On Higher Cheeger Inequalities

A Computational Approach

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What I am and what I am not going to talk about!

There are at least three different approaches to this fascinating subject:

- Geometry of metric-measure spaces,
- Theory of Markov processes,
- Theory of computation.

This talk is about computational aspects of how one may compute or estimate the isoperimetric spectra of graphs and definitely not about the computational consequences of constructing graphs with a relatively high isoperimetric constant! (i.e. theory and applications of expander graphs)

OUTLINE

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2. Computational Hardness of The Normalized Cut Problem
3. Approximations of The Isoperimetry Problem
   - Real relaxation: Federer-Fleming-type results
   - Computational hardness of the isoperimetry problem
   - Euclidean $\|.\|_2$ setting: eigenstructure of the Laplacian
   - Metric embedding, Clustering, and More

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   - Localization and Dirichlet eigenvalues
   - Nodal domains
   - Trees and cycles
   - General weighted graphs

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5. Epilogue

Cheeger constant of a Riemannian Manifold

Cheeger constant of a (compact) n-dimensional Riemannian manifold G:

$$\iota_2^M(G) \stackrel{\text{def}}{=} \inf_A \max\left\{\frac{\mu_{n-1}(\partial A)}{\mu_n(A)}, \frac{\mu_{n-1}(\partial A)}{\mu_n(A^c)}\right\}$$

A runs over open subsets of M.

μ_n: n-dimensional measure, ∂A: the boundary of A

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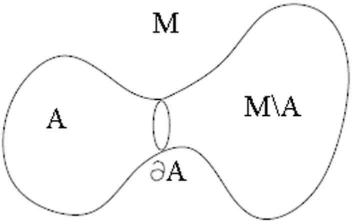
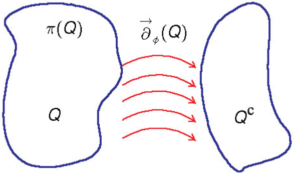




The case of simple graphs

For a simple graph $G = (V, E)$:

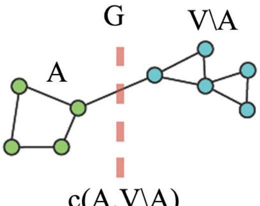
The max version: (Cheeger constant or edge expansion)

$$\iota_2^M(G) \stackrel{\text{def}}{=} \min_{A \subseteq V(G)} \max \left\{ \frac{|E(A, A^c)|}{|A|}, \frac{|E(A, A^c)|}{|A^c|} \right\}$$

The mean version: (scaled uniform sparsest cut or normalized cut)

$$\iota_2^m(G) \stackrel{\text{def}}{=} \min_{A \subseteq V(G)} \frac{1}{2} \left(\frac{|E(A, A^c)|}{|A|} + \frac{|E(A, A^c)|}{|A^c|} \right)$$

2-Isoperimetry Problem: Finding a 2-partition (A, A^c) of $V(G)$ attaining the edge expansion of G.



Measures of connectivity

A classic theorem [PERRON-FROBENIUS]

Let G be a graph with the adjacency matrix C, the diagonal degree matrix D and let $L = D - C$ be its combinatorial Laplacian. Then,

- The number of connected components of G is equal to the number of 0-eigenvalues of L.
- If G is connected then the 0-eigenvalue is simple and its eigenvector does not change its sign (of course in this case the eigenspace is generated by the constant vector $\mathbf{1}$!)

Is it possible to generalize such a theorem?

Let's talk about this!

Importance!

The subject is central and has connections to many different fields of study as Geometric Analysis, Stochastic Processes, Representation theory and Harmonic Analysis, Graph theory and Combinatorial Optimization, Theoretical CS, Mathematical Physics, Signal Processing and AI.

GENERAL REFERENCES FOR FURTHER READING:

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Continuous versus discrete settings

Although the problem and the intuitions around it are essentially the same in both continuous and discrete settings, motivations are quite different.
In the continuous setting Cheeger's constant is used to obtain information about the eigenstructure of the Laplacian and hence the geometry of the manifold which is essentially hard to estimate.
In the discrete setting the eigenstructure of the Laplacian as an easy to compute concept is used to approximate Cheeger's constant (expansion) which is an important algorithmic and geometric hard to compute parameter.
In what follows we try to delve into the details of this scenario.

Weighted graphs

Model: (A finite weighted graph) A simple graph $G=(V,E)$ together with two weight functions $w: V \to \mathbb{R}^+$ and $c: E \to \mathbb{R}^+$.

Notations: For every $x \in V$ and $A \subseteq V$,

$$deg(x) \stackrel{\text{def}}{=} \sum_{y \sim x} c(xy).$$

$$E(A,B) \stackrel{\text{def}}{=} \{e = uv \in E : \; u \in A, v \in B\},$$

$$w(A) \stackrel{\text{def}}{=} \sum_{u \in A} w(u), \quad c(A) \stackrel{\text{def}}{=} \sum_{e \in E(A,A^c)} c(e).$$

$\mathcal{P}_k(V)$: The set of k-partitions of V.

For the case of weighted graphs with potentials see [R. JAVADI PHD THESIS 2011]

Clustering

In general, clustering is the problem of partitioning a data-set into subpartitions in a way that each subpartition contains data-points of similar type.

Computationally, one may consider the problem of finding an optimal, or a suboptimal (sub)partition or one may consider the problem of finding the optimal or a suboptimal overall-similarity cost function.

These alternatives will give rise to a variety of interesting practical and theoretical problems!

A naive generalization: the normalized cut problem

Given a weighted graph $G = (V, E, c, w)$ and integer k ($2 \leq k \leq |V|$), find a k-partition of $V(G)$, $(A_1, \ldots, A_k)$ that attains the following parameters:

A naive generalization of Cheeger's constant (a $\|.\|_\infty$ version):

$$\tilde{\iota}_k^M(G) \stackrel{\text{def}}{=} \min_{\{A_i\}_1^k \in \mathcal{P}_k(V)} \max_{1 \leq i \leq k} \frac{c(A_i)}{w(A_i)}.$$

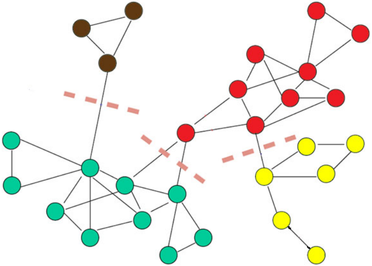
The normalized cut cost function (a $\|.\|_1$ version):
[Shi, Malik 1997-2000] (# citation > 2000!)

$$\tilde{\iota}_k^m(G) \stackrel{\text{def}}{=} \min_{\{A_i\}_1^k \in \mathcal{P}_k(V)} \frac{1}{k} \left(\sum_{i=1}^{k} \frac{c(A_i)}{w(A_i)} \right),$$

We refer to both problems as the normalized cut problem.

An example

All edge and vertex weights are equal to 1, $k = 4$.



$$\tilde{\iota}_4^M = \max\{\frac{1}{3}, \frac{3}{10}, \frac{3}{9}, \frac{1}{6}\} = \frac{1}{3}.$$

$$\tilde{\iota}_4^m = \frac{1}{4}\left(\frac{1}{3} + \frac{3}{10} + \frac{3}{9} + \frac{1}{6}\right) = \frac{17}{60}.$$

Some notations

Acronyms:

NCP : The Normalized Cut Problem.

Superscript m (resp. M): mean (resp. max) version.

Subscript k: appears when k is a constant,
disappears when k is part of the input.

Some notations

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Example:

NCP_k^m :

CONSTANTS: An integer k.
INPUTS: A weighted graph $G = (V, E, w, c)$ and a positive integer N.
QUERY: Is it true that $\tilde{\iota}_k^m(G) \leq N$?

