Do Graphs Admit Topological Field Theories?

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1 The background and motivations

The contents of this talk is related to what I have dwelled upon and tried to digest during the past twenty years. My motivations for sharing my views with the audience at this moment are twofold. First, I feel that the combinatorial aspect of the subject has reached at a level that deserves notification and contains rich and feasible questions to be thought about and answered. Second, I have to confess that I have been astonished by the number of interested and knowledgable students participating in our former combinatorial gatherings and I feel that the subject and its problems can be quite interesting for the mature and developing community of our combinatorial society.

Also, I ought to apologize for not providing any reference in this extended abstract, which is essentially a consequence of what can be described as a process of packing an elephant in a matchbox! The subject I am hinting at is quite huge in literature, contains tons of written articles and can be measured as a large part of the current mathematical sciences being investigated. Hence, to be honest and fair I just suggest using a search-engine and the keywords appearing in what follows. I hope that you can find your own rout in this ocean. A short overview of mine is as follows:

1.1 Episode 1: a meta-mathematical theory of everything

This starts with the ambitious project of finding a *theory of everything* pursued by physicists with the help of mathematicians. This has also proved to have been quite fruitful for mathematics too as a source of nice structures and deep problems. What follows is a very concise and informal description of basics of *topological field theory*[1] viewed from a mathematical side. Naturally, the presentation has an emphasis on what we are going to talk about.

Of course, the beginning of the story (at least the mathematical parts) goes back to all mathematicians who used to think about modeling physical theories. This, at least for the modern mathematical ages, goes back to H. Poincaré and J. Von Neumann, however, our story mostly starts from M. Atiyah's formulation which was mainly based on G. Segal's formulation of conformal field theory. Strictly speaking, every physical theory deals with *space-times*, and in the modern setting we may just think of a *space* as an n-dimensional (nice enough!) manifold G without boundary, where the evolution of such a space in *time* is modeled by a cylinder-like object that can be modeled as an $(n+1)$-dimensional manifold $H = (G, G')$ with boundaries G and G' chosen appropriately (e.g with opposite orientations at boundaries). Such data is usually modeled in a category whose objects are spaces and its morphisms are the space-times called *cobordisms* equipped with *gluing* at their bases. If everything is modeled properly, then it turns out that this category, $\mathrm{Cob}(n)$, is structured enough and can be dealt with as a nice *monoidal category*. We do not talk about how

[1] The word *quantum* is deliberately avoided in this talk!

physical theories crop up in this setting by defining suitable *action functionals* for *fields* and finding their critical points, where this in it is most general form is not well-understood nowadays and is related to a mathematically sound theory of Feynman integrals which is under investigation.

To handle the complexity of this very hard problem, mathematicians have decided to forget about the geometric (i.e metric) properties of spaces and try to understand at least the topological properties at first. This is the approach that gives rise to the concept of an n-dimensional *topological field theory*, rigorously defined by Atiyah (based on the contributions of Segal and others) as a monoidal functor from the category $\mathrm{Cob}(n)$ to the category Vect_K of vector spaces (on K).

It turns out that this investigation has its own importance in the mathematical world. The details of these byproducts are quite beyond the scope of this talk and this speaker's knowledge, however, the importance of these results are related to contributions of Witten, Drinfel'd, Reshetikhin, Turaev, Kapranov, Voevodsky and Kontsevich as tips of some icebergs.

Also, being more mathematical, it is worth mentioning the abstract study of the theory that boils down to the theory of monoidal categories. This can also be used to model a meta-mathematical input-output theory of systems as well as what is already known for the dynamical case through mathematical physics and abstract field theories. These structures can be traced back to the almost 600-page letter of Grothendieck to Quillen (entitled *pursuing of stacks*) and seems to be one of the most important structures to be studied in mathematics of 21st century.

1.2 Episode 2: One-way functions and unique solutions

In the last section I talked about a very fundamental problem of human thinking which is *modeling natural phenomena*. In this section I want to touch upon another deep problem in mathematical sciences which is mostly related to one other fundamental problem of human history, namely *solving equations*. Needless to say, this problem is of a computational nature and we should look at it from a computational point of view. Fortunately, there has been a great boost in theory and results of *theory of computation* during the past thirty years and the subject has become one of the most active parts of computer science and mathematics in 21st century.

There are a number of connections between physics, topology and theory of computation (e.g. see M. Freedman's talk at ICM98), but what I am going to talk about in this section is from one other angle and is based on connections between the $NP \stackrel{?}{=} P$ problem and *one-way functions*.

