Advanced Quantum Field Theory - Fall 1402: PSet 5 TA: Hossein Mohammadi
Subject: Regularization schemes and Vacuum Polarization diagram
Instructor: Dr. Amin Faraji Due Date: dd/mm/yyyy

Problem 1: Regularization Schemes

As you know there are many regularization procedures in evaluating loop diagrams, like dimensional regularization (D.R.), Pauli-Villars regularization (P.V.), lattice regularization, etc. Additionally, there are many techniques that we have to utilize to evaluate one-loop diagrams. In this problem, our goal is to cover all such techniques.

The gist of all these schemes is to extract to divergent part of the integral on the loop momentum. Although there are different ways to regulate divergent amplitudes, all of these will agree on the observations, which means that none of the interpretations that we're going to discuss should be taken seriously. Therefore, the running of parameters, dependence of amplitudes on the energy scale, and other observables are scheme-independent.

(a) [- points] **Pauli-Villars Regularization:**

The idea is very simple, we just subtract a term from the divergent part, so that the final contribution is rendered finite. This term is called the "Pauli-Villars" term which is interpreted as a ghost particle with mass $\Lambda \gg m$, with the wrong term kinetic sign in the Lagrangian.

Let's consider the following example:

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\epsilon)^2}$$

In the P.V. scheme, the idea is to replace it with:

$$\int \frac{d^4k}{(2\pi)^4} \Big[\frac{1}{(k^2 - m^2 + i\epsilon)^2} - \frac{1}{(k^2 - \Lambda^2 + i\epsilon)^2} \Big] \tag{1}$$

This ghost term would cancel the $\frac{1}{k^4}$ contribution which leaves us with a finite value (of course Λ-dependent.)

Aside: The P.V. technique breaks the gauge invariance at the loop level[1], and it's not very convenient when dealing with multi-loop amplitude Besides, it gets very complicated when several propagators are involved in the loop.

(i) Write the measure d^4k in the spherical coordinates, then isolate the angular part. Finally, do a Wick rotation to translate the integral in the usual Euclidean signature.(Use $\int d\Omega_d = \Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ for angular part.)

[1] Remember that gauge-invariance in amplitudes translates into Ward identity. So the regularized amplitude $\mathcal{M}^{(reg)}_{\mu\nu}$ has the property that $p^\mu \mathcal{M}^{(reg)}_{\mu\nu} \neq 0$. That's unfortunate, we do change our procedure to a more gauge-friendly one.

(ii) Evaluate the integral in (1); this is a simple integral that requires a change of variable to evaluate. The final result is:

$$\int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{(k^2-m^2+i\epsilon)^2}-\frac{1}{(k^2-\Lambda^2+i\epsilon)^2}\right]=\frac{i}{16\pi^2}\ln\left(\frac{\Lambda^2}{m^2}\right)$$

(b) [- points] **Feynman parameters:**

We prove a simple integral identity that helps complete square the loop integrals' denominator. There's another way to attack loop integrals, which is Schwinger[2] parametrization

Prove the following identity by integration:

$$\frac{1}{AB}=\int_0^1 dx\frac{1}{(A+(B-A)x)^2}$$

This identity is briefly showcased when we encounter such integral $\int \frac{d^dk}{(2\pi)^4}\frac{1}{k^2}\frac{1}{(k-p)^2}$. By proper and obvious definition of A and B, the denominator square to the form $\int_0^1 dx \int \frac{d^dk}{(2\pi)^4}\frac{1}{((k-px)^2-\Delta)^2}$, where $\Delta=-p^2x(1-x)$.

Aside: Another useful relation is

$$\frac{1}{ABC}=\int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{(xA+yB+zC)^3}$$

which is very useful in calculating QED vertex correction, see figure 1.

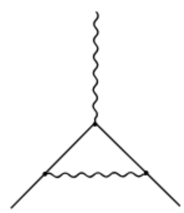

Figure 1: Diagram contributing to QED vertex correction.

(c) [- points] **Dimensional Regularization:** The goal of this problem is to work out the following integral:

$$\int \frac{d^dk}{(2\pi)^4}\frac{k^2a}{(k^2-\Delta)^b}$$

. If we have such a powerful result at our disposal, we only need to use Feynman parametrization to complete-square the denominator and use this formula.

The dimensional regularization scheme, as its name suggests, treats the dimension of the spacetime as a parameter to render finite amplitudes. In the end, we extract the divergent term by a limiting process.

Working out this integral also requires knowledge about $\beta-$function.

[2]Sometimes called Schwinger proper time since the integral was first used in a pertinent content.

