# Machine learning theory

## Kernel methods

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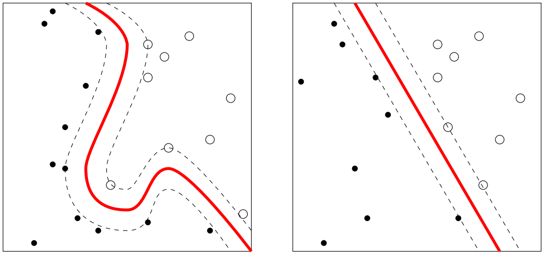
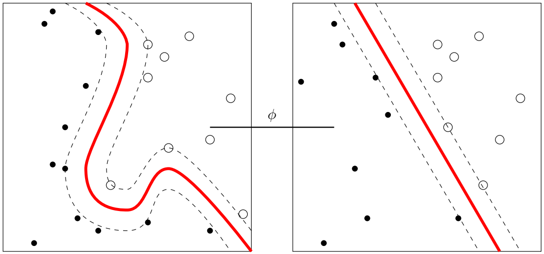
# Motivation

- Most of learning algorithms are linear and are not able to classify non-linearly-separable data.
- How do you separate these two classes?



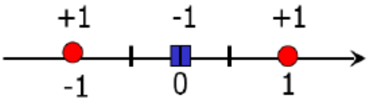
- Linear separation impossible in most problems.
- Non-linear mapping from input space to high-dimensional feature space: $\phi : \mathcal{X} \mapsto \mathbb{H}$.



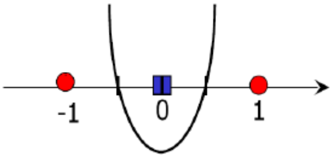

- Generalization ability: independent of $dim(\mathbb{H})$, depends only on $\rho$ and $m$.

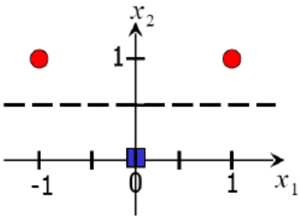
# Kernel methods

- Most datasets are not linearly separable, for example



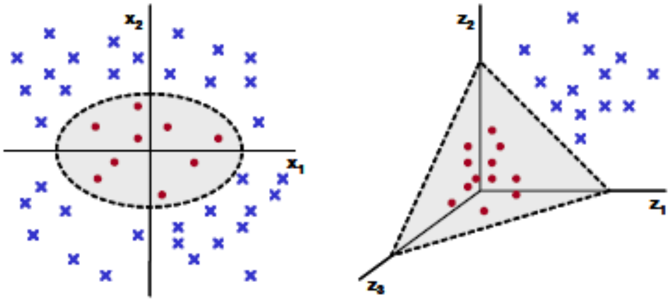

- Instances that are not linearly separable in $\mathbb{R}$, may be linearly separable in $\mathbb{R}^2$ by using mapping $\phi(x) = (x, x^2)$.




- In this case, we have two solutions
  - Increase dimensionality of data set by introducing mapping $\phi$.
  - Use a more complex model for classifier.

- To classify the non-linearly separable dataset, we use mapping $\phi$.
- For example, let $\mathbf{x} = (x_1, x_2)^T$, $\mathbf{z} = (z_1, z_2.z_3)^T$, and $\phi : \mathbb{R}^2 \to \mathbb{R}^3$.
- If we use mapping $\mathbf{z} = \phi(x) = (x_1^2, \sqrt{2}x_1x_2, x_2^2)^T$, the dataset will be linearly separable in $\mathbb{R}^3$.




- Mapping dataset to higher dimensions has two major problems.
  - In high dimensions, there is risk of over-fitting.
  - In high dimensions, we have more computational cost.
- The generalization capability in higher dimension is ensured by using large margin classifiers.
- The mapping is an implicit mapping not explicit.