Computational Hardness of the Normalized Cut Problem

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Known Complexity Results

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[JAVADI, D. 2010]

NCP_k (for both max and mean versions) is *NP*-complete for unweighted (simple) graphs.

NCP^M is *NP*-complete for unweighted trees.

Known Complexity Results

A problem with numerical parameters is said to be *NP*-complete in the strong sense if it is so, even when all of its numerical parameters are bounded by a polynomial in the length of the input.
In other words, a problem that is *NP*-complete even when the inputs are given in unary codes (instead of binary codes).

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[JAVADI, D. 2010] For weighted trees:
NCP^m is *NP*-complete.
NCP^M is *NP*-complete in the strong sense.
$\tilde{\iota}_k^M$ is computable in time $O(n^{2k^2-6k-3})$.

A couple of open problems

- Is it true that NCP_k^m is polynomial time solvable for weighted trees?
- What can we say about the strong *NP*-completeness of NCP^m?

Approximations of the Isoperimetry Problem

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Real relaxation: Federer-Fleming-type results

The gradient operator

Let $\mathcal{F}_w(G)$ and $\mathcal{F}_c(G)$ be the set of all real functions on $V(G)$ and $E(G)$, respectively, equipped with the corresponding weighted inner-products. Define the gradient as

$$\nabla : \mathcal{F}_w(G) \longrightarrow \mathcal{F}_c(G), \quad \nabla f(uv) \stackrel{\text{def}}{=} f(v) - f(u).$$

Gradient of characteristic functions

If $f = \frac{1}{w(A)}\chi_A$ is the normalized characteristic function of a subset $A \subseteq V(G)$ then

$$\|\nabla f\|_{1,c} = \frac{c(A)}{w(A)} = \frac{\|\nabla \chi_A\|_{1,c}}{\|\chi_A\|_{1,w}}.$$

A real relaxation of parameters

Define

$$\mathcal{O}_k^+(G) \stackrel{\text{def}}{=} \left\{ \{f_i\}_1^k \;\middle|\; \{f_i\}_1^k \text{ is positive orthonormal} \right\},$$

$$\tilde{\mathcal{O}}_k^+(G) \stackrel{\text{def}}{=} \left\{ \{f_i\}_1^k \in \mathcal{O}_k^+(G) \;\middle|\; \{\text{supp}(f_i)\}_1^k \in \mathcal{P}_k(G) \right\}.$$

and the relaxed parameters,

$$\gamma_k^m(G) \stackrel{\text{def}}{=} \inf_{\{f_i\}_1^k \in \mathcal{O}_k^+(G)} \frac{1}{k} \left(\sum_{i=1}^{k} \|\nabla f_i\|_{1,c} \right),$$

$$\tilde{\gamma}_k^m(G) \stackrel{\text{def}}{=} \inf_{\{f_i\}_1^k \in \tilde{\mathcal{O}}_k^+(G)} \frac{1}{k} \left(\sum_{i=1}^{k} \|\nabla f_i\|_{1,c} \right).$$

$\gamma_k^M(G)$ and $\tilde{\gamma}_k^M(G)$ are defined similarly!

The isoperimetric constants

A k-subpartition consists of k nonempty and disjoint subsets of $V(G)$.

Let $\mathcal{D}_k(G)$ be the class of all k-subpartitions of $V(G)$.

Define the kth isoperimetric constants of G as,

The maximum (i.e. $\|.\|_\infty$) version:

$$\iota_k^M(G) \stackrel{\text{def}}{=} \min_{\{A_i\}_1^k \in \mathcal{D}_k(V)} \max_{1 \le i \le k} \frac{c(A_i)}{w(A_i)}.$$

The mean (i.e. $\|.\|_1$) version:

$$\iota_k^m(G) \stackrel{\text{def}}{=} \min_{\{A_i\}_1^k \in \mathcal{D}_k(V)} \frac{1}{k} \left(\sum_{i=1}^k \frac{c(A_i)}{w(A_i)} \right),$$

We use the acronym IPP for the corresponding problems.

Justifications for definitions

[Javadi, Hajiabolhassan, D. 2010]

For both max and mean versions, $\gamma_k(G) = \tilde{\gamma}_k(G) = \iota_k(G)$.

The intrinsic inequality

By definitions, in general, we have $\iota_k(G) \leq \tilde{\iota}_k(G)$, where the inequality can be strict (in both maximum and mean versions)!

To the best of my knowledge, the correctness of definitions for subpartitions has been first indipendently observed in [Miclo 2007], [Hajiabolhassan, D. 2008], and [Helffer, T. Hoffmann-Ostenhof, Terracini 2008].

Justifications for definitions

Test function approximation

The equality $\gamma_k(G) = \tilde{\gamma}_k(G) = \iota_k(G)$ shows that $\iota_k(G)$ can be effectively approximated by test functions.

Subpartitions are richer

Computationally, a move from partitions to subpartitions usually makes the problem easier!
(e.g. the polynomial time algorithm for minimum k-subpartition problem [Nagamochi, Kamidoi 2007])

Subpartition residues [Javadi, Shariatrazavi, D. 2011]

There is evidence supporting the fact that subpartition residues contain nontrivial information. Hence, the subpartition setup makes it possible to gain more information in an easier way!

The isoperimetric spectrum

The isoperimetric spectrum of a graph G is defined as

$$0 = \iota_1(G) \leq \iota_2(G) \leq \ldots \leq \iota_{|V(G)|}(G).$$

[Javadi, D. 2010]

- For every weighted graph G we have $\iota_2(G) = \tilde{\iota}_2(G)$.
- For every connected weighted graph G and each $3 \leq k \leq |V|$,

$$\begin{aligned} \iota_k^M(G) &\leq \tilde{\iota}_k^M(G) < (k-1)\, \iota_k^M(G), \\ \iota_k^m(G) &\leq \tilde{\iota}_k^m(G) < 2(1 - \frac{1}{k})\, \iota_k^m(G). \end{aligned}$$

Geometric graphs

Geometric graphs

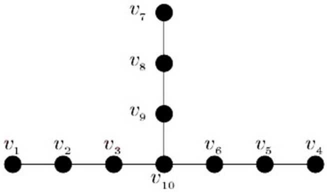
A graph G is said to be *k-geometric*, if $\iota_k(G,K) = \tilde{\iota}_k(G,K)$.
A graph G is said to be *supergeometric*, if it is k-geometric for every $2 \leq k \leq |V(G)|$.

[R. Javadi] (personal communication)

A couple of partial results are available. Characterization of supergeometric graphs is essentially an open problem (in both maximum and mean cases)!

A minimal example

(All edge and vertex weights are equal to 1.)



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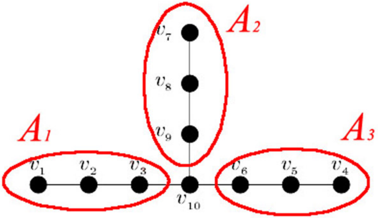
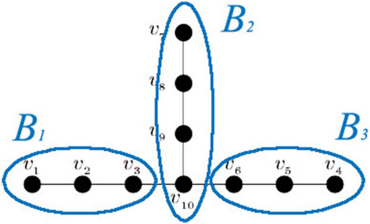


$$\iota_3^m(G) = \frac{1}{3}\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) = \frac{1}{3}$$

A minimal example

(All edge and vertex weights are equal to 1.)



$$\tilde{\iota}_3^m(G) = \frac{1}{3}\left(\frac{1}{3} + \frac{2}{4} + \frac{1}{3}\right) = \frac{14}{24}$$

Computational hardness of the isoperimetry problem

Known Complexity Results

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[JAVADI, D. 2010]

IPP_k is *NP*-complete for unweighted simple graphs.

IPP^m is *NP*-complete for weighted trees.