(i) Take $\beta(a,b) = \int_0^1 dx x^{a-1}(1-x)^{b-1}$ as the definition of beta function. Do two change of variables to show that it is equal to:

   (a) $\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

   (b) $\int_0^\infty ds \frac{s^{a-1}}{(s+1)^{a+b}}$.(In this case take $x = \frac{s}{s+1}$.)

(ii) In the original integral, rewrite the measure in spherical coordinates and do a Wick rotation, as you did in part (a).

(iii) Now compute the $\int dk_E \frac{k_E^{2a}}{(k_E^2+\Delta)^b}$ by using identities of the beta function.

(iv) Put all things together to find:

$$\int \frac{d^dk}{(2\pi)^4}\frac{k^2a}{(k^2-\Delta)^b} = i(-1)^{a-b}\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{b-a-\frac{d}{2}}}\frac{\Gamma(a+\frac{d}{2})\Gamma(b-a-\frac{d}{2})}{\Gamma(b)\Gamma(\frac{d}{2})} \tag{2}$$

Now, finding the divergent part of the amplitudes boils down to knowing the divergences of $\Gamma(z)$ function. For our purposes, only $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$ is enough[3].

The regularization procedure is to take $d = 4-\epsilon$ in the (2), and extract the divergent term by the expansion of the Gamma function around zero. As an example let's regularize $\int \frac{d^dk}{(2\pi)^4}\frac{1}{(k^2-\Delta+i\epsilon)^2}$, which is equal to $\frac{i}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{2-\frac{d}{2}}}\Gamma(\frac{4-d}{2})$. Expanding $d = 4-\epsilon$ give the following result:

$$\frac{i}{16\pi^2}(\frac{4\pi}{\Delta})^{\frac{\epsilon}{2}}(\frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon))$$

Remember that $a^\epsilon = e^{\epsilon\ln(a)} = 1 + \epsilon\ln(a)$, as $\epsilon \to 0$, hence we get:

$$\frac{i}{16\pi^2}\frac{1}{\epsilon} + \frac{i}{16\pi^2}\ln\left(\frac{4\pi e^{-\gamma_E}}{\Delta}\right) + \mathcal{O}(\epsilon)$$

as our divergent and convergent parts respectively, and the ϵ-dependent parts vanish in the limit.

Aside: Some useful identities that we encounter frequently are:

$$\int \frac{d^dk}{(2\pi)^4}\frac{1}{(k^2-\Delta+i\epsilon)^2} = \frac{i}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{2-\frac{d}{2}}}\Gamma(\frac{4-d}{2})$$

$$\int \frac{d^dk}{(2\pi)^4}\frac{k^2}{(k^2-\Delta+i\epsilon)^2} = -\frac{d}{2}\frac{i}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{1-\frac{d}{2}}}\Gamma(\frac{2-d}{2})$$

$$\int \frac{d^dk}{(2\pi)^4}\frac{k^2}{(k^2-\Delta+i\epsilon)^3} = \frac{d}{4}\frac{i}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{2-\frac{d}{2}}}\Gamma(\frac{4-d}{2})$$

(d) [- points] **Dimensional Regularization Subtlties:**

Since we change the dimension of the spacetime, so the dimension of the fields and the couplings in the problem should also change.

[3] γ_E is the Euler-Mascheroni constant which is about 0.577 .

(i) In QED Lagrangian, find the mass dimension of A_μ, Ψ, m, e in d-dimensional spacetime.

(ii) To make coupling constant e dimensionless, show that we have to change $e \to \mu^{\frac{4-d}{2}} e$, where μ is an arbitrary scale (not infinite!). We should justify later that the observables are independent of this scale[4].

[4] The dependence of regularized amplitude to this scale is like $\ln\left(\frac{4\pi e^{-\gamma_E}\mu^2}{\Delta}\right)$.

Problem 2: Vacuum Polarization Diagram

Vacuum polarization involves a process in the vacuum of the theory where a pair of charged particles (e.g. e^-e^+ particles in QED)is created and annihilated immediately. This virtual dipole is of significant importance in observations. Here, we work out the QED vacuum polarization diagram compeletely.

(a) [- points] **The amplitude:**

Write down the amplitude of the vacuum polarization diagram according to QED Feynman rules in momentum space. (The figure 2 shows the Feynman diagram.)

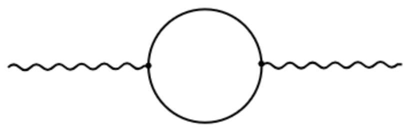

Figure 2: Vacuum Polarization diagram in QED.