- Kernel methods avoid explicitly transforming each point $\mathbf{x}$ in the input space into the mapped point $\phi(x)$ in the feature space.
- Instead, the inputs are represented via their $m \times m$ pairwise similarity values.
- The similarity function, called a **kernel**, is chosen so that it represents a dot product in some high-dimensional feature space.
- The kernel can be computed without directly constructing $\phi$.
- The pairwise similarity values between points in $S$ represented via the $m \times m$ kernel matrix, defined as

$$\mathbf{K} = \begin{pmatrix} k(\mathbf{x}_1, \mathbf{x}_1) & k(\mathbf{x}_1, \mathbf{x}_2) & \cdots & k(\mathbf{x}_1, \mathbf{x}_m) \\ k(\mathbf{x}_2, \mathbf{x}_1) & k(\mathbf{x}_2, \mathbf{x}_2) & \cdots & k(\mathbf{x}_2, \mathbf{x}_m) \\ \vdots & \vdots & \ddots & \vdots \\ k(\mathbf{x}_m, \mathbf{x}_1) & k(\mathbf{x}_m, \mathbf{x}_2) & \cdots & k(\mathbf{x}_m, \mathbf{x}_m) \end{pmatrix}$$

- Function $K(\mathbf{x}_i, \mathbf{x}_j)$ is called kernel function and defined as

**Definition (Kernel)**

Function $K : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ is a kernel if

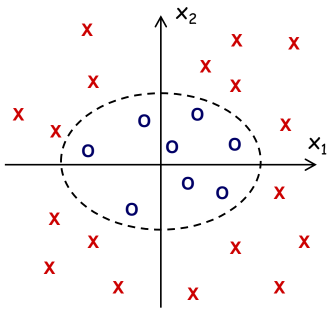
1. $\exists \phi : \mathcal{X} \mapsto \mathbb{R}^N$ such that $K(\mathbf{x}, \mathbf{y}) = \langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle$.
2. Range of $\phi$ is called the feature space.
3. $N$ can be very large.

## Kernels (example)

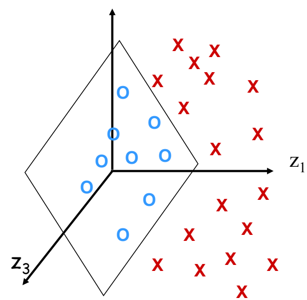
- Let $\phi : \mathbb{R}^2 \mapsto \mathbb{R}^3$ be defined as $\phi(\mathbf{x}) = (x_1^2, x_2^2, \sqrt{2}x_1 x_2)$.
- Then $\langle \phi(\mathbf{x}), \phi(\mathbf{z}) \rangle$ equals to

$$\begin{aligned} \langle \phi(\mathbf{x}), \phi(\mathbf{z}) \rangle &= \left\langle (x_1^2, x_2^2, \sqrt{2}x_1 x_2), (z_1^2, z_2^2, \sqrt{2}z_1 z_2) \right\rangle \\ &= (x_1 z_1 + x_2 z_2)^2 \\ &= (\langle \mathbf{x}, \mathbf{z} \rangle)^2 \\ &= K(\mathbf{x}, \mathbf{z}). \end{aligned}$$

- The above mapping can be described




Input space




feature space

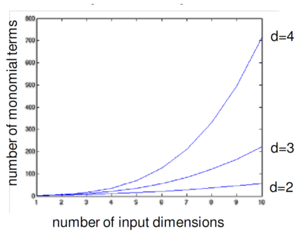
- Let $\phi_1 : \mathbb{R}^2 \mapsto \mathbb{R}^3$ be defined as $\phi(\mathbf{x}) = (x_1^2, x_2^2, \sqrt{2}x_1x_2)$.
- Then $\langle\phi_1(\mathbf{x}), \phi_1(\mathbf{z})\rangle$ equals to

$$\begin{aligned}\langle\phi_1(\mathbf{x}), \phi_1(\mathbf{z})\rangle &= \left\langle(x_1^2, x_2^2, \sqrt{2}x_1x_2), (z_1^2, z_2^2, \sqrt{2}z_1z_2)\right\rangle \\ &= (x_1z_1 + x_2z_2)^2 \\ &= (\langle\mathbf{x}, \mathbf{z}\rangle)^2 = K(\mathbf{x}, \mathbf{z}).\end{aligned}$$

