Large Array Null Steering Using Compressed Sensing

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***Abstract*—In this paper, array null steering is formulated as a sparse recovery problem. In addition, a novel null steering scheme for large arrays by perturbing only a few elements is presented. To achieve this goal, compressed sensing (CS) is used to exploit the sparsity of the perturbed elements. The advantages of the proposed scheme are a significant reduction in hardware cost, and lower power consumption, as well as less aging of the elements, and faster response. Simulation results show that the CS-based method could be efficiently used to generate wide nulls using at least two elements. The interference rejection ratio (IRR) achieved by the proposed method is 10-20 dB better than the existing solutions. The performance of different null generation methods in terms of peak-to-sidelobe ratio (PSLR), pointing error, and beam-width is compared, as well. The proposed algorithm is a prominent solution for the future 5G base stations, where a fast and low-cost beam shaping algorithm is required.**

***Index Terms*—5G, beamforming, compressed sensing, interference suppression, null steering, phased array, sparse recovery.**

I. Introduction

INTERFERENCE rejection is a critical task in many phased array applications, such as radar [1], multi-user communication systems [2], cognitive radio [3], satellite communication [4], and biomedical applications. Particularly, in future 5G base stations where from 64 to 256 antennas are to be used, null steering is vital to increase throughput and decrease overall interference level [5]. Undesired signals are rejected by generating nulls in the array pattern in the corresponding directions, which in practice is achieved by applying a set of phase and/or amplitude perturbations to the pre-configured element weightings.

The conventional method for null steering requires full control of amplitude and phase of all elements, which is costly and relatively slow [6]. Generally, nulls are placed in the interferers directions by minimum perturbation of the element excitation coefficients (weights) in a mean-squared sense. This problem is equivalent to the minimization of mean square of the pattern deviation due to weight perturbation, that is why this method is also known as minimum mean-squared error (MMSE) [6]. Another method is null steering by control of phase only [7]. Using the small phase approximation, the problem reduces to a linear one with an analytic solution [7]. Null steering could also be performed by changing the element positions. In [8], a technique is proposed to change the position of a subset of elements to steer a null, while in [9] synthesizing the array pattern only by optimizing the element positions is implemented. Such methods add to the complexity of the system by using servo motors.

If the interferer is not a point source but distributed in space, one null is not sufficient to reject the incoming signal, so wide-null steering is necessary. In [10], a scheme is proposed to produce a wide null by imposing several equidistant nulls over the desired pattern sector, whereas in the method developed in [11], a wide null is generated by proper control of the weights of the two edge elements of the array. Wide-null steering using higher order nulls is proposed in [12]. However, it is less effective than the multiple nulling method [10].

Null steering, in both cases of narrow and wide nulls, could be performed with some limited degrees of freedom; since, in many cases, the number of required nulls is much smaller than the number of array elements. In these cases, null steering can be performed by using a reduced number of elements. This could be realized in two ways: 1) partitioning the array into subarrays and controlling the subarray weights, or 2) using only a subset of element weights [13]. In both cases, if the elements (or subarrays) to be used for null steering are already specified, the problem can be classified as an under-determined system of linear equations, which can be solved using a least squared solution. Performance of these methods is highly dependent on how elements (or subarrays) are selected. Therefore, intelligent selection of the elements is of high importance in such methods [13].

Compressed sensing (CS), a relatively new signal processing paradigm, is recently used in various fields of engineering [14]. For example, an adaptive digital beamforming technique with CS is proposed in [15] that exploits the angular sparsity of arriving signals to reduce the number of the array elements. Oliveri *et al.*, in [16] and [17], proposed methods for the synthesis of maximally sparse linear arrays based on the bayesian compressive sampling, while a versatile multi-task bayesian compressive sensing strategy is used in [18], for sparsening of conformal arrays. A novel compressive sensing reconstruction approach for correlated images is proposed in [19]. Moreover, in [20], CS is exploited for channel estimation in OFDM systems.

