Problem 1: Casimir Force - Concept and Regularization

The problem of zero-point energy of quantum field theories[1] is not severe since it's not observable at all, as in classical physics, where only the difference in energy or potential levels could be measured or affect the degrees of freedom (D.O.Fs).

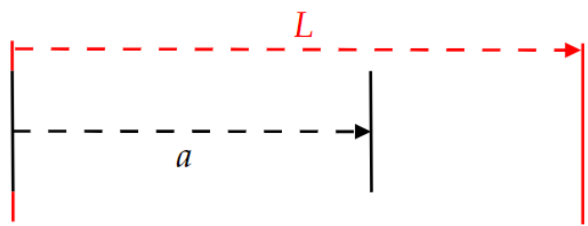

Figure 1: The two nested one-dimensional boxes.

But if this energy somehow depends on the problem's characteristic scale, then the derivative of the Casimir energy with respect to that scale can have physical significance. Here we obtain the Casimir force for a simple one-dimensional setup.

(a) [- points] **The setup:**

Consider a one-dimensional box of length a, inside another box of length L, where $a < L$. The outer box is required to bypass some conceptual issues, and in the end, we take $L \to \infty$ limit.

Upon quantization, the discrete frequencies are $\omega_n = \frac{\pi}{a} n$ and $n \in \mathbb{N}$. Casimir energy is the sum of the energies of all the quantum states. Note that you have to sum energies associated with two boxes due to the presence of two boxes. One is for the original box of length a; the other one is for the remaining space, which has length $L - a$, since we've accounted for the D.O.Fs between length $[0, a]$.

Write $E_{tot}(a)$, show that Casimir force, $F_{Casimir} = -\frac{dE_{tot}}{da}$ diverges.

(b) [- points] **Regularization of force via hard cut-off**

This divergence arises partly because of our ignorance of super high-frequency modes. Walls are made of atoms, and super high-frequency modes will penetrate the small gap between the atoms; thereby it's not viable to sum over all energy states. Whether what modes will penetrate the wall is not our interest because it involves providing a theory for the wall's

[1] Of course, coupling to gravity is not assumed

atoms. The crucial point is that we have to do the summation in part (a) up to modes that penetrate the wall.

We employ a high-frequency cut-off Λ, $\omega < \pi\Lambda$. Then the allowable modes will have $n < n_{max} = [\Lambda a]$.[2] Now repeat the calculation in part (a), but instead up to $n = n_{max}$. Use $x = \Lambda a - [\Lambda a]$ to rewrite your expression in terms of x and Λa. (Suppose $[\Lambda a]$ is not an integer.)

(c) [- points] **Averaging x variable:**

There still remains x, but it's finite and lies in interval $[0, 1)$. We can eliminate this variable by averaging it. Do this averaging.

(d) [- points] **The Casimir energy in the hard cut-off scheme:**

Finally, take derivative with respect to a. And after derivation, you can take the $L \to \infty$ without any concern. What's the value of the Casimir force?

[2] $[X]$ denotes the greatest integer less than or equal to X.

Problem 2: Regularization Schemes

As its name suggests, the model provided in the previous problem cuts off the UV modes hardly. Its form is $\theta(\pi\Lambda - \omega)$. We can forget the model and recruit other regularization schemes to find Casimir force.

Notice that we can use these schemes in calculating the determinant of operators, as you've seen in the class and gained familiarity in applying them. Here, we want you to reproduce the result of problem 1 in different schemes.

(a) [- points] **Gaussian regularization:**

Use Gaussian Kernel to evaluate the zero-point energy and Casimir force.

$$E(r) = \frac{1}{2}\sum_n \omega_n e^{-(\frac{\omega_n}{\pi\Lambda})^2}$$

(b) [- points] **Zeta function regularization:** Use Zeta function Kernel to evaluate the zero-point energy and Casimir force.

$$E(r) = \frac{1}{2}\sum_n \omega_n (\frac{\omega_n}{\mu})^{-s}$$

Aside: One can prove that the results are regulator-independent as long as regulators sastisfy certain criteria, namely:

$$\lim_{x\to\infty} x f^j(x) = 0, \qquad\qquad f(0) = 1$$

We can discuss them in the class if you'd like to.

Problem 3: Point Particle Quantization by three different methods: Path Integral Labor - Faddeev Popov - BRST

As you know, the relativistic free-particle action

$$S_0 = -m \int d\tau (-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu)^{-\frac{1}{2}}$$

suffers several ailments. So, we define another action involving einbein.

$$S = \frac{1}{2}\int d\tau (e^{-1}(\tau)\dot{x}.\dot{x} - e(\tau)m^2)$$

This formulation comes from putting a metric on the world line, a one-dimensional metric, with $g_{\tau\tau} = e^2(\tau)$. We can call it one-dimensional Polyakov action.