- Let $\phi_2 : \mathbb{R}^2 \mapsto \mathbb{R}^4$ be defined as $\phi(x) = (x_1^2, x_2^2, x_1x_2, x_2x_1)$.
- Then $\langle\phi_2(\mathbf{x}), \phi_2(\mathbf{z})\rangle$ equals to

$$\begin{aligned}\langle\phi_2(\mathbf{x}), \phi_2(\mathbf{z})\rangle &= \left\langle(x_1^2, x_2^2, x_1x_2, x_2x_1), (z_1^2, z_2^2, z_1z_2, z_2z_1)\right\rangle \\ &= (\langle\mathbf{x}, \mathbf{z}\rangle)^2 = K(\mathbf{x}, \mathbf{z}).\end{aligned}$$

- Feature space can grow really large and really quickly.
- Let $K$ be a kernel $\mathbf{K}(\mathbf{x}, \mathbf{z}) = (\langle\mathbf{x}, \mathbf{z}\rangle)^d = \langle\phi(\mathbf{x}), \phi(\mathbf{z})\rangle$
- The dimension of feature space equals to $\binom{d+n-1}{d}$.
- Let $n = 100$, $d = 6$, there are 1.6 billion terms.



- The kernel methods have the following benefits.
  **Efficiency:** $K$ is often more efficient to compute than $\phi$ and the dot product.
  **Flexibility:** $K$ can be chosen arbitrarily so long as the existence of $\phi$ is guaranteed (Mercer's condition).

**Theorem (Mercer's condition)**

*For all functions $c$ that are square integrable (i.e., $\int c(x)^2 dx < \infty$), other than the zero function, the following property holds:*

$$\int \int c(x)K(x,z)c(z)dxdz \geq 0.$$

- This Theorem states that $K : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ is a kernel if matrix $\mathbf{K}$ is positive semi-definite (PSD).
- Suppose $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$ and consider the following kernel

$$K(\mathbf{x}, \mathbf{z}) = (\langle \mathbf{x}, \mathbf{z} \rangle)^2$$

- It is a valid kernel because

$$\begin{aligned} K(\mathbf{x}, \mathbf{z}) &= \left(\sum_{i=1}^{n} x_i z_i\right)\left(\sum_{j=1}^{n} x_j z_j\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} (x_i x_j)(z_i z_j) = \langle \phi(\mathbf{x}), \phi(\mathbf{z}) \rangle \end{aligned}$$

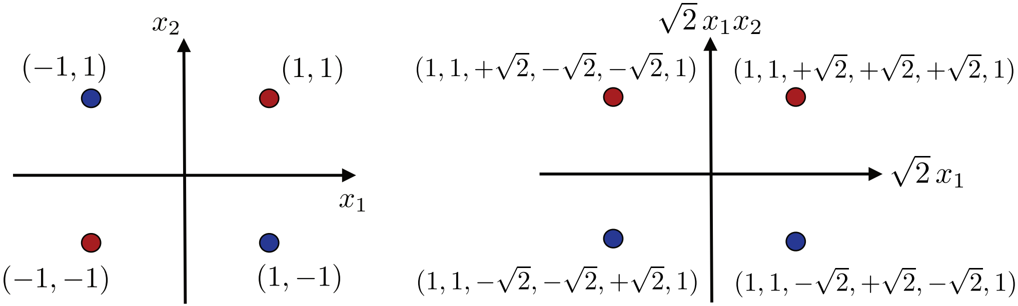
  where the mapping $\phi$ for $n = 2$ is

$$\phi(\mathbf{x}) = (x_1x_1, x_1x_2, x_2x_1, x_2x_2)^T$$

- Consider the polynomial kernel $K(x, z) = (\langle \mathbf{x}, \mathbf{z} \rangle + c)^d$ for all $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n$.
- For $n = 2$ and $d = 2$,

$$K(\mathbf{x}, \mathbf{z}) = (x_1 z_1 + x_2 y_2 + c)^2$$
$$= \left\langle \left[x_1^2, x_2^2, \sqrt{2}x_1x_2, \sqrt{2}cx_1, \sqrt{2}cx_2, c\right], \left[z_1^2, z_2^2, \sqrt{2}z_1z_2, \sqrt{2}cz_1, \sqrt{2}cz_2, c\right] \right\rangle$$

- Using second-degree polynomial kernel with $c = 1$:




- The left data is not linearly separable but the right one is.