In this paper, array null steering is formulated as a sparse recovery problem. In addition, CS is used to perform null steering using only a few elements. A novel algorithm is proposed which starts by the pre-determined number of elements dedicated to null steering, and returns the indices of the most effective elements and their corresponding weights. Using this method one can use a partially adaptive phased array instead of a fully adaptive one. It should be noted that implementation of a fully adaptive phased array has several disadvantages. The extra hardware adds to the complexity, size, cost, and power consumption of the system, and increases

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the response time of the system to interference [13]. Even if a fully adaptive array is available, the proposed method is still beneficial to decrease the response time of the array, and increase the mean time before failure (MTBF) of the active array components contributing to overall reliability of the system. Moreover, perturbing less number of elements may result in less gain drop (higher array directivity), and lower peak-to-sidelobe ratio (PSLR) compared to the conventional null steering methods as shown in Section IV.

The rest of the paper is organized as follows. In Section II, null steering is formulated as a sparse recovery problem. Section III presents the proposed scheme. Simulation results are demonstrated in Section IV. Finally, Section V concludes the paper. The notations used in this work are listed in Table I.

TABLE I
NOTATIONS AND THEIR DEFINITION

Symbol	Meaning
$(.)^T$	transpose
$(.)^H$	hermitian
$\lvert . \rvert$	absolute value
$\lVert . \rVert_p$	ℓ_p-norm

II. Problem Statement

In this section the null steering is formulated as a sparse recovery problem. Fig. 1 illustrates a uniform linear array (ULA) of N elements placed along the x-axis. To steer the main beam toward the desired angle, θ_s, the conventional weight for element n is given by

$$w_{0n} = a_{0n} e^{jkd_n u_s} \tag{1}$$

where $u_s = \sin\theta_s$, and a_{0n} and k denote the element excitation amplitude, and the wave-number, respectively. Besides,

$$d_n = (n - \frac{N+1}{2})d, \quad n = 1, 2, \cdots, N \tag{2}$$

indicates the element positions along the x-axis, where d is the element spacing. Hence, the conventional radiation pattern is given by

$$B_c(u) = \sum_{n=1}^{N} w_{0n} e^{-jkd_n u} = \sum_{n=1}^{N} a_{0n} e^{-jkd_n (u-u_s)} \tag{3}$$

where $u = \sin\theta$ and θ is the angle measured from broadside to the array as illustrated in Fig. 1.

To generate M nulls in the array pattern in the directions of interference sources located at $\theta_{i_1}, \theta_{i_2} \cdots, \theta_{i_M}$ (as shown in Fig. 1), element weights can be perturbed from their conventional values as

$$w_n = w_{0n} + x_n \tag{4}$$

to produce a new pattern, $B(u)$, under M constraints:

$$B(u_m) = \sum_{n=1}^{N} w_n e^{-jkd_n u_m} \tag{5}$$

$$= B_c(u_m) + \sum_{n=1}^{N} x_n e^{-jkd_n u_m} = 0, \quad m = 1, 2, \cdots, M$$

All M constraints in (5) can be represented as a linear set of equations:

$$\mathbf{Ax} = \mathbf{y} \tag{6}$$

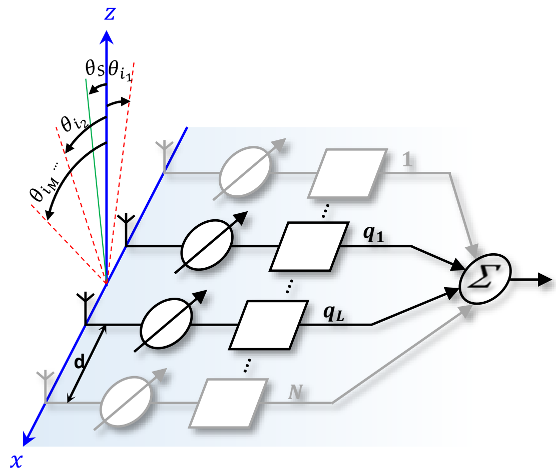


Fig. 1. Uniform linear array configuration and coordinates.