It is well-known and not hard to see that the $NP \stackrel{?}{=} P$ problem and other similar problems as $NP - BPP \stackrel{?}{=} \emptyset$ are deeply related to finding functions that are easy to compute but hard to invert, known as *one-way functions*. In other words, finding a function $f : \Sigma^* \to \Sigma^*$ that is easy to compute (in a predefined acceptable way), but can not be inverted in the sense that finding an element in $f^{-1}(w)$ is not easy (in the same computational setup), is a very fundamental problem in theory of computation.

Informally, let (θ, x) be a set of data such that x satisfies θ in a predefined setup. As a couple of examples you may think of θ as some sort of *equation* and x as its *solution*. For instance, you may think of θ as a Boolean expression on n variables (given in a predefined standard form) and let x be an assignment of variables that makes θ true. The corresponding decision problem in the latter case is usually referred to as the SAT problem and is known to be NP-complete. Now, consider the function $\sigma : (\theta, x) \mapsto \theta$ that *forgets* the solution and ask *"Is this a one-way function?"* I should add that the mentioned setup is not as restricted as it may look because of the *dichotomy conjecture* for *constraint satisfaction problems*.

The existence problem for one-way functions is also deeply related to real world applications which are far from what a physicist might be interested in. As a matter of fact, a negative solution to this problem will rock the whole security of our communications (at least from a theoretical point of view) and make many people as bankers and governors quite nervous!

It is clear that if one looks for a one-way function that admits an inverse *function* in our *satisfaction* setup, then one should impose the condition that x is the *unique* solution of θ. Note that finding unique solutions is also quite important from the eyes of a physicist, since usually the most important problem for such a scientist is to describe a system in terms of some equations derived from the fundamental laws of the theory and then find the solution that describes the evolution of the system in time. It is not surprising if one emphasizes that finding conditions for having a unique solution is a physical condition that is usually posed to mathematicians to handle.

The easy fact that, *"verifying solutions is quite easy while finding them is generically hard"*, is even understandable from the point of view of a high school student. Hence, there is a good chance that the proposed setup actually gives rise to a one-way function! Also, one may check that the decision problem related to the inversion of this candidate function reduces to solving *unique solution* promise problem which seems to be essentially harder than the decision problem of some other well-known candidates as the RSA function (note: recently it is proved that PRIMES is in P). Therefore, our next important question is *"How hard are unique solution promise problems?"*

For instance, the unique solution promise problem for the case of Boolean satisfaction is known as USAT and it is quite easy to see that this problem Karp-reduces to SAT. Consequently, one last question related to the hardness of USAT can be whether SAT also Karp-reduces to USAT. Unfortunately, this is not established yet, but a very interesting result of L. G. Valiant and V. V. Vazirani states that actually a randomized reduction do exist! Hence, although it is not clear whether polynomial-time computability of USAT implies $NP = P$, but we know that it definitely implies $NP = RP$ (consequences as $USAT \in BPP_{promise} \Rightarrow NP - BPP = \emptyset$ can also be verified which is widely believed to be an evidence for the hardness of the USAT problem).

1.3 Episode 3: Graph colorings and graph grammars

As I believe in a *discrete* world, I am interested in discrete models and discrete dynamics. Hence, my objects of interest are discrete sets of points! Now, it is more or less well understood that one may look at a graph as a discrete set in which vicinity is modeled by *adjacency*. Consequently, one may think of a graph as a discrete set of points, each equipped with a *tangent space* of its outgoing edges that determines nearness, exactly as a manifold that can be described in the same way[2].

Now, following our physical approach, we should be interested in dynamics on these objects and try to formulate them in a meta-mathematical world of monoidal categories. It is interesting that this not only is important since it is in coherence with the setup presented in Episode 1, but also it is again important since the state space of any algorithm (hard or soft) is a graph and this setup is also important for its consequences in theory of computation and analysis of algorithms (e.g. for more on this one may refer to the theory of *combinatorial landscapes*)!

On the other hand, considering classical dynamics on graphs, as discrete diffusions, Schrödinger operators, percolation or games, one comes to the conclusion that, as in the case of manifolds, there is a close relation between the dynamics and the graph itself if the dynamics is chosen in a suitable and natural sort of way. For instance, there is a quite well established theory of electrical networks that speaks about potentials, currents and

[2]I prefer not to talk about graphs as one dimensional manifolds with singularities here.

Kirchhoff's conservation laws. Fortunately, the very well understood theory of *potentials* and its probabilistic counterpart for *martingales* puts forward a very well-founded base for a research on analysis of graphs through a study of function spaces on their set of vertices (i.e. points).