In this problem, we approach its quantization with three methods. All of which are the same in their nature. To be more specific, the Faddeev-Popov method accounts for the gauge group volume by inserting a functional "1:, which is a functional representation of the Dirac delta. BRST tries to address this issue by introducing new fields to the problem.

Note that the problem may be challenging to solve. I, myself, even struggle with some parts of it! Please don't worry and proceed as much as you can.

(a) [- points] **Warm-up:**

This action is reparametrization invariant, as suggested in your notes. Work out δX^μ and δe under infinitesimal transformation $\tau \to \tau + \xi(\tau)$.

Then find the Equation of Motion (E.O.M) of $e(\tau)$. Find E.O.M of x by varying action with respect to it. Show that by replacing back e into E.O.M of x^μ, you end up with the same E.O.M derived from the old action S_0.[Which is $\partial_\tau(\frac{m\dot{x}_\mu}{\sqrt{-\dot{X}^2}}) = 0$].

This means that reparametrization serves as a gauge transformation for the action S, while it was hidden in S_0.

(b) [- points] **Path integral method - direct computation:**

We evaluate the path integral by discretization. Notice that it's similar to the exercises that were solved beforehand. Except that you have to manage to integrate over gauge D.O.Fs properly!

Let's just focus on the propagator. Once we tackle the problem, it's not hard to generalize the problem.

$$\mathcal{N}\int_{x(0)=x}^{x(1)=x'} \mathcal{D}e\mathcal{D}x^\mu e^{-\frac{1}{2}\int_0^1(\frac{1}{e}\dot{x}^2 - em^2)d\tau}$$

(i) As you saw in (a), $\delta e = \partial_\tau(\xi e)$. To fix the gauge to a constant value, we choose $L = \int_0^1 \sqrt{g^{\tau\tau}}d\tau = \int_0^1 e(\tau)d\tau$. Argue why it must be fixed like this. By which I mean it should be equal to the length of the path and not any other constant.

(ii) To proceed further, the integration over e, which is integration over metric parameters, has been vastly studied during World War II.[3] These parameters are called Teichmuller parameters. Fortunately, in one-dimensional metrics, it's easy to classify all the metrics and define an integration measure on the space of all metrics.

The e-integration involves both scaling(constant modes) and reparametrization modes. We've removed the reparametrization invariance by the choice of the gauge (L), and there just remains to integrate over the scaling modes. This is realized via

$$\mathcal{N}\int_0^\infty dL \int_{x(0)=x}^{x(1)=x'} \mathcal{D}e\mathcal{D}x^\mu e^{-\frac{1}{2}\int_0^1(\frac{1}{L}\dot{x}^2-Lm^2)d\tau} \tag{1}$$

Up to now, I just motivated the above formula, now do the following tasks:

(iii) Exapnd around classical path, and by working out $||\delta x^\mu||$(as done in your notes), conclude that measure would be $\mathcal{D}x^\mu = \prod \sqrt{L}d\delta x^\mu(\tau)$.

(iv) Discretize and evaluate the path integral.(You've to end up to something proportional to $\det\left(-\frac{\partial_\tau^2}{L^2}\right)^{-2}$).

(v) By spectral decomposition, we find that $\det\left(-\frac{\partial_\tau^2}{L^2}\right) = \prod_{n=1}^\infty \frac{n^2}{L^2}$. In this part, just regularize this infinite product via Zeta function regulator.

(vi) **Optional!** Plug back into path integral (1) and do a fourier transformation to find the usual Feynman propagator.

[- points] **Faddeev-Popov procedure**

As mentioned, it involves adding a functional "1". This functional "1", magically takes gauge group volume into consideration, as you will see in this part.

(i) Define $1 = \Delta(e)\int \mathcal{D}\xi\delta(e-L[\xi])$, where $L[\xi]$ is value of the constant einbein after gauge transformation. Now just plug this into path integral (1).

(ii) Do the integral over e, now that there is a functional Dirac delta function, you can do this very very easy!

(iii) Since the integrand is gauge invariant, do a gauge transformation to drop $\xi-$dependence of your exponential part and $L[\xi]$. Then factor out the path integral on $\mathcal{D}\xi$, which produces the volume of gauge group, and accounts for gauge D.O.Fs. (This part is rather formal, no need to write a solution, just explain the ellimination of ξ.)

(iv) Now try to solve for $\Delta(L)$ from the part (i). Use integral representation of Dirac delta function. At the end you should plug this into the result of part (iii), and end up with propagator's form.

[- points] **BRST quantization**

We're not going to delve into this completely, but it's going to be a very long introduction to BRST.

[3] Even in the front of the war!

The BRST approach compensates for the gauge D.O.Fs in the Lagrangian by another method, adding new auxiliary fields with (anti-)commuting Grassmann fields. This approach is very powerful to fix gauge in ANY gauge thoery to ANY function. It involves promoting symmetry of the Lagrangian to a larger symmetry called BRST symmetry. This symmetry is crucial to show the independence of physical results from different gauges and construct consistent Hilbert space of the theory.