where

$$\mathbf{y} = [-B_c(u_1), -B_c(u_2), \cdots, -B_c(u_M)]^T, \tag{7}$$

$\mathbf{x} = [x_1, x_2, \cdots, x_N]^T$, $u_m = \sin\theta_{i_m}$, and

$$\mathbf{A} = \begin{bmatrix} e^{-jkd_1u_1} & e^{-jkd_2u_1} & \cdots & e^{-jkd_Nu_1} \\ e^{-jkd_1u_2} & e^{-jkd_2u_2} & \cdots & e^{-jkd_Nu_2} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-jkd_1u_M} & e^{-jkd_2u_M} & \cdots & e^{-jkd_Nu_M} \end{bmatrix}_{M\times N} \tag{8}$$

As long as $M < N$, Eq. (6) indicates an under-determined system of linear equations, which has infinitely many solutions. The conventional full phase/amplitude control method solves this problem by means of least squared error approximation [21] which is equivalent to minimizing the squared error between the conventional pattern and the constrained pattern. It is shown in [21] that the problem reduces to:

$$\min_{\mathbf{x}} \|\mathbf{x}\|_2 \quad \text{subject to} \quad \mathbf{Ax} = \mathbf{y} \tag{9}$$

whose solution is:

$$\mathbf{x}_{\text{MMSE}} = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{y}. \tag{10}$$

In this work, we are interested in the sparse solutions, where the values of most of entries are zero, because such solutions impose the least perturbation to the elements weighting. This is equivalent to null steering using only a few elements with all benefits demonstrated in Section IV. Hence, the problem can be formulated as a so-called constrained ℓ_0-minimization problem:

$$\min_{\mathbf{x}} \|\mathbf{x}\|_0 \quad \text{subject to} \quad \mathbf{Ax} = \mathbf{y} \tag{11}$$

where $\|\mathbf{x}\|_0$, i.e. ℓ_0-norm of $\mathbf{x}$, denotes the number of non-zero entries of $\mathbf{x}$. In the CS literature, this problem is known as sparse recovery problem, in which $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{y}$ are called measurement matrix, unknown sparse signal, and measurement vector, respectively.

The best solution to ℓ_0-minimization problem found so far is the combinatorial search, which is computationally intractable for large values of N [22], since all possible k-sparse vectors ($k \leq M$) should be tested, in the worst case (a vector is k-sparse if it contains at most k non-zero entries.). Hence, several sparse recovery algorithms, such as orthogonal matching pursuit [23], iterative hard threshold

[24], YALL1 [25], and smoothed ℓ_0-norm (SL0) [26] are introduced to solve the problem with a reasonable complexity. After investigating different sparse recovery algorithms, we concluded that SL0 and YALL1 show desirable performance in solving this problem. Thus, they are used as sparse recovery algorithms to perform the simulation in the rest of the work.

In this section, we showed that CS could be used to reduce the number of perturbed elements required for null steering. The proposed scheme is presented in the next section.

III. Proposed Null Steering Scheme

Assuming that only L out of N array elements are used for null steering, a two-step scheme is proposed. The first step is devoted to finding L elements that are capable of steering the nulls, while perturbing the array pattern as slightly as possible. In the second step, the weight perturbation of these L elements for null steering is estimated assuming no weight perturbation for the remaining $N-L$ elements.

In the following, the proposed scheme is explained in detail. The goal of the first step is to find the support of the weight perturbation vector, $\mathbf{x}$, i.e. the set of indices of the elements to be used for null steering, (see Fig. 1):

$$\mathcal{S} = \{i \in \{1, 2, \cdots, N\} | x_i \neq 0\}. \tag{12}$$

To do so, initially a sparse recovery algorithm should be applied to obtain a solution to (11), $\mathbf{x}^{(0)}$. However, some sparse recovery algorithms (such as SL0 and YALL1) do not return an exactly sparse solution (i.e. most entries are insignificant, but not zero). Setting $N-L$ least significant entries of $\mathbf{x}^{(0)}$ to zero yields an L-sparse vector, but this solution does not necessarily satisfy $\mathbf{Ax} = \mathbf{y}$. So, we set the indices of the L most significant entries of $\mathbf{x}^{(0)}$ as the support of the solution. Note that this pruning process is not required when sparse recovery algorithms such as CoSaMP that inherently perform the pruning are used. The first step is now successfully done.