IPP^M is polynomial (actually linear) time solvable for weighted trees (even with potentials)!.

Known Complexity Results

[JAVADI, D. 2010]

There exists an algorithm that computes ι_k^m for every weighted tree, in time $O(n^{\lfloor (3k-3)/2 \rfloor})$.

[JAVADI, SHARIATRAZAVI, D. 2011]

There exists an algorithm that given a weighted tree with rational weights (and potentials!) on n vertices and an integer k, computes ι_k^M and a minimizer in $(n \log n)$-time.

Question!

- What can we say about the strong NP-completeness of IPP^m?
- Can you find a fast algorithm to compute ι_k^m as well as a minimizer at the same time for weighted trees?

Euclidean $\|.\|_2$ setting: eigenstructure of the Laplacian

A second step for approximation

Considering $\iota_k = \gamma_k$ one may be curious about the following,

Question

Is it possible to approximate $\frac{\|\nabla f\|_{1,c}}{\|f\|_{1,w}}$ by the normalized energy-form $\frac{\|\nabla f\|_{2,c}}{\|f\|_{2,w}}$?

An answer

The miracle of Laplacian provides an affirmative answer to this question!

The symmetrization technique

Divergence as the adjoint of the gradient is defined as

$$(\nabla^* f)(x) \stackrel{\text{def}}{=} \frac{1}{w(x)} \left(\sum_{e:(e^+=x)} c(e) f(e) - \sum_{e:(e^-=x)} c(e) f(e) \right).$$

Then the Laplacian is defined as

$$(Lf)(x) \stackrel{\text{def}}{=} (\nabla^* \nabla f)(x) = \frac{1}{w(x)} \sum_{y:(x\sim y)} c(xy)(f(x) - f(y)).$$

i.e. $L = W^{-1}(D - C)$.

And we have the Green formula as

$$\|\nabla f\|_{2,c}^2 = \langle \nabla f, \nabla f \rangle_c = \langle \nabla^* \nabla f, f \rangle_w = \langle Lf, f \rangle_w.$$

Approximating eigenvalues using energy-forms

Courant-Fischer-Weyl min-max principle:

Let

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n,$$

be the eigenvalues of L. Then, for any $0 \leq k < n$,

$$\lambda_k = \min_{W \in \mathcal{W}_k} \max_{0 \neq f \in W} \left\{ \frac{\langle Lf, f \rangle_w}{\|f\|_{2,w}^2} \right\} = \max_{W \in \mathcal{W}_{k-1}^{\perp}} \min_{0 \neq f \in W} \left\{ \frac{\langle Lf, f \rangle_w}{\|f\|_{2,w}^2} \right\},$$

in which

$$\mathcal{W}_k \stackrel{\text{def}}{=} \{ W \leq L^2(w) \mid \dim(W) \geq k \},$$

$$\mathcal{W}_k^{\perp} \stackrel{\text{def}}{=} \{ W \leq L^2(w) \mid \dim(W^{\perp}) \leq k \}.$$

Approximating eigenvalues using energy-forms

Ky-Fan-Wielandt's min-max principle:

Let

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n,$$

be the eigenvalues of L. Then, for any $0 \leq k < n$,

$$\overline{\lambda_k} \stackrel{\text{def}}{=} \frac{1}{k}\sum_{i=1}^{k}\lambda_i = \frac{1}{k} \min_{U\in\mathcal{M}_{n\times k}(U^*U=I_k)} tr(U^*LU)$$

$$= \frac{1}{k} \min_{\{f_i\}_{i=1}^{k}\text{(orthonormal)}} \sum_{i=1}^{k} \langle Lf_i, f_i\rangle_w$$

$$= \frac{1}{k} \min_{\{f_i\}_{i=1}^{k}\text{(orthogonal)}} \sum_{i=1}^{k} \frac{\langle Lf_i, f_i\rangle_w}{\|f\|_{2,w}^2}.$$

Some classical consequences!

$|L| \stackrel{\text{def}}{=} \max_x L(x,x) = \max_x \frac{deg(x)}{w(x)}$ is the normalized maximum degree.

A couple of basic norm inequalities

- $\|\nabla f\|_{1,c} \leq \|c\|^{\frac{1}{2}} \|\nabla f\|_{2,c}$
- $\frac{\|\nabla f^2\|_{1,c}}{\|f^2\|_{1,w}} \leq \sqrt{2|L|} \frac{\|\nabla f\|_{2,c}}{\|f\|_{2,w}}$.

Now, using the min-max principles one may easily verify that,

$$\frac{1}{2}\lambda_k \leq \iota_k^M \quad \text{and} \quad \overline{\lambda}_k \leq \iota_k^m.$$

The miracle of determinants

Determinants makes it possible to compute the eigenstructure of a matrix effectively in polynomial time!

Cheeger's inequality

Therefore, the following inequality can be considered as an effective approximation of the isoperimetric constant $\iota_2^M = \tilde{\iota}_2^M$.

Classical Cheeger's inequality 1969 (this version [ALON 1984] and [LAWLER, SOKAL 1988])

$$\frac{\lambda_2}{2} \leq \iota_2^M \leq \sqrt{2|L|\lambda_2}$$

(For an improved version (factor 2 in rhs removed!) with a different proof see [MONTENEGRO AND TETALI 2006].)

Question!

Can we similarly approximate higher isoperimetric constants for each $k > 2$?

Clustering And More

Isoperimetry: a global picture

Data

A set V along with a measure $w : V \to \mathbb{R}^+$ and a shortest path metric whose data is given by edge weights of a graph structure G as $c : E \longrightarrow \mathbb{R}^+$.

(Gromov-Milman:The whole story is told in the universe of metric measure spaces and what we are going to do is to compare the metric and the measure appropriately!)

The maps

One may define $\mathcal{I}_k^M : \mathcal{D}_k(V) \to \mathbb{R}^+$ as

$$\mathcal{I}_k^M(\{A_i\}_1^k) \stackrel{\text{def}}{=} \max_{1\leq i\leq k} \frac{c(A_i)}{w(A_i)}.$$

Also, define $\mathcal{I}_k^m$ similarly.

Isoperimetry: a global picture

Observations

- The metric appears within this comparison through its atomic presentations as edge-weights indirectly.
- These maps are too nonsmooth which make the minimization problems extremely hard.

Questions

- Is it possible to change the metric measure space as $\sigma : V \to X$ in a way that the minimum is almost preserved, leading to a simpler optimization problem in each case?

We are going to discuss some aspects of this!

Generalized Rayleigh quotient

Given $(V, \tilde{c}, \tilde{w})$ and a map $\sigma : V \to \mathbb{R}^n$, define

The Rayleigh quotient of σ w.r.t $A \subseteq V$

$$\mathcal{R}_A(\sigma) \stackrel{\text{def}}{=} \frac{\sum\limits_{x,y \in A} \tilde{c}(xy) \|\sigma(x) - \sigma(y)\|_2^2}{\sum\limits_{x \in A} \tilde{w}(x) \|\sigma(x)\|_2^2}$$

We will see that different choices of $\tilde{c}(xy)$ will give rise to many interesting results!

A special case

$\tilde{c}(xy) = w(x)w(y)$

Given $\sigma : V \to \mathbb{R}^n$, $w : V \to \mathbb{R}^+$ and $A \subseteq V$, let $\tilde{c}(xy) \stackrel{\text{def}}{=} w(x)w(y)$ and

$$\mathbf{m} \stackrel{\text{def}}{=} \frac{\sum_{x\in A} w(x)\sigma(x)}{\sum_{x\in A} w(x)}.$$

Then,

$$\sum_{x\in A} w(x)\|\sigma(x) - \mathbf{m}\|_2^2 = \frac{1}{2w(A)} \sum_{x,y\in A} w(x)w(y)\|\sigma(x) - \sigma(y)\|_2^2.$$

The case of unit sphere!

