Free Boundary Problem for a Sublinear Elliptic Equation

Morteza Fotouhi

Sharif University of Technology

August 16, 2021

Hilbert's 19^{th} Problem

Are the minimizers of $\int_\Omega F(\nabla u, u, x)dx$ analytic?

Hilbert's 19th Problem

Are the minimizers of $\int_\Omega F(\nabla u, u, x)dx$ analytic?

Remarkable Results:

- S. Bernstein (1904): C^3 solutions are analytic.

Hilbert's 19^{th} Problem

Are the minimizers of $\int_\Omega F(\nabla u, u, x)dx$ analytic?

Remarkable Results:

- S. Bernstein (1904): C^3 solutions are analytic.
- I.G. Petrowsky (1939): C^1 solutions are analytic.

Hilbert's 19^{th} Problem

Are the minimizers of $\int_\Omega F(\nabla u, u, x)dx$ analytic?

Remarkable Results:

- S. Bernstein (1904): C^3 solutions are analytic.
- I.G. Petrowsky (1939): C^1 solutions are analytic.
- E. De Giorgi - J. Nash (1957): With merely assumptions the solutions are $C^{1,\alpha}$.

Main Questions

$$u \leftarrow \min \int_{\Omega} |\nabla u|^2 + 2F(x,u)dx$$

The minimizers solve the semilinear problem

$$\Delta u = f(x,u)$$

- Regularity of the minimizers?
- Regularity of the level sets $\{u = c\}$?

Classical Results

- If $\Delta u \in \mathcal{L}^p$ and $1 \leq p < \infty$ then $u \in W^{2,p}$. (Not true for $p = \infty$)

Classical Results

- If $\Delta u \in \mathcal{L}^p$ and $1 \leq p < \infty$ then $u \in W^{2,p}$. (Not true for $p = \infty$)
- If $\Delta u \in C^\alpha$ and $0 < \alpha < 1$ then $u \in C^{2,\alpha}$. (Not true for $\alpha = 0, 1$)

Classical Results

- If $\Delta u \in \mathcal{L}^p$ and $1 \leq p < \infty$ then $u \in W^{2,p}$. (Not true for $p = \infty$)
- If $\Delta u \in C^\alpha$ and $0 < \alpha < 1$ then $u \in C^{2,\alpha}$. (Not true for $\alpha = 0, 1$)

Counterexample:

$$u(x) = x_1 x_2 (\log|x|)^\alpha \notin C^{1,1}, \text{ for } 0 < \alpha \leq 1 \text{ and } |x| \leq 1.$$

If $\alpha = 1$ then $\Delta u \in \mathcal{L}^\infty$.
If $0 < \alpha < 1$ then $\Delta u \in C^0$.

Classical Results

- If $\Delta u \in \mathcal{L}^p$ and $1 \leq p < \infty$ then $u \in W^{2,p}$. (Not true for $p = \infty$)
- If $\Delta u \in C^\alpha$ and $0 < \alpha < 1$ then $u \in C^{2,\alpha}$. (Not true for $\alpha = 0, 1$)
- If Δu is *Dini then* $u \in C^2$.

Counterexample:

$$u(x) = x_1 x_2 (\log |x|)^\alpha \notin C^{1,1}, \text{ for } 0 < \alpha \leq 1 \text{ and } |x| \leq 1.$$

If $\alpha = 1$ then $\Delta u \in \mathcal{L}^\infty$.
If $0 < \alpha < 1$ then $\Delta u \in C^0$.

Regularity Results for Semilinear Equations

- (Shahgholian, 2003) In semilinear equation $\Delta u = f(x, u)$, if $f_u \geq -C$ (in weak sense) and $|f_x| \leq M$ then $u \in C^{1,1}$.

Regularity Results for Semilinear Equations

- (Shahgholian, 2003) In semilinear equation $\Delta u = f(x, u)$, if $f_u \geq -C$ (in weak sense) and $|f_x| \leq M$ then $u \in C^{1,1}$.
- (Nadirashvili, 2013) Let $u \in W^{2,p}$ for $p > n$ be a bounded solution of the semilinear equation $\Delta u = f(u)$ such that $|\nabla u| > c > 0$ then the level sets are C^∞-smooth. Moreover, if $f \in C(\mathbb{R})$ (or $f \in \mathcal{L}^\infty$) then $u \in C^2$ (or $u \in C^{1,1}$).

Regularity Results for Semilinear Equations

- (Shahgholian, 2003) In semilinear equation $\Delta u = f(x, u)$, if $f_u \geq -C$ (in weak sense) and $|f_x| \leq M$ then $u \in C^{1,1}$.
- (Nadirashvili, 2013) Let $u \in W^{2,p}$ for $p > n$ be a bounded solution of the semilinear equation $\Delta u = f(u)$ such that $|\nabla u| > c > 0$ then the level sets are C^∞-smooth. Moreover, if $f \in C(\mathbb{R})$ (or $f \in \mathcal{L}^\infty$) then $u \in C^2$ (or $u \in C^{1,1}$).