- Some valid kernel functions
  - **Polynomial kernels** consider the kernel defined by

$$K(\mathbf{x}, \mathbf{z}) = (\langle \mathbf{x}, \mathbf{z} \rangle + c)^d$$

    $d$ is the degree of the polynomial and specified by the user and $c$ is a constant.
  - **Radial basis function kernels** consider the kernel defined by

$$K(\mathbf{x}, \mathbf{z}) = \exp\left(-\frac{\|\mathbf{x} - \mathbf{z}\|^2}{2\sigma^2}\right)$$

    The width $\sigma$ is specified by the user. This kernel corresponds to an infinite dimensional mapping $\phi$ .
  - **Sigmoid kernel** consider the kernel defined by

$$K(\mathbf{x}, \mathbf{z}) = \tanh\left(\beta_0 \langle \mathbf{x}, \mathbf{z} \rangle + \beta_1\right)$$

    This kernel only meets Mercer's condition for certain values of $\beta_0$ and $\beta_1$.
- **Homework:** Please prove VC-dimension of the above kernels.

- We give the crucial property of PDS kernels, which is to induce an inner product in a Hilbert space.

**Lemma (Cauchy-Schwarz inequality for PDS kernels)**

*Let $\mathbf{K}$ be a PDS kernel matrix. Then, for any $\mathbf{x}, \mathbf{z} \in \mathcal{X}$,*

$$K(\mathbf{x}, \mathbf{z})^2 \leq K(\mathbf{x}, \mathbf{x})K(\mathbf{z}, \mathbf{z})$$

**Proof.**

1. Consider the kernel matrx $\mathbf{K} = \begin{pmatrix} K(x,x) & K(x,x') \\ K(x',x) & K(x'x') \end{pmatrix}$.
2. By definition, if $K$ is PDS, then $\mathbf{K}$ is SPSD for all $x, x' \in \mathcal{X}$.
3. Then, the product of the eigenvalues of $\mathbf{K}$, $\det(\mathbf{K})$, must be non-negative.
4. Using $K(x,x') = K(x',x)$, we have $\det(\mathbf{K}) = K(x,x)K(x'x') - K(x,x')^2 \geq 0$.

□

**Theorem (Reproducing kernel Hilbert space (RKHS))**

*Let $K : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ be a PDS kernel. Then, there exists a Hilbert space $\mathbb{H}$ and a mapping $\phi$ from $\mathcal{X}$ to $\mathbb{H}$ such that for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$*

$$K(\mathbf{x}, \mathbf{y}) = \langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle .$$

- This Theorem implies that PDS kernels can be used to implicitly define a feature space.

- For any kernel $\mathbf{K}$, we can associate a normalized kernel $\mathbf{K}_n$ defined by

$$K_n(\mathbf{x},\mathbf{z}) = \begin{cases} 0 & \text{if } ((K(\mathbf{x},\mathbf{x})=0) \vee (K(\mathbf{z},\mathbf{z})=0)) \\ \dfrac{K(\mathbf{x},\mathbf{z})}{\sqrt{K(\mathbf{x},\mathbf{x})K(\mathbf{z},\mathbf{z})}} & \text{otherwise} \end{cases}$$

**Lemma (Normalized PDS kernels)**

*Let $K$ be a PDS kernel. Then, the normalized kernel $K_n$ associated to $K$ is PDS.*