An important observation!
[LEE, OVEIS GHARAN, TREVISAN 2012]:

On the unite sphere i.e., when

$$\forall\, x \quad \|\sigma(x)\|_2 = 1,$$

we have,

$$\sum_{x\in A} w(x)\|\sigma(x) - \mathbf{m}\|_2^2 = \frac{1}{2w(A)}\sum_{x,y\in A} w(x)w(y)\|\sigma(x) - \sigma(y)\|_2^2$$

$$= \frac{\sum_{x,y\in A} w(x)w(y)\|\sigma(x) - \sigma(y)\|_2^2}{2\sum_{x\in A} w(x)\|\sigma(x)\|_2^2} = \frac{1}{2}\mathcal{R}_A(\sigma).$$

A symmetrization

Let's discuss an embedding of the graph structure $G(c,w)$ on a vertex set V of size n into $\mathbb{R}^n$ with Euclidean inner-product.

Define the positive semidefinite matrix $\Phi \overset{\text{def}}{=} (\lambda^* I - L)W^{-1}$ where λ^* is the greatest eigenvalue of the Laplacian L and W is the diagonal weight matrix. Then we have

$$\Phi(x,y) = \begin{cases} \frac{c(xy)}{w(x)w(y)} & x \sim y, \\ \frac{\lambda^*}{w(x)} - \frac{deg(x)}{w(x)^2} & x = y, \\ 0 & \text{otherwise.} \end{cases}$$

The embedding

Hence, since Φ is symmetric and positive semidefinite there is a factorization $\Phi = P^t P$ and one may define an embedding $\sigma : V \to \mathbb{R}^n$ for which $\sigma(x)$ is the xth column of P.

An embedding

Observation! ($\tilde{c}(xy) = w(x)w(y)$)

For this embedding σ based on $\Phi \stackrel{\text{def}}{=} (\lambda^* I - L)W^{-1} = P^t P$,

$$\sum_{x \in A} w(x)\|\sigma(x) - \mathbf{m}\|_2^2 = \sum_{x \in A} w(x)\|\sigma(x)\|_2^2 - \frac{1}{w(A)} \sum_{x \in A} w(x)^2 \|\sigma(x)\|_2^2$$

$$-\frac{1}{w(A)} \sum_{x \neq y \in A} w(x)w(y)\langle \sigma(x), \sigma(y)\rangle = \lambda^*(|A| - 1) - tr_A(L) + \frac{c(A)}{w(A)}.$$

Definition: k-means cost function (A^* is the residual of $\{A_i\}_1^k$.)

$$C_k^{\sigma,w}(\{A_i\}_1^k) \stackrel{\text{def}}{=} \frac{1}{k} \left(\sum_{i=1}^{k} \sum_{x \in A_i} w(x)\|\sigma(x) - \mathbf{m}_i\|_2^2 + \sum_{x \in A^*} w(x)\|\sigma(x)\|_2^2 \right).$$

Isoperimetry vs. k-means

Given $\sigma : V \to \mathbb{R}^n$ based on $\Phi \stackrel{\text{def}}{=} (\lambda^* I - L)W^{-1} = P^t P$ define $\mathcal{C}_k^{\sigma,w} : \mathcal{D}_k(V) \to \mathbb{R}^+$ as

Definition

$$\mu_k(\sigma, w) \stackrel{\text{def}}{=} \min_{\{A_i\}_1^k \in \mathcal{D}_k(V)} \mathcal{C}_k^{\sigma,w}(\{A_i\}_1^k),$$

$$\tilde{\mu}_k(\sigma, w) \stackrel{\text{def}}{=} \min_{\{A_i\}_1^k \in \mathcal{P}_k(V)} \mathcal{C}_k^{\sigma,w}(\{A_i\}_1^k).$$

Easy to verify!

- $\tilde{\mu}_k(\sigma, w) = (\frac{n}{k} - 1)\lambda^* - \frac{tr(L)}{k} + \tilde{\iota}_k^m$.
- $\mu_k(\sigma, w) = (\frac{n}{k} - 1)\lambda^* - \frac{tr(L)}{k} + \iota_k^m$.

Therefore, not only k-means is equivalent to normalized cut but its relaxed version (call it k-means with outliers) is equivalent to mean isoperimetry!

Mystery of [Ng, Jordan, Weiss 2002] algorithm!

[Ng, Jordan, Weiss 2002] algorithm

- Solve the generalized eigenvalue problem $(D - C)f = \lambda W f$ (eq. $Lf = W^{-1}(D - C)f = \lambda f$),
- Choose $f_1, \ldots f_k$ corresponding to the first k smallest eigenvalues, and define $F(x) \stackrel{\text{def}}{=} (\sqrt{w(x)} f_1(x), \ldots, \sqrt{w(x)} f_k(x))$,
- Define the normalized embedding as $\sigma : V \to \mathbb{S}^{k-1} \subset \mathbb{R}^k$ as $\sigma(x) \stackrel{\text{def}}{=} \frac{F(x)}{\|F(x)\|_2}$,
- Apply the k-means algorithm to σ.

Questions!

- Can we theoretically justify this algorithm?
- Can we use the idea of this algorithm to approximate the maximum version parameters ι^M and $\tilde{\iota}^M$?

Mystery of [Ng, Jordan, Weiss 2002] algorithm!

Remarks!

- $(D-C)f = \lambda W f \;\Leftrightarrow\; Lf = W^{-1}(D-C)f = \lambda f$
  $\Leftrightarrow\; W^{-1/2}(D-C)W^{-1/2}g = \lambda g, \quad (g = W^{1/2}f)$
- Let U be the matrix whose columns are the eigenfunctions f_x and also let Λ be the diagonal matrix of the corresponding eigenvalues. Then, $LW^{-1} = (WU)\Lambda(WU)^{-1}$.

Question!

What are the connections between the standard embedding in terms of P using $\Phi = P^tP = (\lambda^* I - L)W^{-1}$ and the NJW embedding in terms of U?

Mystery of [Ng, Jordan, Weiss 2002] algorithm!

An important experimental observation [Ng, Jordan, Weiss 2002]

If you have a good embedding $\sigma : V \to \mathbb{R}^d$, then projecting all vectors to the unit sphere (i.e. normalizing all vectors to have unit length) must still work!

We already know that on the unit sphere the k-means cost function coincides with the Rayleigh quotient, i.e.,

$$\sum_{x \in A} w(x) \left\| \frac{\sigma(x)}{\|\sigma(x)\|_2} - \mathbf{m} \right\|_2^2 = \frac{\sum_{x,y \in A} w(x)w(y) \left\| \frac{\sigma(x)}{\|\sigma(x)\|_2} - \frac{\sigma(y)}{\|\sigma(y)\|_2} \right\|_2^2}{2 \sum_{x \in A} w(x) \left\| \frac{\sigma(x)}{\|\sigma(x)\|_2} \right\|_2^2}$$

$$= \frac{1}{2} \mathcal{R}_A(\sigma).$$

[LEE, OVEIS GHARAN, TREVISAN 2012]

This gives rise to the following important observation:

[LEE, OVEIS GHARAN, TREVISAN 2012]

Study the distance

$$\left\| \frac{\sigma(x)}{\|\sigma(x)\|_2} - \frac{\sigma(y)}{\|\sigma(y)\|_2} \right\|_2$$

and the generalized Rayleigh quotient in $\mathbb{R}^d$.

We will come back to this subject!

Metric Embedding And Approximations

Approximating uniform sparsest cut

Isoperimetric (Cheeger) inequalities again

Since the eigenstructure of a matrix is polynomially computable any Cheeger-type inequality provides a polynomial $O(\frac{1}{\sqrt{\iota}})$ approximation algorithm!

