Advanced Quantum Field Theory - Fall 1402: PSet 4 TA: Hossein Mohammadi

Subject: Effective Action and Its Relation to $W[J]$

Instructor: Dr. Amin Faraji Due Date: dd/mm/yyyy

Problem 1: Effective Action Rudiments

The idea of effective action is to replace the full quantum theory with a classical action $\Gamma[\phi]$, which carries the same data about the amplitudes essentially. Since quantum theories deviate from classical counterparts at loops, the classical theory $\Gamma[\phi]$ should encode all loops in its tree-level

For example, QED's effective action is:

$$\Gamma_{QED} = \int d^4x \left[\bar{\psi}\left(i\partial\!\!\!/ - m + \Sigma(i\partial\!\!\!/)\right)\psi - e\bar{\psi}\Gamma^{\mu}(i\partial\!\!\!/)A_{\mu}\psi - \frac{1}{2}A_{\mu}\left(\partial^{\mu}\partial^{\nu} - \Box\eta^{\mu\nu}\right)\left(1 - \Pi(-\Box)\right)A_{\nu} + \dots\right]$$

Where $\Sigma(p\!\!\!/)$, $\Gamma^{\mu}(p\!\!\!/)$, and $\Pi(p^2)$ are 1PI graphs contributing to electron self-energy, vertex correction, and vacuum polarization respectively. Don't worry about this Lagrangian and vague expressions; you'll see them later.

(a) [- points] **1PI Diagrams:**

Draw the first few 1PI diagrams contributing to $\Sigma(p\!\!\!/)$, $\Gamma^{\mu}(p\!\!\!/)$, and $\Pi(p^2)$. More specifically, complete the diagrams below by adding suitable components. Also, mention the order in perturbation theory.

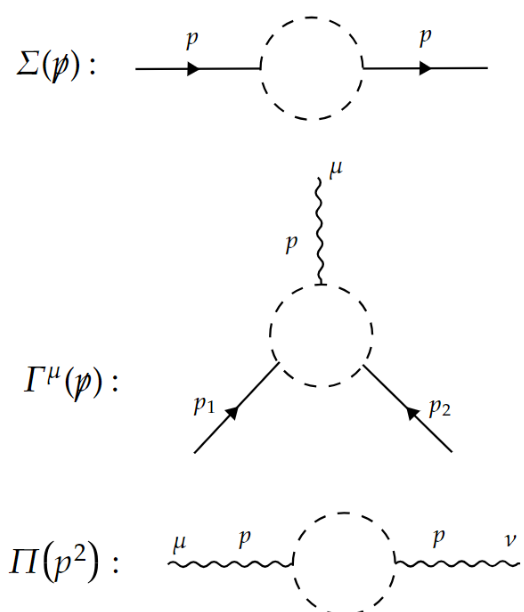

Figure 1: 1PI contributions to QED Lagrangian.

(b) [- points] $W[J]$ **and its interpretation**

As you know, $\Gamma[\phi]$ is the Legendre transformation of the generating functional of connected diagrams, $W[J]$. In this part of the problem, we want to establish their equivalence.

(i) To begin with, let's check if $W[J]$ generates connected diagrams.

$$(-i\hbar)^n \frac{\partial^n W[J]}{\partial J(x_1)\dots\partial J(x_n)} = -i\hbar \langle J| T\phi(x_1)\dots\phi(x_m) |J\rangle_{Connected}$$

For $n = 2, 3$, show that it, indeed, generates connected Feynman diagrams. (Just plug back $W[J] = -i\hbar \ln(Z[J])$ into the above relation and take the derivatives.)

For general n, argue that the expression gives the connected n-point functions.

(c) [- points] **The relation between** $\Gamma[\phi]$ **and** $W[J]$

The ultimate result is that

$$\Gamma[\phi] = W[J_\phi] - \int d^4x\ J_\phi(x)\phi(x)$$

Where $J_\phi(x)$ is an implicit functional, $\frac{\partial W[J]}{\partial J(x)}\Big|_{J=J_\phi} = \phi(x)$.

Compute $\frac{\partial \Gamma[\phi]}{\partial \phi(x)}$ by chain rule and define the inverse transformation.

(d) [- points, **Optional, just read it.**] **How is this possible?**

You may ask why effective action has such a simple relation with $W[J]$? Here's a simple proof.

As you know, all connected diagrams can be found either by $W[J]$ or by classical action, which is $\Gamma[\phi]$. In the path integral, classical stuff means take $\hbar \to 0$ limit.

$$W[J] \equiv \lim_{\hbar\to 0}(-i\hbar)\ln\Big(\int \mathcal{D}\phi \exp\Big\{\frac{i}{\hbar}\big(\Gamma[\phi] + \int d^4x J(x)\phi(x)\big)\Big\}\Big)$$

Taking this limit means that only classical field (that exteremizes the exponential) will survive. The exteremum occurs at

$$\phi_J = \frac{\partial \Gamma}{\partial \phi}\Big|_{\phi=\phi_J} = -J$$

By substituting back into the above relation:

$$W[J] = \Gamma[\phi_J] + \int d^4x J(x)\phi_J(x)$$

Seems familiar.

(e) [- points] **Check the free theory**

Let's check this formal procedure on $\mathscr{L} = -\frac{1}{2}\phi(\Box + m^2)\phi$

(i) Calculate $W[J]$.

(ii) Find J_ϕ.

(iii) Find $\Gamma[\phi]$. Express the moral lesson you've got from this exercise. (Hint: What's the relation between quantum action and effective action of a free theory?)

(iv) Show that $\Gamma[\phi]$ is minimized by $<\phi>$, which is the expectation value of the field operator on the vacuum $|0\rangle$.

Aside: As I told earlier, I prefer not to explore advanced stuff like Yang-Mills and their quantization. Although it's the true playground for both the Faddeev-Popov procedure and BRST symmetry, it's better not to open up a new topic[1]. Soon, when we start renormalization, you'll be bombarded with different exercises and calculations. So it's fair to have a rest for now.

[1] One can utilize these to quantize string theory. We can talk about it if you would like to.