Sharif University of Technology
Department of Mathematical Sciences
Final Exam of Homogenization Theory
Date, Time Limit: June 2018, 3 Hourse
Name: ....

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1. (20 points) Let Ω denote an open set in $\mathbb{R}^N$, $\partial\Omega$ is Lipschitz continuous and $A \in M(\alpha, \beta, \Omega)$ for the positive constants α and β such that $0 < \alpha < \beta$. Show that for any $f \in L^2(\Omega)$ and for any $g \in H^{-\frac{1}{2}}(\partial\Omega)$, the problem

$$\begin{cases} -\text{div}(A\nabla u) + u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = g & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution.

2. (20 points) Let $a : \mathbb{R} \longrightarrow \mathbb{R}$ be a periodic function satisfying $a(x) = a(x+1)$ for all $x \in \mathbb{R}$, and for which there exists $m, M > 0$ such that $m \leq a(x) \leq M$ for all $x \in \Omega$. Further, let ϵ be a small, positive number and for $f \in L^2(0,1)$ consider the boundary value problem:

$$(P^\epsilon) : \begin{cases} -\left(a_\epsilon u^\epsilon_x\right)_x = f(x) & x \in (0,1) \\ u^\epsilon(0) = 0, \quad u^\epsilon(1) = 0 \end{cases}$$

where $a_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $a_\epsilon(x) = a\left(\dfrac{x}{\epsilon}\right)$ for all $x \in \mathbb{R}$. Derive the homogenized problem corresponding to Problem P^ϵ. Then determine explicitly the homogenized diffusion coefficient for the case $a_\epsilon(x) = (2 + \sin(2\pi x/\epsilon))^{-1}$

3. (30 points) Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary and ϵ is a positive parameter. Consider the problem:

$$\begin{cases} -\text{div}(A^\epsilon \nabla u^\epsilon) = f & \text{in } \Omega \\ u^\epsilon = 0 & \text{on } \Omega \end{cases}$$

where $f \in H^{-1}(\Omega)$, and the matrix A^ϵ is defined by

$$A^\epsilon(x) = A\left(\frac{x}{\epsilon}\right) \qquad \text{a.e. on } \mathbb{R} \tag{1}$$

where

$$\begin{cases} A \text{ is } Y\text{-periodic} \\ A \in M(\alpha, \beta, Y) \end{cases} \tag{2}$$

with $Y = (0,1)^N$ and $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < \beta$.

(a) Show that as $\epsilon \longrightarrow 0$ the following limits are hold:

$$\begin{cases} (i) \quad u^\epsilon \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega) \\ (ii) \quad A^\epsilon \nabla u^\epsilon \rightharpoonup A^0 \nabla u^0 \quad \text{weakly in } (L^2(\Omega))^N \end{cases}$$

where u^0 is the unique solution of the homogenized problem:

$$\begin{cases} -\text{div}(A^0 \nabla u^0) = f & \text{in } \Omega \\ u^0 = 0 & \text{on } \partial\Omega \end{cases}$$

(b) Represent a explicit formula for the homogenized(effective) matrix A^0.

(c) Show that as $\epsilon \longrightarrow 0$:

$$\int_\Omega A^\epsilon \nabla u^\epsilon \nabla u^\epsilon \, dx \longrightarrow \int_\Omega A^0 \nabla u^0 \nabla u^0 \, dx$$

4. (15 points) Using the formal asymptotic expansion, derive the macroscopic equation and corresponding effective diffusion matrix for the following microscopic problem:

$$\begin{cases} u_t^\epsilon - \text{div}(A^\epsilon \nabla u^\epsilon) = f & \text{in } \Omega \times (0,T) \\ u^\epsilon(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u^\epsilon(x,0) = g(x) & \text{in } \Omega \end{cases}$$

where Ω be a bounded open subset of $\mathbb{R}^N$, $f \in L^2(\Omega \times (0,T))$, $g \in L^2(\Omega)$, $T > 0$, and matrix A^ϵ satisfies the conditions in the previous problem (equation 1 and equation 2).

5. (15 points) Let $Y = (0,1)^N$ and $\{v^\epsilon\}$ be a bounded sequence in $L^2(\Omega)$. Prove that there exists a subsequence $\{v^{\epsilon'}\}$ and a function $v^0 \in L^2(\Omega \times Y)$ such that $\{v^{\epsilon'}\}$ two-scale converges to v^0.