Another fascinating aspect of the subject is that graph partitioning (in a general sense) is among problems which are computationally not quite well-understood and resists approximations!

In this talk we concentrate on the case of uniform sparsest cut (USC for short) since it is

- among the most simple cases
- is still intriguing!
- has influenced the main ideas and best results so far.

Approximating uniform sparsest cut

Given a simple graph $G(V,E)$ (i.e. all weights are equal to 1), recall the definition of the uniform sparsest cut as

$$\Phi^* \stackrel{\text{def}}{=} \min_{A\subseteq V(G)} \frac{|E(A,A^c)|}{|A||A^c|}$$

$$= \min_{A\subseteq V(G)} \frac{1}{n}\left(\frac{|E(A,A^c)|}{|A|} + \frac{|E(A,A^c)}{|A^c|}\right) = \frac{2}{n}\iota_2^m(G)$$

[Papadimitrou 1997]

NCP$_k$ (for both max and mean versions) is *NP*-complete for unweighted (simple) graphs.

What about approximations?!

An L_1 relaxation

L_1 relaxation of the USC

$$\Phi^* \stackrel{\text{def}}{=} \min_{A \subseteq V(G)} \frac{|E(A, A^c)|}{|A||A^c|} = \min_{\{f_x \in L_1(\Omega)\,:\, x \in V\}} \frac{\sum_{xy \in E} \|f_x - f_y\|_1}{\sum_{x,y \in V} \|f_x - f_y\|_1}$$

Here Ω can be a finite set or $[0, 1]$ with the Lebesgue measure.

LP Approximation (first step!) (also see [Leighton, Rao 1999])

Using homogeneity relax to the case of semimetrics as

$$M^* \stackrel{\text{def}}{=} \min \sum_{xy \in E} d_{xy}$$

$$\text{s.t. } \sum_{x,y \in V} d_{xy} = 1, \quad \forall\, x, y \;\; d_{xx} = 0, \;\; d_{xy} \geq 0, \;\; d_{xy} = d_{yx},$$

$$\forall\, x, y, z \;\; d_{xy} \leq d_{xz} + d_{zy}.$$

An L_1 relaxation

LP Approximation (first step!)

Clearly $M^* \leq \Phi^*$ and can be computed in polynomial time!

What is the approximation factor?

Clearly we have to somehow relate the solution of the LP relaxation d^* to the L_1 distance function, i.e., we must seek a relation such as

$$\forall\, y, z \quad d^* \lesssim \|f_y - f_z\|_1 \lesssim C d^*,$$

for some set $\{f_x \in L_1(\Omega) \ : \ x \in V\}$ and a constant C that turns out to be the approximation factor!

Metric embeddings

Embeddings and distortion

A metric space (X, d_X) is said to bi-Lipschitz embed with distortion $C \geq 1$ into a metric space (Y, d_Y) if there exists a mapping $\sigma : X \to Y$ and a scaling factor $s \geq 0$ such that

$$\forall\, y, z \in X \quad s\, d_X(y,z) \leq d_Y(\sigma(y), \sigma(z)) \leq C\, s\, d_X(y,z)$$

Also $c_Y(X) \stackrel{\text{def}}{=} \inf C$ where the infimum is taken over all bi-Lipschitz embeddings of X into Y with distortion C.

$L_1(\Omega, \mu)$ has a nice metric structure!

The space $L_1(\Omega, \mu)$ with the metric $\sqrt{\|f - g\|_{L_1(\Omega,\mu)}}$ is isometric to the Hilbert space $L_2(\Omega \times \mathbb{R}, \mu \times \lambda)$ where λ is the Lebesgue measure.

Bourgain's embedding theorem

[Bourgain 1985]

For every finite metric space (X, d_X) with n points we have $c_2(X) \lesssim \log n$.

Note: The embedding is effectively constructible.

[Dvoretzky 1961]

For every infinite dimensional Banach space Y and every $n \in \mathbb{N}$ we have $c_Y(\ell_2^n) = 1$.

Corollary: For every finite metric space (X, d_X) with n points we have $c_Y(X) \le c_2(X) \lesssim \log n$.

[Linial, London, Rabinovich 1995]

Even the weaker inequality $c_1(X) \lesssim \log n$ is asymptotically sharp!

Note: Leaves no hope for better LP-based approximation c_1!

SDP approximation (second step!)

have to add more feasible constraints!

Metrics of negative type

A metric space (X, d_X) is said to be of negative type if the metric space $(X, \sqrt{d_X})$ admits an isometric embedding into a Hilbert space.

Note: See [SCHOENBERG 1938] for the terminology.

Note: $L_1(\Omega, \mu)$ is of negative type!

Idea: [GOEMANS AND LINIAL 1997,2002]

$$M^{**} \stackrel{\text{def}}{=} \min \sum_{xy \in E} d_{xy}$$

s.t. $\sum_{x,y \in V} d_{xy} = 1$, and d is a semimetric of negative type, i.e.,

$\forall\, x, y, z \;\; d_{xx} = 0, \;\; d_{xy} \geq 0, \;\; d_{xy} = d_{yx}, \;\; d_{xy} \leq d_{xz} + d_{zy},$

$\exists\, A$ symmetric positive semidefinite s.t. $d_{xy} = a_{xx} + a_{yy} - 2a_{xy}$.

SDP approximation (second step!)

Clearly $M^* \leq M^{**} \leq \Phi^*$.

It works! [Arora, Rao, Vazirani 2004]

$\frac{\Phi^*}{M^{**}} \lesssim \sqrt{\log n}$. No simple proof yet!

Note: see [Naor, Rabani, Sinclair 2005] for a more structured proof.

A lower bound! [Devanur, Khot, Saket, Vishnoi 2006]

$(\log \log n) \lesssim \frac{\Phi^*}{M^{**}} \lesssim \sqrt{\log n}$.

Note: Also see [Naor ICM2010] for the history.

An inapproximability result

[Khot and Vishnoi 2005]
[Chawla, Krauthgamer, Kumar, Rabani, Sivakumar 2006]

If there exists a polynomial constant factor approximation algorithm for the (general) sparsest cut problem then the Unique Games Conjecture is not true!

Note: Also see [Khot ICM2010] for more on this subject.

Higher Isoperimetric Inequalities

GENERAL REFERENCES FOR FURTHER READING:

- Chung, Fan; Grigor'yan, Alexander; Yau, Shing-Tung, *Higher eigenvalues and isoperimetric inequalities on Riemannian manifolds and graphs*, Comm. Anal. Geom. **8** (2000), no. 5, 969-1026.
- Daneshgar, Amir; Hajiabolhassan, Hossein; Javadi, Ramin, *On the isoperimetric spectrum of graphs and its approximations*, J. Combin. Theory Ser. B **100** (2010), no. 4, 390-412.
- Daneshgar, Amir; Javadi, Ramin; Miclo, Laurent, *On nodal domains and higher-order Cheeger inequalities of finite reversible Markov processes*, Stochastic Process. Appl. **122** (2012), no. 4, 1748-1776.
- Lee, James R.; Oveis Gharan, Shayan; Trevisan, Luca, *Multi-way spectral partitioning and higher-order Cheeger inequalities*, STOC'12-Proceedings of the 2012 ACM Symposium on Theory of Computing, 1117-1130, ACM, New York, 2012. `http://arxiv.org/abs/1111.1055`
- Tanaka, Mamoru, *Multi-way expansion constants and partitions of a graph*, `http://arxiv.org/abs/1112.3434`.