Counterexample: (Andersson-Weiss, 2006) There is a solution for $\Delta u = -\chi_{\{u>0\}}$ which $u \notin C^{1,1}$.

Example: Obstacle Problem

$$u \leftarrow \min_{u \geq 0} \int |\nabla u|^2 + 2u \, dx \xrightarrow{\text{satisfies}} \Delta u = \chi_{\{u>0\}}$$

Example: Obstacle Problem

$$u \leftarrow \min_{u\geq 0} \int |\nabla u|^2 + 2u\, dx \xrightarrow{\text{satisfies}} \Delta u = \chi_{\{u>0\}}$$

H. Brezis - D. Kinderlehrer (1973): Solutions have optimal regularity $u \in C^{1,1}$.

Example: Obstacle Problem

$$u \leftarrow \min_{u\geq 0} \int |\nabla u|^2 + 2u\, dx \xrightarrow{\text{satisfies}} \Delta u = \chi_{\{u>0\}}$$

H. Brezis - D. Kinderlehrer (1973): Solutions have optimal regularity $u \in C^{1,1}$.
L. Caffarelli (1977): Regular part of Free boundary $\partial\{u>0\}$ is analytic.

Example: Obstacle Problem

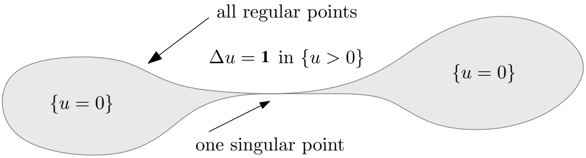
$$u \leftarrow \min_{u \geq 0} \int |\nabla u|^2 + 2u\, dx \xrightarrow{\text{satisfies}} \Delta u = \chi_{\{u>0\}}$$

H. Brezis - D. Kinderlehrer (1973): Solutions have optimal regularity $u \in C^{1,1}$.
L. Caffarelli (1977): Regular part of Free boundary $\partial\{u > 0\}$ is analytic.
A. Figalli - J. Serra (2019): Singular part of Free boundary is at least C^1.

Example: Obstacle Problem

$$u \leftarrow \min_{u \geq 0} \int |\nabla u|^2 + 2u\, dx \xrightarrow{\text{satisfies}} \Delta u = \chi_{\{u>0\}}$$

H. Brezis - D. Kinderlehrer (1973): Solutions have optimal regularity $u \in C^{1,1}$.
L. Caffarelli (1977): Regular part of Free boundary $\partial\{u > 0\}$ is analytic.
A. Figalli - J. Serra (2019): Singular part of Free boundary is at least C^1.



Higher Regularity in a Semilinear Problem ($0 < q < 1$)

$$\min \int |\nabla u|^2 + \frac{2}{1+q}|u|^{1+q}\, dx \overset{\text{solves}}{\Longrightarrow} \Delta u = |u|^{q-1}u$$

Priori Regularity: At least $u \in C^{2,q}$. Moreover, $u \in C^\infty$ in $\{u \neq 0\}$.

Higher Regularity in a Semilinear Problem $(0 < q < 1)$

$$\min \int |\nabla u|^2 + \frac{2}{1+q}|u|^{1+q}\,dx \xrightarrow{\text{solves}} \Delta u = |u|^{q-1}u$$

Priori Regularity: At least $u \in C^{2,q}$. Moreover, $u \in C^\infty$ in $\{u \neq 0\}$.

Question: Which regularity for solutions and free boundaries $\partial\{u > 0\}$ and $\partial\{u < 0\}$ do we expect?

Special Case when $q = 0$

$$\Delta u = \chi_{\{u>0\}} - \chi_{\{u<0\}}$$

Special Case when $q = 0$

$$\Delta u = \chi_{\{u>0\}} - \chi_{\{u<0\}}$$

- N. Uraltseva (2001): Optimal regularity is $u \in C^{1,1}$.

Special Case when $q = 0$

$$\Delta u = \chi_{\{u>0\}} - \chi_{\{u<0\}}$$

- N. Uraltseva (2001): Optimal regularity is $u \in C^{1,1}$.
- H. Shahgholian - N. Uraltseva - G. Weiss (2007): Free boundary is C^1.

Higher Regularity ($0 < q < 1$)

$$\Delta u = |u|^{q-1}u = (u_+)^q - (u_-)^q$$

- M. F. - H. Shahgholian (2017): The optimal regularity of solution on the free boundary $\partial\{u > 0\}$ is $C^{[\kappa],\kappa-[\kappa]}$, where $\kappa = 2/(1-q)$.

Higher Regularity ($0 < q < 1$)

$$\Delta u = |u|^{q-1}u = (u_+)^q - (u_-)^q$$

- M. F. - H. Shahgholian (2017): The optimal regularity of solution on the free boundary $\partial\{u > 0\}$ is $C^{[\kappa],\kappa-[\kappa]}$, where $\kappa = 2/(1-q)$.
- M. F. - H. Shahgholian - G.S. Weiss (2021): The regular part of free boundary is $C^{1,\alpha}$.

Thank you for your attention.