**Proof.**

1. Let $\{x_1,\ldots,x_m\} \subseteq \mathcal{X}$ and let $\mathbf{c}$ be an arbitrary vector in $\mathbb{R}^n$.
2. We will show that $\sum_{i,j=1}^{m} c_i c_j K_n(\mathbf{x}_i,\mathbf{x}_j) \geq 0$.
3. By Lemma Cauchy-Schwarz inequality for PDS kernels, if $K(\mathbf{x}_i,\mathbf{x}_i)=0$, then $K(\mathbf{x}_i,\mathbf{x}_j)=0$ and thus $K_n(\mathbf{x}_i,\mathbf{x}_i)=0$ for all $j \in \{1,2,\ldots,m\}$.
4. We can assume that $K(\mathbf{x}_i,\mathbf{x}_i) > 0$ for all $i \in \{1,2,\ldots,m\}$.
5. Then, the sum can be rewritten as follows:

$$\sum_{i,j=1}^{m} c_i c_j K_n(\mathbf{x}_i,\mathbf{x}_j) = \sum_{i,j=1}^{m} \frac{c_i c_j K(\mathbf{x}_i,\mathbf{x}_j)}{\sqrt{K(\mathbf{x}_i,\mathbf{x}_i)K(\mathbf{x}_j,\mathbf{x}_j)}} = \sum_{i,j=1}^{m} \frac{c_i c_j \langle \phi(\mathbf{x}_i), \phi(\mathbf{x}_j)\rangle}{\|\phi(\mathbf{x}_i)\|_{\mathbb{H}} \cdot \|\phi(\mathbf{x}_j)\|_{\mathbb{H}}} = \left\| \sum_{i=1}^{m} \frac{c_i \phi(\mathbf{x}_i)}{\|\phi(\mathbf{x}_i)\|_{\mathbb{H}}} \right\|_{\mathbb{H}}^2 \geq 0.$$

□

- The following theorem provides closure guarantees for all of these operations.

**Theorem (Closure properties of PDS kernels)**

*PDS kernels are closed under*

1. *sum*
2. *product*
3. *tensor product*
4. *pointwise limit*
5. *composition with a power series* $\sum_{k=1}^{\infty} a_k x^k$ *with* $a_k \geq 0$ *for all* $k \in \mathbb{N}$.

**Proof.**

We only proof the closeness under sum. Consider two valid kernel matrices $\mathbf{K}_1$ and $\mathbf{K}_2$.

1. For any $\mathbf{c} \in \mathbb{R}^m$, we have $\mathbf{c}^T\mathbf{K}_1\mathbf{c} \geq 0$ and $\mathbf{c}^T\mathbf{K}_2\mathbf{c} \geq 0$.
2. This implies that $\mathbf{c}^T\mathbf{K}_1\mathbf{c} + \mathbf{c}^T\mathbf{K}_2\mathbf{c} \geq 0$.
3. Hence, we have $\mathbf{c}^T(\mathbf{K}_1 + \mathbf{K}_2)\mathbf{c} \geq 0$.
4. Let $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2$, which is a valid kernel.

□

- **Homework:** Please prove other closure properties of PDS kernels.

## Basic kernel operations in feature space

- **Norm of a point:** we can compute the norm of a point $\phi(\mathbf{x})$ in feature space as

$$\|\phi(\mathbf{x})\|^2 = \langle \phi(\mathbf{x}), \phi(x) \rangle = K(\mathbf{x}, \mathbf{x}),$$

which implies that $\|\phi(\mathbf{x})\| = \sqrt{K(\mathbf{x}, \mathbf{x})}$.

- **Distance between Points:** the distance between two points $\phi(\mathbf{x}_i)$ and $\phi(\mathbf{x}_j)$ can be computed as

$$\begin{aligned} \|\phi(\mathbf{x}_i) - \phi(\mathbf{x}_j)\|^2 &= \|\phi(\mathbf{x}_i)\|^2 + \|\phi(\mathbf{x}_j)\|^2 - 2\langle \phi(\mathbf{x}_i), \phi(\mathbf{x}_j) \rangle \\ &= K(\mathbf{x}_i, \mathbf{x}_i) + K(\mathbf{x}_j, \mathbf{x}_j) - 2K(\mathbf{x}_i, \mathbf{x}_j), \end{aligned}$$

which implies that $\|\phi(\mathbf{x}_i) - \phi(\mathbf{x}_j)\| = \sqrt{K(\mathbf{x}_i, \mathbf{x}_i) + K(\mathbf{x}_j, \mathbf{x}_j) - 2K(\mathbf{x}_i, \mathbf{x}_j)}$.