Basic objective

Our basic objective is the following:

Higher Cheeger (isoperimetric) inequalities

- $\frac{1}{2}\lambda_k \leq \iota_k^M \leq \sqrt{\tau(k)\ |L|\ \lambda_k}$ [LEE, OVEIS GHARAN, TREVISAN 2012]
- $\overline{\lambda}_k \leq \iota_k^m \leq \sqrt{\xi(k)\ |L|\ \overline{\lambda}_k}$ STILL OPEN!

in which $\tau(k)$ and $\xi(k)$ are constants only depending on k.
The conjecture/theorem is deeply related to the geometry of metric-measure spaces and theory of computation.

Note that lower bounds are easy! In what follows we concentrate on the existing techniques and results in relation to the proof of the max-version's upper bound.

Basic objective

Our basic objective is to explain the following:

Two different approaches

There are at least two different approaches leading to higher Cheeger-type inequalities:

- Miclo's approach: Define intermediate parameters depending on subdomains, then handle singularities by moving to a continuous setting and proving the inequality for the generic case.
- Lee-Oveis Gharan-Trevisan's approach: Use embedding and dimension reduction (as in NJW algorithm) and then try to construct a suitable test function using the first k eigenvectors.

Both of these approaches somehow depend on localizing the problem on subdomains. Hence, we first review the basics of this approach.

Localization and Dirichlet eigenvalues

A localization procedure: discrete case

- Let $A \subset V$ be a subset with the vertex-boundary $\delta(A)$, and L be the Laplacian operator. Then a pair $(\lambda, f \neq 0)$ satisfies the Dirichlet boundary problem

$$\begin{cases} (Lf)(x) = \lambda f(x) & \forall\, x \in A, \\ f(x) = 0 & \forall\, x \in \delta(A), \end{cases}$$

  if and only if $(\lambda, f \neq 0)$ is an eigenpair of $L|_A$.

- Also, variational principles are generally valid by adding the restriction that all functions must be equal to zero outside A, specially

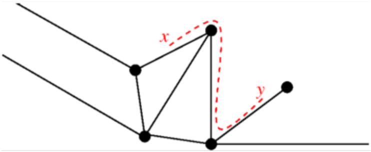
$$\lambda_1(A) = \min_{0 \neq f = 0 \ on \ A^c} \left\{ \frac{\langle Lf, f \rangle_w}{\|f\|_{2,w}^2} \right\}.$$

The continuous setting: quantum graphs

Given a weighted graph $G = (V, E, w, c)$

Metric graph: Each edge $e = xy \in E$ is replaced by a segment $[x, y]$ of length $1/c(xy)$.

Quantum graph: A metric graph along with a natural Laplacian operator L.



We use notations $\mathcal{G}, f, \mathcal{A}$ for metric setting in contrast to G, f, A for discrete setting.

Measures on a quantum graph

$\rho_{x,y}$: Natural Lebegue measure on $[x, y]$.

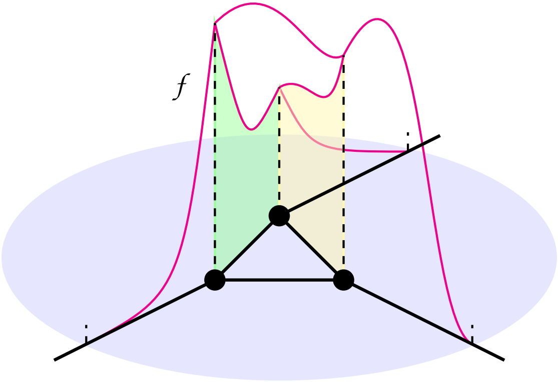
$$\rho := \sum_{xy \in E} \rho_{x,y}.$$

We have $\rho([x, y]) = 1/c(xy)$, so we call ρ the resistance measure.

Also define an atomic measures on $\mathcal{G}$ as follows

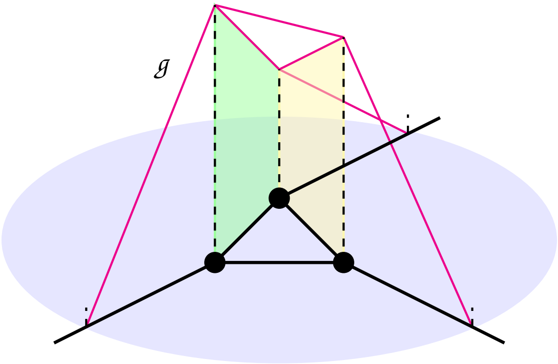
$$\omega := \sum_{x \in V} \omega(x)\ \delta_x.$$

$\mathscr{F}_0(\mathcal{G})$: Set of all absolutely continuous real functions on $\mathcal{G}$.



$$\forall f \in \mathscr{F}_0(\mathcal{G}), \quad \mathcal{E}(f) := \int (f')^2 \, d\rho.$$

$\mathscr{F}_0(\mathcal{G})$: Set of all absolutely continuous real functions on $\mathcal{G}$.



$$\mathcal{E}(g) = \sum_{xy \in E} c(xy)\, |g(x) - g(y)|^2.$$

Dirichlet eigenvalues

For $\mathcal{A} \subset \mathcal{G}$, the principal Dirichlet eigenvalue is defined as,

$$\lambda_1(\mathcal{A}) := \inf_{\substack{f \in \mathscr{F}_0(\mathcal{A}) \\ \|f\|_{2,\omega}^2 \neq 0}} \frac{\mathcal{E}(f)}{\|f\|_{2,\omega}^2}.$$

Dirichlet eigenvalues

For $\mathcal{A} \subset \mathcal{G}$, the principal Dirichlet eigenvalue is defined as,

$$\lambda_1(\mathcal{A}) := \inf_{\substack{f \in \mathscr{F}_0(\mathcal{A}) \\ \|f\|_{2,\omega}^2 \neq 0}} \frac{\mathcal{E}(f)}{\|f\|_{2,\omega}^2}.$$

We denote the minimizer by $f_{\mathcal{A}}$ which is unique up to a factor, provided $\mathcal{A}$ is connected.

$$\mathscr{D}_1(\mathcal{G}) := \{\mathcal{A} \subset \mathcal{G} : \ \mathcal{A} \text{ is open and connected }, \mathcal{A} \cap V \neq 0\},$$

$$\mathscr{D}_k(\mathcal{G}) := \{\{\mathcal{A}_1, \ldots, \mathcal{A}_k\} : \ \mathcal{A}_i \in \mathscr{D}_1(\mathcal{G}), \ \mathcal{A}_i \cap \mathcal{A}_j = \emptyset\}.$$

Weights and the Laplacian in the continuous setting

Define $\tilde{c} : V \times \mathcal{G} \to \mathbb{R}$ as

$$\tilde{c}(x,a) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{\rho([x,a])} & a \neq x \;\&\; \exists\, xy \in E : a \in [x,y], \\ 0 & \text{otherwise.} \end{cases}$$

Now, for any $\mathcal{A} \in \mathscr{D}_1$ define $A \stackrel{\text{def}}{=} V \cap \mathcal{A}$ and

$$\forall\, x,y \in A \quad \hat{L}_{\mathcal{A}}(x,y) \stackrel{\text{def}}{=} \begin{cases} L(x,y) & x \neq y, \\ \frac{1}{w(x)} \left(\sum_{a \in A \cap \partial \mathcal{A}} \tilde{c}(x,a) \right) & x = y. \end{cases}$$

A localization procedure: continuous case

- For any $\mathcal{A} \in \mathscr{D}_1$ there is a unique and positive function $f_{\mathcal{A}}$ that attains the minimum

$$\inf_{\substack{f \in \mathscr{F}_0(\mathcal{A}) \\ \|f\|_{2,\omega}^2 \neq 0}} \frac{\mathcal{E}(f)}{\|f\|_{2,\omega}^2} = \lambda_1(\mathcal{A})$$

with $\|f\|_{2,\omega}^2 = 1$. Also, $\lambda_1(\mathcal{A})$ is the smallest eigenvalue of $\hat{L}_{\mathcal{A}}$ with $\hat{L}_{\mathcal{A}} f_{\mathcal{A}} = \lambda_1(\mathcal{A}) f_{\mathcal{A}}$ where $f_{\mathcal{A}} = f_{\mathcal{A}}|_A$. Moreover, $f_{\mathcal{A}}$ is the affine extension of $f_{\mathcal{A}}$ on $\mathcal{A}$.