From this point of view, it is interesting to compare our situation when we are dealing with function spaces on graphs and manifolds. Note that if one imposes natural smoothness conditions on a manifold structure to use analytic methods then one benefits from very strong regularity conditions as the basic *intermediate value theorem* stating that positive and negative points are separated by zero points. However, such hypothesis and regularities are not applicable in the discrete case, where on the other hand, we have other simplicities at hand when, for example, we are dealing with *finite simple* graphs which makes the techniques of finite combinatorics applicable. These facts, in my opinion, shows that the case of *smooth manifolds* and *finite simple graphs* are the two extreme cases with different natures that one must understand before one is going to deal with the most difficult case of *directed infinite graphs.*

Therefore, let us think of nice potentials on a finite simple graph related to a well-defined setup as an electrical network. Then it is clear that such functions satisfy the basic property that they take *different* values on an edge with a *nonzero* current and resistance. Naturally, the discrete version of this problem also becomes important both from dynamical and computational points of view. As a matter of fact, this problem usually referred to as the *graph coloring* problem and its dual called the *integer flow* problem are quite well known and extensively studied in graph theory and combinatorics. Let me note that from a computational point of view we know that this is one of the hardest problems human can think of, since by a consequence of the well-known PCP theorem, (arbitrary) *approximability* of the *graph least coloring problem* implies that $NP = RP$ (note: compare with USAT results). Also, by the existence of arbitrary sparse graphs with a given chromatic number, it is known that the graph coloring problem is hard because the chromatic number is a *global* property of a graph that can not be studied or approximated through *local* information.

To this end, let me somehow sum up what we have talked about so far in the context of the coloring problem. Let (θ, x) be a pair consisting of a *uniquely colorable* graph and its unique coloring x. Then on the one hand, our fundamental one-wayness question reduces to *"How one may check the unique colorability and find the unique coloring of a given uniquely colorable graph if it exists?"* Definitely, from what we have talked about we know that this can not be done locally, and the best one may expect is to categorize graphs based on the computational complexity of their coloring structure. Hence, one may ask *"How one can construct uniquely colorable graphs with controlled but arbitrarily complex coloring structure?"*

Also, by what we learnt from topological field theories, we understand that if we are interested in categorizing graphs based on the complexity of their topological structure then we should be interested in some sort of cylindrical construction which is also linked to a well-behaved dynamics. Consequently, if this dynamic is naturally related to the function space of potentials, this can also be related to the original coloring problem! I should note that the fundamental coloring problem is closely linked to the most fundamental problem in the general category of graphs, in the sense that it is a special case of the *homomorphism problem* when the model space is the complete graph, and also, the relationship of the homomorphism problem to graph dynamics is more or less understood along the same line of thought one thinks about rigidity and nonembedding results for Riemannian manifolds or comparison theorems for Markov chains.

It came as a surprise to me that both ways of thinking just converges to the same question of *"How one may construct graphs through cylinders?"* Naturally, this question must be tackled using constructions and amalgams in the category of graphs, and again this is a bit of a surprise that such a language already exists in the literature of *graph grammars* that is developed mainly because of its importance and applications in formal treatment and verification of discrete dynamics described in graphical models. Also, it is interesting to note that contrary to the case of manifolds, there is a *trivial* construction for any graph through *edge-cylinders*! We follow more complex amalgam constructions in the next section.

2 The cylindrical construction

In this section I am just going to state a bunch of claims! I apologize for not being rigorous and explicit. Hopefully, the article containing all the details will appear on `arXiv.org` soon.

Following fundamental contributions of P. Hell and J. Nešetřil, my first claim is that there exists a very general *cylindrical construction* for graphs that I denote by $G \boxtimes C$. I will try to explain some simple cases and examples through my slides. Also, I claim that this construction is *tensorial* in the sense that there exists a Hom construction, $[C, G]$, called the *exponential graph* construction such that both constructions are functorial and satisfy the following fundamental property.

Theorem 1. *Let C be a cylinder and G be a graph. Then for any graph H,*

a) $Hom(G \boxtimes C, H) \neq \emptyset \quad \Leftrightarrow \quad Hom(G, [C, H]) \neq \emptyset.$

b) *There exist a retraction*

$$r_{_{G,H}} : Hom(G \boxtimes C, H) \to Hom(G, [C, H])$$

and a section

$$s_{_{G,H}} : Hom(G, [C, H]) \to Hom(G \boxtimes C, H)$$

such that $r_{_{G,H}} \circ s_{_{G,H}} = \mathbf{1}$, where $\mathbf{1}$ is the identity mapping.

I will try to show that many standard constructions in graph theory are among the special cases of these constructions. As examples I will try to show that the *indicator construction*, the *power graph construction* and the *looped line graph construction* are exponential graphs, and on the other hand, I will show that most of tensorial constructions as *voltage graph, NEPS, zig-zag* and *replacement* constructions are cylindrical.

At the end of this talk I will try to show how adjunctions can be considered as special cases of reductions for the homomorphism problem and in this regard I will try to formulate a couple of problems that I believe are of fundamental importance as characterization of *openness* and *closedness* of graphs (i.e. with respect to a given cylinder). Definitely, these are just the first steps!!