Problem 1: Grassmannian Path Integrals

In this problem, we will review some essential concepts regarding non-commutative path integrals.

(a) [- points, 9.5 Peskin] **Basic properties of Grassmann numbers:**

The basic fact about these numbers is that they anti-commute, $\eta\theta = -\theta\eta$. Hence they square to zero, $\theta^2 = 0$. Argue that any function of Grassmann number is a linear function, $f(\eta) = A\eta + B$.

Aside: In light of $\theta^2 = 0$, can we conclude that terms such as $\bar{\psi}\psi\bar{\psi}\psi$ or $(\bar{\psi}\psi)^2$ vanish in Lagrangian? As an instance, 4-Fermi theory has the former term.

(b) [- points, 9.5 Peskin] **Integration of Grassmann valued functions:**

We define $\int d\theta(A + B\theta) = B$. Justify this definition by considering a change of variable $\theta \to \theta + \eta$.

(c) [- points, 9.5 Peskin] **A simple Integration:**

Let's play more with this interesting variables.

(i) To warm up, compute $\int d\bar{\theta}d\theta e^{-\bar{\theta}b\theta}$, where b is an ordinary number.[Just expand this integral and use the $\theta^2 = \bar{\theta}^2 = 0$ property.]

(ii) Compute $\int d\bar{\theta}d\theta\ \bar{\theta}\theta e^{-\bar{\theta}b\theta}$, where b is an ordinary number.[Notice the importance of ordering of Grassmann variables, unless we will have a minus ambiguity.]

(iii) To generalize to multi-variable case, consider the following integral:

$$\int d\bar{\theta}_1 \dots d\bar{\theta}_n d\theta_1 \dots d\theta_n e^{-\bar{\theta}_i A_{ij} \theta_j}$$

Where A_{ij} is a $n \times n$ matrix. Work out this formula. [Expand the exponential and decide which terms survive. You must end up to $\det(A)$.]

(iv) [Optional] Another way of doing this, as Peskin does, is utilizing defintion of Jacobian of transformation in the Grassmann variables. This is a simple exercise that could be accompllished.

Problem 2: Path Integral Manual Dexterity

In this problem, we use all our knowledge to quantize elementary theories by path integrals. Scalar QED and establishing the equivalence of correlation functions in second-quantized and path integral quantizations, are subjects of this exercise.

(a) [- points, Peskin Problem 9.1] **Scalar QED:**

This problem concerns the scalar QED theory of a complex field ϕ interacting with the electromagnetic potential A^μ. The lagrangian is

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}^2 + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi$$

Where $D_\mu = \partial_\mu + ieA_\mu$.

Use the functional methods discussed in Section 9.2, show that the following are Feynmann rules of this theory.

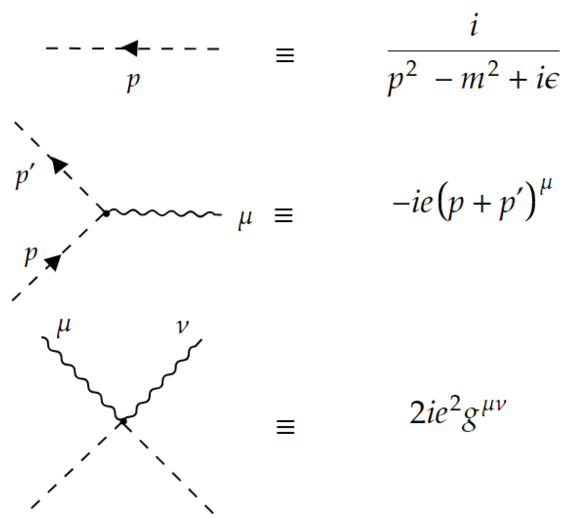

Figure 1: Scalar QED basic Feynman diagrams.

(b) [- points] **Correlation functions in different quantization schemes**

We want to establish the equivalence between n-point functions in second-quantization and path integal quantization.

(i) Using what we've learned in the classroom about generating functionals and sources J, prove that second quantized n-point function

$$\langle\Omega|\, T\phi(x_1)\ldots\phi(x_n)\,|\Omega\rangle = \frac{\langle 0|\, T\{\phi_0(x_1)\ldots\phi_0(x_n)e^{i\int d^4x\mathscr{L}_{int}[\phi]}\}\,|0\rangle}{\langle 0|\, T\{e^{i\int d^4x\mathscr{L}_{int}[\phi]}\}\,|0\rangle}$$

, is equivalent to

$$(-i)^n \frac{1}{Z[0]} \frac{\partial^n Z}{\partial J(x_1) \ldots \partial J(x_n)}\Big|_{J=0}$$

. In first relation $\phi_0(x)$ is the Heisenberg picture fields in free-theory and in the second, $Z[J]$ is generating functional of the full theory.

[Despite its formidable appearance, it's a trivial question. Don't be afraid and start by divding full Lagrangian into $\mathscr{L} = \mathscr{L}_{free} + \mathscr{L}_{int}$, then plug it into path integral and use the relation between path-integral and n-point function in "Free Theories".]

(ii) Now that you've proved the most general case, let's verify it for ϕ^3 theory, where $\mathscr{L}_{int} = \frac{g}{3!}\phi^3$. Expand $e^{i\int d^4 \mathscr{L}_{int}}$ in the path integral and conclude that two point function in this interactive theory gives the same result as Feynman rules up to order g^4.

Problem 3: Quantum Statistical Mechanics

We work out the world's most famous problem in the path integral formalism. These problems are 9.2 (a) and (b), Peskin.

(a) [- points, Peskin Problem 9.2 (a)] **Quantum Statistical Mechanics**

Solve problem 9.2 (a), which is a review of quantum mechanical path integral, but in Euclidean signature.

(b) [- points, Peskin Problem 9.2 (b)] **Harmonic Oscillator Path Integral**

Now work out the part (b). Use the fourier expansion suggested, plug into path integral. The integration is a bit baffling, but pay close attention to (9.23) to figure out how to do these integrals. You will eventally end up with an infinite product, which turns out to be $\sinh(z)$, as suggested in the question. This gives the correct partition function for quantum mechanical oscillators, one that you've read in advanced statistical mechanics courses.

REFLECTIONS: There are many interesting ideas in path integrals:

- Schwinger Dyson equations: this equations govern the expectation values of the field operators, and bear a close resemblance to classical counterparts up to contact terms.
- Noether current and other symmetries: this is surprising how a simple change of variables in path integral could give us Noether current and Ward-Takahashi identity in QED.
- Quantization of QED: this is also a general procedure to quantize spin-1 fields, discussed in your textbook, I hope you cover these parts too.