- λ_1 is strictly decreasing on $\mathscr{D}_1$, i.e.,

$$\mathcal{A}, \mathcal{B} \in \mathscr{D}_1(\mathcal{G}) \ \&\ \mathcal{A} \subsetneq \mathcal{B} \ \Rightarrow \ \lambda_1(\mathcal{B}) < \lambda_1(\mathcal{A}).$$

Dirichlet connectivity spectrum

For any quantum graph $(\mathcal{G}, L)$ define,

Dirichlet connectivity constants [Miclo 2007]

$$\underline{\Lambda}_k \overset{\text{def}}{=} \inf_{\{\mathcal{A}_i\}_1^k \in \mathcal{D}_k(\mathcal{G})} \left(\max_{1 \leq j \leq k} \lambda_1(\mathcal{A}_j) \right).$$

Using the variational principle for eigenvalues and norm inequalities one can prove that

- For any quantum graph $(\mathcal{G}, L)$ and any k we have $\lambda_k \leq \underline{\Lambda}_k$.
- $\frac{1}{2|L|}(\iota_k^M)^2 \leq \underline{\Lambda}_k$.

Miclo's conjecture

Miclo's conjecture [Miclo 2007]

For each k, there exists a universal constant $\tau(k) > 0$ such that for all weighted graphs,

$$\underline{\Lambda}_k \leq \frac{1}{2}\tau(k)\,\lambda_k.$$

Note that,

Miclo implies Cheeger

Assuming that Miclo's conjecture is true,

$$\frac{1}{|L|}(\iota_k^M)^2 \leq \underline{\Lambda}_k \leq \tau(k)\,\lambda_k$$

Which is the hard side of Cheeger's inequality!

Nodal domains

Miclo's approach

Hence, in Miclo's approach one should study the minimizers of the map

$$\mathscr{D}_k(\mathcal{G}) \ni \{\mathcal{A}_i\}_1^k \longmapsto \max_{1\leq j\leq k} \lambda_1(\mathcal{A}_j)$$

and compare them to λ_k.

[Javadi, Miclo, D. 2012]

To prove Miclo's and Cheeger's conjectures it is sufficient to prove them for connected graphs.

A natural question!

Is it possible to find a subset $\mathcal{A}$ for which $\lambda_1(\mathcal{A}) = \lambda_k$?

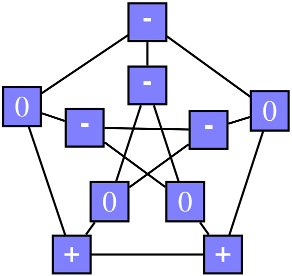
The answer is Yes and this is the main motivation to study nodal domains! The ideas leading to this concept are quite old and goes back to Hilbert and Courant (1943).

(strong) Nodal domains: discrete case

Definition

For a graph (G, V), a strong positive nodal domain of a function f is a connected component of the set $\{x \in V \ : \ f(x) > 0\}$.

Strong negative nodal domains are defined similarly. The total number of strong nodal domains are denoted by $\mathfrak{S}$.



The sign pattern of an eigenfunction corresponding to λ_2.

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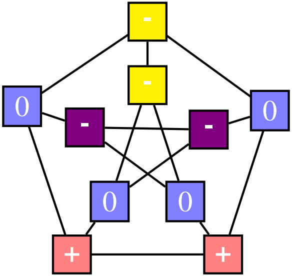

The sign pattern of an eigenfunction corresponding to λ_2.

Nodal domains: continuous case

Definition

For a metric graph $\mathcal{G}$,

- A nodal domain of a function f is a connected component of the set $\mathcal{G} - \{x \in \mathcal{G} \ : \ f(x) = 0\}$.
- A continuous nodal domain of a discrete function f defined on V is a nodal domain of the affine extension of f on $\mathcal{G}$.
- The number of continuous nodal domain of a discrete function f is denoted by $\mathfrak{N}$.

Continuous nodal domains are natural!

Let $f \neq 0$ be an eigenfunction of L for the eigenvalue λ and let $\mathcal{A}$ be one of its continuous nodal domains. Then $\lambda_1(\mathcal{A}) = \lambda$ and f is proportional to the minimizer $f_{\mathcal{A}}$ on $A = \mathcal{A} \cap V$.

A strategy that fails!

Strategy!

For every given graph G and any integer $k \leq |V|$, if one is able to find a function f with $\mathfrak{N}(f) \geq k$ then Miclo's conjecture is positively solved!

Hence the number of nodal domains is an extremely important subject to be studied!

Simple observations

If f_1 and f_2 are, respectively, the first and the second eigenfunctions of the Laplacian of a connected graph G, by Perron-Frobenius theorem we have

$$\mathfrak{N}(f_1) = 1, \quad \mathfrak{N}(f_2) \geq 2.$$

This is the verification of conjectures for the case $k = 2$!

A strategy that fails!

What about $k > 2$?

On the number of nodal domains [Courant-Hilbert 1943, many others!]

If f is an eigenfunction of the kth eigenvalue λ_k of L with multiplicity r, then $\mathfrak{S}(f) \leq k + r - 1$.

On the number of nodal domains [Javadi, D. 2011]

Given a graph G with a Laplacian L whose cycle-space is d dimensional, if $d \leq k \leq |V|$ and f is a nowherezero eigenfunction of the kth eigenvalue λ_k, then $k - d \leq \mathfrak{S}(f) \leq k$.

The result has first appeared in [Berkolaiko 2008] for simple eigenvalues.

Trees and cycles

Nodal domains of trees

Since a tree T has no cycle we have $\mathfrak{S}(f) = k$ for any nowherezero eigenfunction. In this case we have a slightly better result.

[Biyikoğlu 2003]

If f is a nowherezero eigenfunction of the kth eigenvalue of the Laplacian matrix of a tree, then the eigenvalue is simple and $\mathfrak{S}(f) = k$.

Hence, Miclo's and consequently Cheegre's higher inequalities are valid on a tree when we have a nowherezero eigenfunction of a simple eigenvalue. Hence, we have to handle two problems:

Two majour problems

- Study the structure of nodal domains of nowherezero eigenfunctions (and generalize if possible!).
- Handle the case of eigenvalues with multiplicities.

Handy subpartitions

A k-subpartition $(\mathcal{A}_1, \ldots, \mathcal{A}_k) \in \mathscr{D}_k(\mathcal{G})$ is said to be handy, if

$$\forall i \neq j, \quad a \in \partial\mathcal{A}_i \cap \partial\mathcal{A}_j \Rightarrow \deg(a) \leq 2.$$

We are not going to delve into the details, but let's just point out that:

Comments!

- L is said to be handy if any eigenvalue with multiplicity m admits m independent handy eigenfunctions. For instance, if there exists a complete set of orthogonal eigenfunctions that do not vanish on V, then all eigenvalues are simple and L is handy!
- Fix a weight function w and a simple graph G. Consider the open and convex set of w-selfadjoint operators $\mathcal{L}(G, w)$ whose graph is G with the pointwise topology. We say that a property is generically true if it is true for a dense subset of $\mathcal{L}(G, w)$.

What we know!