Let $\mathcal{S} = \{q_1, q_2, \cdots, q_L\} \subseteq \{1, \cdots, N\}$ to be the support, ruling out the zero entries of $\mathbf{x}$ from (11), reduces it to:

$$\min_{\mathbf{x}_\mathcal{S}} \|\mathbf{x}_\mathcal{S}\|_0 \quad \text{s.t.} \quad \mathbf{A}_\mathcal{S}\mathbf{x}_\mathcal{S} = \mathbf{y} \tag{13}$$

where

$$\mathbf{A}_\mathcal{S} = \begin{bmatrix} e^{-jkd_{q_1}u_1} & e^{-jkd_{q_2}u_1} & \cdots & e^{-jkd_{q_L}u_1} \\ e^{-jkd_{q_1}u_2} & e^{-jkd_{q_2}u_2} & \cdots & e^{-jkd_{q_L}u_2} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-jkd_{q_1}u_M} & e^{-jkd_{q_2}u_M} & \cdots & e^{-jkd_{q_L}u_M} \end{bmatrix} \tag{14}$$

is an $M \times L$ sub-matrix of $\mathbf{A}$ and $\mathbf{x}_\mathcal{S} = [x_{q_1}, x_{q_2}, \cdots, x_{q_L}]^T$ is a vector containing non-zero entries of $\mathbf{x}$. This problem is similar to (11), but the dimension is reduced from N to L where $L \ll N$ and $L > M$.

Next, $\mathbf{A}_\mathcal{S}$ is calculated according to (14) which is followed by solving (13) using a sparse recovery algorithm to obtain $\mathbf{x}_\mathcal{S}$. Note that, this time one can use a sparse recovery algorithm different from that used to solve (11). Ultimately, $\mathbf{x}_{opt}$ is an $N \times 1$ L-sparse vector with support $\mathcal{S}$, whose non-zero entries are entries of $\mathbf{x}_\mathcal{S}$. The entire process of estimating $\mathbf{x}_{opt}$ is summarized in Algorithm 1.

Algorithm 1: Weight Perturbation Estimation

Input: $N, L, d, \lambda, \theta_s, \{\theta_{i_1}, \theta_{i_2}, \cdots, \theta_{i_M}\}$
Output: $\mathbf{x}_{opt}$

1: **Calculate A** and **y** according to (7) and (8)
2: $\mathbf{x}^{(0)} =$ **Sparse_Recovery1**($\mathbf{A}, \mathbf{y}$) (solution of (11))
3: $\mathcal{S} =$ indices of L most significant entries of $\mathbf{x}^{(0)}$
4: **Calculate** $\mathbf{A}_\mathcal{S}$ according to (14)
5: $\mathbf{x}_\mathcal{S} =$ **Sparse_Recovery2**($\mathbf{A}_\mathcal{S}, \mathbf{y}$) (solution of (13))
6: $\mathbf{x}_{opt} =$ the $N \times 1$ vector with support $\mathcal{S}$, whose non-zero entries are entries of $\mathbf{x}_\mathcal{S}$

Note that using the proposed scheme, one can arbitrarily choose any number of elements to perform the null steering. In the next section, the performance of the different methods in generating wide nulls is analyzed.