Define BRST transformations by:

$$\begin{aligned} \delta_{BRST}(x^\mu) &= \Lambda c e^{-1} \dot{x}^\mu \\ \delta_{BRST}(e) &= \Lambda \dot{c} \\ \delta_{BRST}(c) &= 0 \\ \delta_{BRST}(b) &= i\Lambda\pi \\ \delta_{BRST}(\pi) &= 0 \end{aligned} \tag{2}$$

Where $c(\tau)$ is an auxiliary ghost, real Grassmann field. Λ is pure imaginary Grassmann number (independent of τ) called BRST parameter. $b(\tau)$ is anti-ghost, Grassmann field, and $\pi(\tau)$ is a usual commuting field.

We also add gauge fixing term to action. Consider you want to fix gauge freedom (e) to an arbitrary function $f(x, e, \pi)$, define $\Psi = \int_0^1 d\tau\ b(\tau) f(x, e, \pi)$. Then add $S_{\mathbf{fix}}[x, e, c, b, \pi] = \frac{\delta}{\delta\Lambda}\Psi$ to your action, where $\frac{\delta}{\delta\Lambda}\Psi$ means BRST transformation of Ψ with the parameter Λ removed from the left.

Up to now, your action is $S_q \equiv S[x, e] + S_{\mathbf{fix}}[x, e, c, b, \pi]$. The path integral would be

$$Z = \int \mathcal{D}x\mathcal{D}e\mathcal{D}c\mathcal{D}b\mathcal{D}\pi e^{-S_q}$$

explicitly, well-defined for any choice of $f(x, e, \pi)$. We call S_q gauge-fixed BRST-invariant action.

(i) Show that this is a symmetry of the action S. ($\delta_{BRST}(S) = 0$)

(ii) Fix the freedom with the choice $e = L$ or $f(x, e, c, b, \pi) = L - e$, compute $S_{\mathbf{fix}}$ from the definition of Ψ.

(iii) Write out the path integral expression, you'll see that integral over π results a functional Dirac delta function on e, so that you can evaluate e-integration. This is pretty similar to Faddeev-Popov procedure.

Finally reach at $Z = \int \mathcal{D}x\mathcal{D}c\mathcal{D}b\ e^{-\int_0^1 dx\left(\frac{\dot{x}^2}{2L} + \frac{L}{2}m^2 + b\dot{c}\right)}$. Where the integration on b,c ghosts will produce the same $\det(\partial_\tau)$ on the Faddeev-Popov procedure.

This is the end of the story. You can proceed by this action to do perturbative calculation, including ghost vertices in your Feynman diagrams. All the well-known QFT identites and observations satisfied in any perturbative order.

Problem 4: Free Fermionic Path integral and miscellaneous derivations

We've looked at Fermionic path integrals before. Here, we complete our discussion.

Also, There were some minor derivation steps that I mentioned during the recorded assistant classes, and we investigate some of them here.

(a) **Fermionic Path Integral:**

(i) Generalize the Grassmann integral you've encountered in problem set 2, prove the following relation:

$$\int d\bar{\theta}_1 \dots d\bar{\theta}_n d\theta_1 \dots d\theta_n e^{-\sum_{ij} \bar{\theta}_i A_{ij} \theta_j + \sum_i \bar{\eta}_i \theta_i + \sum_i \bar{\theta}_i \eta_i} = \det(A) e^{\sum ij \bar{\eta}_i A^{-1}_{ij} \eta_j}$$

(ii) Define $(i\gamma^\mu \partial_\mu - m)^{-1}$ by $(i\gamma^\mu \partial_\mu - m)\Pi(x-y) = -i\delta(x-y)$, then do a Fourier transformation and find the usual propagator.

(iii) Now evaluate $Z[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{i\int d^4x \bar{\Psi}(x)(i\gamma^\mu\partial_\mu - m)\Psi + \bar{\eta}\Psi + \bar{\Psi}\eta}$.

(iv) Compute the following four-point function, with the definition of the generating functional.

$$\langle 0| T\{\Psi(x_1)\bar{\Psi}(x_2)\Psi(x_3)\bar{\Psi}(x_4)\} |0\rangle$$

(b) [- points] **Miscellaneous:**

(i) As we saw in the assistant class, an innocent shift in path integral measure will benefit us overwhelmingly! Prove the Schwinger-Dyson equation (pursue the same line followed in the class.)

$$-i\Box_x \frac{\partial Z[J]}{\partial J(x)} = \left\{ \mathscr{L}'_{int}[-i\frac{\partial}{\partial J(x)}] + J(x) \right\} Z[J]$$

(ii) In proving gauge invariance, I showed that adding the $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ doesn't affect correlation functions of a gauge-invariant operator. What about $-\frac{1}{2\xi}(\partial_\mu A^\mu)^4$ or $\xi A^\mu A_\mu$?