambda) - ik) = 0 \text{ for some } k \in \mathbb{R}\}$

$$= \{\lambda \mid Spec(B(\lambda)) \cap i\mathbb{R} \neq \varnothing\}$$

- ✓ *Eigenfunctions are of the form* $u(\xi) = u_{per}(\xi)e^{ik\xi}$

where $u_{per}(\xi + q) = u_{per}(\xi)$

- ✓ *Sectoriality of spectrum is also true in this case.*

CASE 3: FRONT, $V(\xi) \to V_{\pm}$ AS $\xi \to \pm\infty$

$$L = D\partial_{\xi\xi} + c\partial_{\xi} + f_U(V(\xi))$$

$$A(\xi,\lambda) = \begin{bmatrix} 0 & I \\ -D^{-1}[\lambda I - f_U(V(\xi))] & -cD^{-1} \end{bmatrix}$$

$$\lim_{\xi\to\infty} A(\xi,\lambda) = A_{\pm}(\lambda) = \begin{bmatrix} 0 & I \\ -D^{-1}[\lambda I - f_U(V_{\pm})] & -cD^{-1} \end{bmatrix}$$

$$E^s(\xi_0,\lambda) = \left\{ \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathbb{C}^{2n} : \begin{pmatrix} u \\ v \end{pmatrix}(\xi) \to \begin{pmatrix} V_+ \\ 0 \end{pmatrix} \text{as } \xi \to +\infty, \text{ where} \begin{pmatrix} u \\ v \end{pmatrix}(\xi_0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\}$$

$$E^u(\xi_0,\lambda) = \left\{ \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \mathbb{C}^{2n} : \begin{pmatrix} u \\ v \end{pmatrix}(\xi) \to \begin{pmatrix} V_- \\ 0 \end{pmatrix} \text{as } \xi \to -\infty, \text{where} \begin{pmatrix} u \\ v \end{pmatrix}(\xi_0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\}$$

- λ *is in the resolvent set of L if and only if,* $A_{\pm}(\lambda)$ *are both hyperbolic with the same Morse index,*

$$\dim E_{-}^{u}(\lambda) = \dim E_{+}^{u}(\lambda)$$

and

$$E_{-}^{u}(0,\lambda) \oplus E_{+}^{s}(0,\lambda) = \mathbb{C}^{n}$$

- λ *is in the point spectrum* Σ_{pt}, *if and only if,* $A_{\pm}(\lambda)$ *are both hyperbolic with the same Morse index,*

$$\dim E_{-}^{u}(\lambda) = \dim E_{+}^{u}(\lambda)$$

but

$$E_{-}^{u}(0,\lambda) \cap E_{+}^{s}(0,\lambda) \neq \varnothing$$

- λ *is in the essential spectrum* Σ_{ess}, *if either at least one of the two asymptotic matrices* $A_{\pm}(\lambda)$ *is not hyperbolic, or else if it does , but the Morse indices are different.*

Case 4: Pulse, $V(\xi) \to V_0$ as $\xi \to \pm\infty$

Special case of front wave with this different that the Morse indices are always the same.

$$\lim_{\xi \to \pm\infty} A(\xi, \lambda) = A_0(\lambda)$$

Evans function

- Choose analytic bases $\{V_j^u(\lambda)\}_{j=1,\ldots,k}$ and $\{V_j^s(\lambda)\}_{j=1,\ldots,n-k}$ for $E^s(0,\lambda)$ and $E^u(0,\lambda)$, respectively.

$$E(\lambda) = \det[V_1^u(\lambda),\cdots,V_k^u(\lambda),V_1^s(\lambda),\cdots,V_{n-k}^s(\lambda)]$$

Result:

I. $E(\lambda) = 0 \Leftrightarrow E^u(0,\lambda) \cap E^s(0,\lambda) \neq \varnothing \Leftrightarrow \lambda$ is an eigenvalue.

II. The order of λ_* as a zero of the Evans function is equal to the algebraic multiplicity of λ_* as an eigenvalue of L.

EXAMPLE

$$u_t = u_{xx} - u + u^3$$

Stationary solution: $q(x) = \sqrt{2} Sechx$

Linear equation: $v_t = v_{xx} + (3q(x)^2 - 1)v$

$$L = \partial_{xx} + (3q(x)^2 - 1)$$

$$A(x,\lambda) = \begin{bmatrix} 0 & 1 \\ 3q(x)^2 - 1 - \lambda & 0 \end{bmatrix} \rightarrow A_{\pm}(\lambda) = \begin{bmatrix} 0 & 1 \\ -1 - \lambda & 0 \end{bmatrix}$$

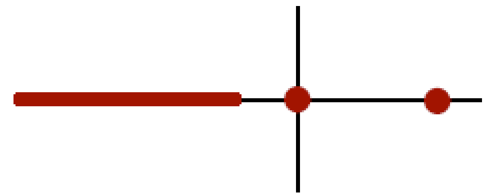
$$\Sigma_{ess} = (-\infty, -1)$$

$$u_-(x,\lambda) = e^{\sqrt{1+\lambda}x}[1 + \frac{\lambda}{3} - \sqrt{1+\lambda}\tanh(x) - Sech^2(x)]$$

$$u_+(x,\lambda) = e^{-\sqrt{1+\lambda}x}[1 + \frac{\lambda}{3} + \sqrt{1+\lambda}\tanh(x) - Sech^2(x)]$$

$$E(\lambda) = \det\begin{pmatrix} u_-(0,\lambda) & u_+(0,\lambda) \\ u_-'(0,\lambda) & u_+'(0,\lambda) \end{pmatrix} = -\frac{2}{9}\lambda(\lambda-3)\sqrt{1+\lambda}$$

$$\Sigma_{pt} = \{-1, 0, 3\}$$



Neural Field: Integro-Differential Equation

$$\frac{1}{\alpha}\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \gamma\int_{-\infty}^{+\infty} w(y)f(u(x-y,t))dy$$

- Rest state: $u(x,t) = \overline{u}$

$$\overline{u} = \gamma f(\overline{u})\int_{-\infty}^{+\infty} w(y)dy$$

- Linear Equation:

$$\frac{1}{\alpha}\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \beta\int_{-\infty}^{+\infty} w(y)u(x-y,t)dy$$

$$\beta = \gamma f'(\overline{u})$$

$$Lu = -u + \beta \int_{-\infty}^{+\infty} w(y)u(x-y)dy$$

- Eigenfunctions: $u(x) = e^{ikx}u_0$

$$\lambda + 1 = \beta \hat{w}(k)$$

- *If we assume that* $w(y) = w(-y)$*, then* $\hat{w}(k)$ *is a real even function of k and the stability condition is*

$$\beta \hat{w}_{\max} < 1$$

Reference

- B. Sandstede, Stability of travelling waves, Handbok of Dynamical Systems II, 2002.
- A. I. Volpert, V. A. Volpert, and V. A. Volpert, *Traveling wave solutions of parabolic systems,* Translations of Mathematical Monographs **140, American Mathematical Society,** 1994.
- L. Zhang
- S. Coombes
- C.R. Laing