(b) [- points] **Trace Technology:**

To simplify the numerator, use these two identities in Clifford algebra: $Tr(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$, $Tr(\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu) = 4\big(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\nu}\eta^{\beta\mu}\big)$. You should end up to:

$$Tr\big[\gamma^\mu(\not{k} - \not{p} + m)\gamma^\nu(\not{k} + m)\big] = 4\big[-p^\mu k^\nu - k^\mu p^\nu + 2k^\mu k^\nu + \eta^{\mu\nu}(-k^2 + p.k + m^2)\big]$$

(c) [- points] **Drop Unnecessary Terms:**

Now justify you can drop the $p^\mu k^\nu$ part of the integral, use odd integrand argument to conclude these terms vanish after integration.

(d) [- points] **Feynman Trick:**

Introduce Feynman parametrization to complete square the denominator, then change the variables of integration according to $k^\mu \to k^\mu + p^\mu(1-x)$. Show that the Jacobian is unit.

You have to have something like this:

$$\Pi_2^{\mu\nu} = 4ie^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^4} \frac{2k^\mu k^\nu - \eta^{\mu\nu}(k^2 - x(1-x)p^2 - m^2)}{(k^2 + p^2x(1-x) - m^2)^2}$$

(e) [- points] **Dimensional Regularization:**

Use D.R. to regularize this amplitude. There's another trick that you should utilize: replace $k^\mu k^\nu$ by $\frac{1}{d}\eta^{\mu\nu}k^2$. [5]

$$\Pi_2^{\mu\nu} = -\frac{e^2}{2\pi^2}p^2\eta^{\mu\nu}\Big(\int_0^1 dx\, x(1-x)\Big[\frac{2}{\varepsilon}\ln\big(\frac{\tilde{\mu}^2}{m^2 - p^2x(1-x)}\big) + \mathcal{O}(\varepsilon)\Big]\Big)$$

[5] Can you justify this innocent trick?

(f) [- points] **Reviving Gauge Invariance:**

As we've promised, D.R. should preserve gauge invariance (or Ward Identity.) But this is not manifest in the regularized amplitude in part (e). The idea is that we hadn't considered the full amplitude yet, which consists of $p^\mu p^\nu$ terms.

Do so and end up with[6]:

$$\Pi_2^{\mu\nu} = -\frac{8e^2}{(4\pi)^{\frac{d}{2}}}(p^2\eta^{\mu\nu} - p^\mu p^\nu)\Gamma(2-\frac{d}{2})\mu^{4-d}\int_0^1 dx(1-x)x(\frac{1}{m^2-p^2x(1-x)})^{2-\frac{d}{2}}$$

So we recover the gauge invariance in the one-loop level.

[6] I know it's tedious, but absolutely necessary!

Problem 3: Physics of the Vacuum Polarization

As we saw,

$$i\Pi_2^{\mu\nu} = -i(p^2\eta^{\mu\nu} - p^\mu p^\nu)e^2\Pi_2(p^2),$$

Where

$$\Pi_2(p^2) = \frac{1}{2\pi^2}\int_0^1 dx(1-x)x\Big(\frac{2}{\epsilon} + \ln\Big(\frac{\tilde{\mu}^2}{m^2 - p^2x(1-x)}\Big)\Big).$$

(a) [- points] **The Dressed Propagator $G^{\mu\nu}$:**

Dressed Propagator could be interpreted as Fourier transform of the corrected Coloumb potential, i.e.

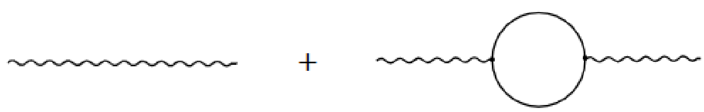

Figure 3: The dressed propagator up to one-loop level.

Conclude that $iG^{\mu\nu} = -\frac{i}{p^2}(1 - e^2\Pi_2(p^2))\eta^{\mu\nu}$.

(b) [- points] **Renormalization condition:**

Now it's time to define a renormalization condition. It's pretty natural to expect that all the quantum effects are in electric charge and the Coloumb potential (in momentum space) is the same as before, with e_R instead of bare e.

$$\tilde{V}(p_0^2) = \frac{e_R^2}{p_0^2}$$

Now solve e_R in terms of e, then find renormalized potential, which is:

$$\tilde{V}_R(p^2) = \frac{e_R^2}{p^2}\Big(1 + \frac{e_R^2}{2\pi^2}\int_0^1 dx(1-x)x\ln\Big(1 - \frac{p^2}{m^2}x(1-x)\Big) + \mathcal{O}(e_R^4)\Big)$$

Aside: The renormalized potential reproduces the Lamb shift in the limit $m^2 \gg |p^2|$(or Hydrogen atom in low-energy.) The process involves calculating integral with this approximation and then do an inverse Fourier transformation to see Dirac delta function in the position space[7].

[7] For more detail, consult to 16.3.1 Schwartz