For each $1 \leq i \leq n$, the function $f_i$ is a $\zeta_i$-excessive (resp. $\zeta_i$-deficient) function (with respect to $\Delta$) on $X$.

For each $1 \leq i \leq n$, the subset $Q_i$ is a nonnegative (resp. nonpositive) bipolar part of $f_i$.

**Theorem 3.** Consider a graph $G$ and let $\Gamma = (\zeta_1, \zeta_2, \ldots, \zeta_n)$. If $F = (f_1, f_2, \ldots, f_n)$ along with $Q = (Q_1, Q_2, \ldots, Q_n)$ is a compatible transverse set of functions for $\Delta$, then

$$2\bar{\zeta}_n \geq \iota_n(G)^2.$$

**Proof.** Let $0 \neq g_i \equiv f_i|_{Q_i}$. Then,

$$\bar{\zeta}_n \geq \frac{1}{n} \sum_{i=1}^{n} \frac{\|\nabla g_i\|_{L^2}}{\|g_i\|_{L^\infty}} \geq \frac{1}{n} \sum_{i=1}^{n} \frac{\|\nabla g_i^2\|^2_{L^2}}{\|g_i\|^2_{L^\infty}} \geq \frac{1}{2} \iota_n(G)^2,$$

where the first and the second inequalities follow from Lemma 6 and 4, respectively, and the third one is a direct application of Cauchy-Schwarz inequality.

It ought to be noted that Theorems 2 and 3 together, can be considered as a generalized Cheeger inequality. In what follows we deduce a special case where one may get an explicit inequality for the mean spectrum.

**Theorem 4.** Consider a kernel $K$ on a base graph $G$. Let $F = (f_2, f_3, \ldots, f_{n+1})$ be a list of eigenfunctions of $\Delta$ for the list of eigenvalues $\Gamma = (\lambda_2, \lambda_3, \ldots, \lambda_{n+1})$, respectively, such that along with $Q = (Q_2, Q_3, \ldots, Q_{n+1})$ form a compatible transverse set of functions for $\Delta$. Then,

$$\lambda_n \leq \iota_n(G) \leq \sqrt{2(n + 1)} \bar{\lambda}_{n+1}.$$  \hspace{1cm} (7)

Moreover, we would like to add that following the same scenario described for the mean version, one may define the $n$th max-isoperimetric constant as

$$\varsigma_n(K, \pi) \equiv \min_{\{Q_i\}_{i=1}^{n}} \left( \max_{1 \leq i \leq n} \frac{\bar{\delta}(Q_i)}{\pi(Q_i)} \right).$$

It is noteworthy that all of the previous mentioned results such as the Federer-Fleming theorem can also be verified for this version with appropriate modifications. For instance, we may state a more standard Cheeger inequality for the max-isoperimetric constant $\varsigma_n$ using Theorems 2 and 3 and their counterparts, along with Courant-Fischer variational theorem as follows.

**Theorem 5.** For a given graph $G$, let $f$ be an eigenfunction of $\Delta$ corresponding to the $n$th eigenvalue $\lambda_n$. Also, let $(Q_1, Q_2, \ldots, Q_n)$ be a list of $n$ disjoint nonempty subsets of $V(G)$ such that for every $1 \leq i \leq n$ we have $f|_{Q_i} \neq 0$ and each $Q_i$ is a nonnegative or nonpositive bipolar part of $f$. Then,

$$\frac{\lambda_n}{2} \leq \varsigma_n(G) \leq \sqrt{2\lambda_n}, \quad \text{and} \quad \bar{\lambda}_n \leq \iota_n(G) \leq \sqrt{2\lambda_n}.$$  \hspace{1cm} (8)
Also, as a corollary of Theorem 5 by considering the fact that always the second eigenvalue
has an eigenfunction with two nodal domains, we obtain the classical Cheeger inequality as,
\[ \frac{\lambda_2}{2} \leq \iota_2(G) \leq \varsigma_2(G) \leq \sqrt{2 \lambda_2}. \]  

(9)

It also must be emphasized that a direct use of eigenvalues and eigenfunctions (not necessarily
tuned with repetition) in Theorem 3 will definitely make a deviation from sharpness
which can be easily verified by a comparison to the classical Cheeger inequality (Inequality (9)). Note that the classical Cheeger inequality is far from being sharp by a recent result
of Montenegro and Tetali [39].
To provide some examples let us recall the following result.

**Theorem B.** [4, 5] Let \( K \) be a kernel on a tree \( T \) and let \( f_n \) be an eigenfunction of \( \Delta \) with
eigenvalue \( \lambda_n \) which does not vanish on any vertex. Then \( \lambda_n \) is simple and \( f_n \) has exactly \( n \) strong nodal domains.

Therefore, a generalized Cheeger inequality is valid for any Markov chain on a tree \( T \) with
a nowherezero eigenfunction \( f_n \) of an eigenvalue \( \lambda_n \), i.e.
\[ \min(\lambda_n, \frac{\lambda_n}{2}) \leq \iota_n(T) \leq \varsigma_n(T) \leq \sqrt{2 \lambda_n}. \]

For more on the extensive literature of Markov chains on trees the interested reader is re-
ferred to [5, 37] and references therein.
On the other hand, it is quite interesting that even for the case of trees we do not know
enough about the behavior of parameters discussed in this article, and as Example 4 shows
one encounters nongeometric trees in very small cases. Hence, we believe that the following
problem can be considered to be a nice starting point for the study of supergeometric graphs.

**Problem 1.** Characterize the class of supergeometric trees.

### 5.3 Algorithmic considerations

In this section we touch on some algorithmic aspects of the isoperimetry problem and we
study its relationships to some well-known concepts as the \( k \)-means algorithm and the
normalized cuts method. This section is mainly influenced by the seminal contribution of
J. Malik and J. Shi [43] (also see [21]) that was brought to our attention after the presenta-
tion of the first two authors’ article on the isoperimetric spectrum of graphs [19].
Following our notations in Section 3, for a set \( X \), \( D_n(X) \) stands for the set of all \( n \)-sets \( \{Q_i\}_1^n \), where \( Q_i \)'s are nonempty disjoint subsets of \( X \). Also \( P_n(X) \subseteq D_n(X) \) consists of all \( n \)-partitions of \( X \).

**Definition 6.** Given a function \( f \in \mathcal{F}(X) \) and a weight function \( \omega : X \rightarrow \mathbb{R}^+ - \{0\} \), for
every \( 1 \leq n \leq |X| \), the cost function \( C_n^f(\omega) : D_n(X) \rightarrow \mathbb{R}^+ \) is defined as follows
\[ M_n^f(\omega)(\{Q_i\}_1^n) \overset{\text{def}}{=} \sum_{i=1}^n \sum_{u \in Q_i} \omega(u)\|f(u) - m_i\|^2, \]  

where \( m_i \overset{\text{def}}{=} \frac{\sum_{u \in Q_i} \omega(u)f(u)}{\sum_{u \in Q_i} \omega(u)} \),
and
\[ C_n^f(\omega)(\{Q_i\}_1^n) \overset{\text{def}}{=} M_n^f(\omega)(\{Q_i\}_1^n) + \sum_{u \in Q^*} \omega(u)\|f(u)\|^2, \]  

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