IV. Simulation Results

The purpose of this section is to compare the performance of the proposed scheme with the existing null steering methods. Throughout this section, a ULA with half-wavelength ($\lambda/2$) spacing is considered. Nevertheless, the results derived in this work can be extended to any element spacing. To compare different algorithms in suppressing a distributed interference, a figure of merit, named interference rejection ratio (IRR), is defined as:

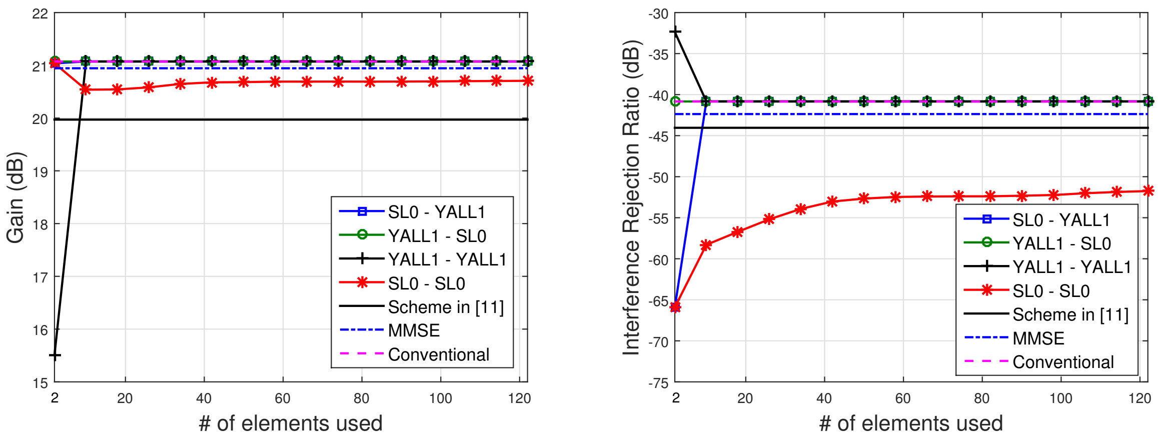
$$\text{IRR} = \frac{1}{B(\theta_s)} \int_{-\pi}^{+\pi} B(\theta).I(\theta) d\theta \tag{15}$$

where $B(\theta)$ is the array radiation pattern, and $I(\theta)$ is a function representing the angular distribution of the interference signal over θ, which is modeled with a Gaussian function:

$$I(\theta) = \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(\theta-\theta_i)^2}{2\sigma_i^2}} \tag{16}$$

where θ_i is the interference center direction and σ_i indicates the standard deviation of the interference angular location. In fact, IRR measures how much the array beam is able to suppress a Gaussian distributed interference. Hence, low values of IRR are desired (corresponding to high suppression of the interference). Note that in this work, array weights are normalized by the maximum value for two reasons: 1) the array gain comparison becomes fair, and 2) the methods that require a higher variation of the weight amplitudes will be distinguished.

In the first simulation, four possible combinations of the two aforementioned sparse recovery algorithms, SL0 and YALL1, as **Sparse_Recovery1** and **Sparse_Recovery2** in **Algorithm 1** are used to compare CS-based null steering with the full amplitude/phase control (MMSE) method [6] and the scheme presented in [11]. Fig. 2 compares the performance of the different schemes in rejecting a distributed interference. In this simulation, the proposed scheme generates two nulls around $\theta_i = 33°$ with the spacing of $0.05°$ to suppress an interference with $\sigma_i = 1.6°$ for $N = 128$, $\theta_s = 10°$. Note that the scheme in [11] attempts to produce a wide null around $\theta_i = 33°$ by perturbing the two side-elements of the array. The proposed



(a) (b)

Fig. 2. Comparison of the different null steering methods in terms of (a) array gain, and (b) IRR for $N = 128$, steering at $10°$, when a distributed interference is located around $\theta_i = 33°$ with $\sigma_i = 1.6°$.