[Javadi, Miclo, D. 2012]

Given a handy k-subpartition $\mathcal{A} \in \mathscr{D}_k(\mathcal{G})$, then it corresponds to the nodal domains of an eigenfunction of L if and only if it is uniform, rectifiable and bipartite.

[Javadi, Miclo, D. 2012]

Let $\mathcal{A} \in \mathscr{D}_k(\mathcal{G})$ be a minimizing subpartition for $\underline{\Lambda}_k$. If $\mathcal{A}$ is handy, then $\mathcal{A}$ is a uniform and rectifiable partition in $\mathscr{P}_k(\mathcal{G})$.

A conjecture

[Miclo 2007]

For any tree and any k we have $\underline{\Lambda}_k = \lambda_k$.
Hence, Miclo's and consequently Cheegre's higher inequalities are valid for trees with the universal constant $\tau(k) = 2$.

Conjecture [Javadi, Miclo, D. 2012]

The following properties are generically true:

- Any minimizing subpartition for $\underline{\Lambda}_k$ is handy.
- Any generator $L \in \mathcal{L}(G, w)$ is handy.

Correctness of these conjectures imply the effectiveness of Miclo's approach!

Cycles

The case of cycles is more intricate!

We are happy that all subpartitions are handy!

One can eventually prove:

[Javadi, Miclo, D. 2012]

When G is a cycle, we have,

$$\underline{\Lambda}_k \leq \begin{cases} \lambda_k & \text{, if } k = 1 \text{ or } k \text{ is even} \\ 24\ \lambda_k & \text{, if } k \geq 3 \text{ is odd.} \end{cases}$$

Hence, Miclo's and consequently Cheegre's higher inequalities are valid for cycles with the universal constant $\tau(k) = 48$.

A Sketch of [LEE, OVEIS GHARAN, TREVISAN 2012]'S Proof

Generalized Rayleigh quotient

Let $\sigma : V \supseteq A \to \mathbb{R}^d$ and definition

The Rayleigh quotient of σ

$$\sigma^{(2)} : A \to \mathbb{R}, \quad \sigma^{(2)}(x) \stackrel{\text{def}}{=} \|\sigma(x)\|_2^2 \stackrel{\text{def}}{=} (|\sigma|(x))^2,$$

$$\nabla\sigma : A \times A \to \mathbb{R}^d, \quad \nabla\sigma(x,y) \stackrel{\text{def}}{=} \sigma(y) - \sigma(x),$$

$$\mathcal{R}_A(\sigma) \stackrel{\text{def}}{=} \frac{\sum\limits_{x,y\in A} \tilde{c}(xy)\|\sigma(x) - \sigma(y)\|_2^2}{\sum\limits_{x\in A} \tilde{w}(x)\|\sigma(x)\|_2^2} = \frac{\||\nabla\sigma|\|_{2,\tilde{c}}^2}{\||\sigma|\|_{2,\tilde{w}}^2},$$

A Generalized Inequality

Some basic facts

- $\exists\, A \subseteq support(\sigma)$,

$$\frac{\tilde{c}(A)}{\tilde{w}(A)} \le \frac{\|\nabla\sigma^{(2)}\|_{1,\tilde{c}}}{\|\sigma^{(2)}\|_{1,\tilde{w}}} \le \sqrt{2|L|}\frac{\||\nabla\sigma|\|_{2,\tilde{c}}}{\||\sigma|\|_{2,\tilde{w}}} = \sqrt{2|L|\mathcal{R}_A(\sigma)},$$

- There exists a coordinate $i \in \{1, \ldots, d\}$ such that for the projection $\sigma_i : A \to \mathbb{R}$ we have

$$\mathcal{R}_A(\sigma_i) \le \mathcal{R}_A(\sigma).$$

Hence, the basic strategy is to control the generalized Rayleigh quotient!

Radial projection distance

[Lee, Oveis Gharan, Trevisan 2012]

Given an embedding $\sigma : V \to \mathbb{R}^d$ study the distance

$$d_\sigma \stackrel{\text{def}}{=} \left\| \frac{\sigma(x)}{\|\sigma(x)\|_2} - \frac{\sigma(y)}{\|\sigma(y)\|_2} \right\|_2$$

and the induced shortest-path pesudo-metric $\hat{d}_\sigma$.

Radial projection distance

[Lee, Oveis Gharan, Trevisan 2012]

Let σ be an $\ell^2(w)$-orthonormal k-dimensional embedding. Then

- $\mathcal{E}_\sigma(V) \stackrel{\text{def}}{=} \sum_{x \in V} w(x) \|\sigma(x)\|_2^2 = k$,
- For any unit vector v we have $\sum_{x \in V} w(x) \langle v, \sigma(x) \rangle^2 = 1$,
- For any $A \subseteq V$ with $diam(A, d_\sigma) \leq \Delta$ we have

$$\mathcal{E}_\sigma(A) \stackrel{\text{def}}{=} \sum_{x \in A} w(x) \|\sigma(x)\|_2^2 \leq \frac{1}{1 - \Delta^2}.$$

Main idea is to control the induced energy on a subset by its diameter with respect to d_σ!

Reduction to partitioning!

[Lee, Oveis Gharan, Trevisan 2012]

Let σ be an $\ell^2(w)$-orthonormal k-dimensional embedding, $\tilde{c} = c$ and $\tilde{w} = w$. Then if for some $\beta, \delta > 0$ and $r \in \mathbb{N}$ there exists r disjoint subsets $A_1, \ldots, A_r$ such that $\hat{d}_\sigma(A_i, A_j) \geq \beta$ for $i \neq j$ and

$$\forall\, 1 \leq i \leq r \quad \mathcal{E}_\sigma(A_i) \geq \delta\, \mathcal{E}_\sigma(V),$$

then there exists disjointly supported real-functions $\psi_1, \ldots, \psi_r$ such that

$$\forall\, 1 \leq i \leq r \quad \mathcal{R}_V(\psi_i) \leq \frac{2}{\delta(r - i + 1)} \left(1 + \frac{4}{\beta}\right)^2 \mathcal{R}_V(\sigma).$$

Hence the problem is reduced to partitioning in the pesudo-metric space $(V, \hat{d}_\sigma)$!

The sketch of proof

[Lee, Oveis Gharan, Trevisan 2012]

- Let σ be the $\ell^2(w)$-orthonormal k-dimensional embedding produced by the eigenstructure of the Laplacian L, $\tilde{c} = c$ and $\tilde{w} = w$.
- Use standard results in random partition theory to obtain suitable subsets $A_1, \ldots, A_r$ in $(V, \hat{d}_\sigma)$.
- Choose the parameters appropriately to get $\mathcal{R}_V(\psi_i) \leq O(k^6)\ \lambda_k$.

[Lee, Oveis Gharan, Trevisan 2012] use a more detailed analysis to show that

$$\mathcal{R}_V(\psi_i) \leq O(k^2)\ \lambda_k.$$

Hence, Miclo's and Cheeger's inequalities are valid in the *Max* i.e. $\|.\|_\infty$ case!

A general setup

Given a weighted graph $G = (V, E, c, w)$, define the maps $\mathcal{I}_k^p : \mathcal{D}_k(V) \to \mathbb{R}^+$ as

$$\mathcal{I}_k^p(\{A_i\}_1^k) \overset{\text{def}}{=} \|\left(\frac{c(A_i)}{w(A_i)}\right)_{i=1}^k\|_p$$

and study their extremal values $\min \mathcal{I}_k^p(\{A_i\}_1^k)$ from a computational point of view. Specially compare these values with $\|(\lambda_i)_{i=1}^k\|_p$ where λ_i's are the eigenvalues of a natural Laplacian on G.

In particular, determine those graphs for which the extremal value can be attained on partitions.

It seems that everything is about estimates of connectedness and density!!!

Thank you!

Comments and Criticisms are Welcomed

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