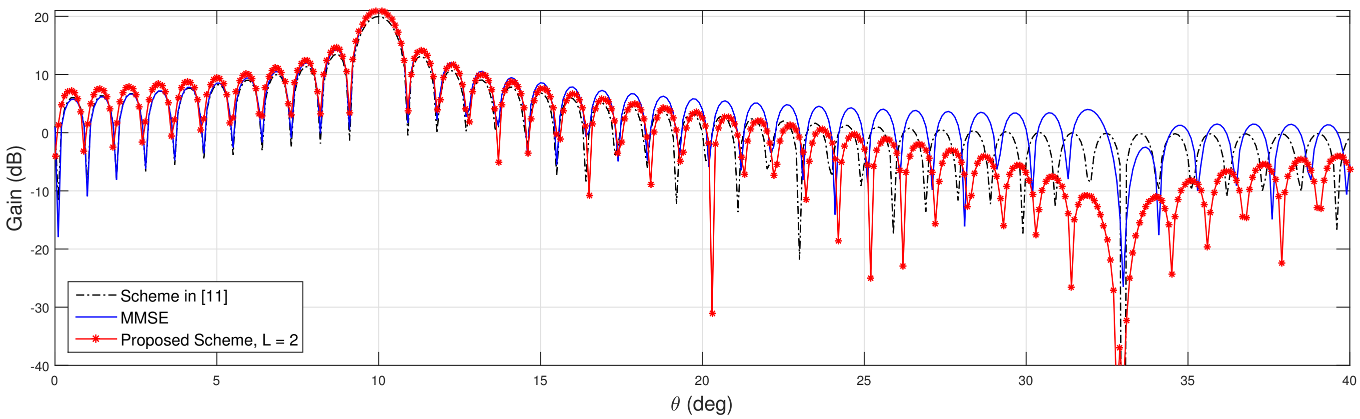


Fig. 3. Array radiation patterns generated by the different null steering schemes for $N = 128$, steering at $10°$, when a distributed interference is located around $\theta_i = 33°$ with $\sigma_i = 1.6°$.

method (SL0-SL0) outperforms the scheme in [11] by $8dB$ to $22dB$ for different values of L, in terms of IRR, while the other CS-based schemes fail to provide nulls with sufficient IRR. Another important result is that in the proposed method, as the number of elements used for null steering decreases, the IRR decreases. The radiation patterns depicted in Fig. 3 indicate that not only the null generated using the proposed method is deeper and wider, but also the sidelobes are lower in the interference region.

The performance of the proposed scheme in rejecting a distributed interference located around $\theta_i = 38°$ with $\sigma_i = 1.6°$ by imposing three nulls with the spacing of $0.05°$ by perturbing three out of 256 elements of a linear array with $\theta_s = 0°$ is compared with the other schemes in Table II. In addition to gain and IRR, other parameters of the beams formed by each method are included in the comparison. It is seen that, not only the proposed method is superior in terms of gain and IRR, but also it does not have any broadening effect on the half power beam-width (HPBW) of the pattern, and does not cause any pointing error.

TABLE II
BEAM PARAMETERS FOR $N = 256$, $L = 3$, NULLS AT $37.95°, 38°, 38.05°$, STEERING AT $0°$.

Feature	This Work	Scheme in [11]	MMSE
IRR (dB)	-79.8	-52.1	-51.5
Gain (dB)	24.04	23.84	24.00
PSLR (dB)	13.57	13.53	13.63
HBPW (°)	0.6	0.6	0.6
Pointing Error (°)	0	0	0

V. CONCLUSION

In this paper, null steering was formulated as sparse recovery problem and compressed sensing was used to perform null steering using only a few elements of a large array. The capability of the proposed scheme in providing wide nulls was investigated. Different combinations of SL0 and YALL1 algorithms were analyzed among which the SL0-SL0 method was superior in terms of the Interference Rejection Ratio (IRR). The IRR is a figure of merit measuring the ability of a pattern to suppress a distributed interference with Gaussian distribution. It was shown that the IRR of the proposed method is 10 to 20 dB better than the existing solutions.

The future work will focus on applying the proposed scheme to 5G base station arrays where multiple wide nulls are required. Deriving an optimal null placement, using higher order nulls, and extending the proposed method to the planar arrays are main steps in future research.

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