- **Mean in feature space:** the mean of the points in feature space is given as

$$\mu_\phi = \frac{1}{m} \sum_{i=1}^{m} \phi(\mathbf{x}_i).$$

Since we haven't access to $\phi(x)$, we cannot explicitly compute the mean point in feature space but we can compute the squared norm of the mean as follows.

$$\begin{aligned} \|\mu_\phi\|^2 &= \langle \mu_\phi, \mu_\phi \rangle \\ &= \left\langle \frac{1}{m} \sum_{i=1}^{m} \phi(\mathbf{x}_i), \frac{1}{m} \sum_{i=1}^{m} \phi(\mathbf{x}_i) \right\rangle \\ &= \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} \langle \phi(\mathbf{x}_i), \phi(\mathbf{x}_j) \rangle = \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} K(\mathbf{x}_i, \mathbf{x}_j). \end{aligned}$$

- **Total variance in feature space:** the squared distance of a point $\phi(x_i)$ to the mean $\mu_\phi$ in feature space:

$$\begin{aligned}\|\phi(\mathbf{x})-\mu_\phi\|^2 &= \|\phi(\mathbf{x}_i)\|^2 - 2\langle\phi(\mathbf{x}_i),\mu_\phi\rangle + \|\mu_\phi\|^2 \\ &= K(\mathbf{x}_i,\mathbf{x}_i) - \frac{2}{m}\sum_{j=1}^{m}K(\mathbf{x}_i,\mathbf{x}_j) + \frac{1}{m^2}\sum_{a=1}^{m}\sum_{b=1}^{m}K(\mathbf{x}_a,\mathbf{x}_b).\end{aligned}$$

The total variance in feature space is obtained by taking the average squared deviation of points from the mean in feature space

$$\begin{aligned}\sigma_\phi^2 &= \frac{1}{m}\sum_{i=1}^{m}\|\phi(\mathbf{x}_i)-\mu_\phi\|^2 \\ &= \frac{1}{m}\sum_{i=1}^{m}\left(K(\mathbf{x}_i,\mathbf{x}_i) - \frac{2}{m}\sum_{j=1}^{m}K(\mathbf{x}_i,\mathbf{x}_j) + \frac{1}{m^2}\sum_{a=1}^{m}\sum_{b=1}^{m}K(\mathbf{x}_a,\mathbf{x}_b)\right) \\ &= \frac{1}{m}\sum_{i=1}^{m}K(\mathbf{x}_i,\mathbf{x}_i) - \frac{2}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}K(\mathbf{x}_i,\mathbf{x}_j) + \frac{1}{m^2}\sum_{a=1}^{m}\sum_{b=1}^{m}K(\mathbf{x}_a,\mathbf{x}_b) \\ &= \frac{1}{m}\sum_{i=1}^{m}K(\mathbf{x}_i,\mathbf{x}_i) - \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}K(\mathbf{x}_i,\mathbf{x}_j) \\ &= \frac{1}{m}\mathrm{Tr}[\mathbf{K}] - \|\mu_\phi\|^2.\end{aligned}$$

- **Centering in feature space:**
  - We can center each point in feature space by subtracting the mean from it
    $$\hat{\phi}(\mathbf{x}_i) = \phi(\mathbf{x}_i) - \mu_\phi.$$
  - We have not $\phi(\mathbf{x}_i)$ and $\mu_\phi$, hence, we cannot explicitly center the points.
  - However, we can still compute the centered kernel matrix $\hat{\mathbf{K}}$, that is, the kernel matrix over centered points.
    $$\begin{aligned}
    \hat{K}(x_i, x_j) &= \left\langle \hat{\phi}(\mathbf{x}_i), \hat{\phi}(\mathbf{x}_j) \right\rangle \\
    &= \langle \phi(\mathbf{x}_i) - \mu_\phi, \phi(\mathbf{x}_j) - \mu_\phi \rangle \\
    &= \langle \phi(\mathbf{x}_i), \phi(\mathbf{x}_j) \rangle - \langle \phi(\mathbf{x}_i), \mu_\phi \rangle - \langle \phi(\mathbf{x}_j), \mu_\phi \rangle + \langle \mu_\phi, \mu_\phi \rangle \\
    &= K(\mathbf{x}_i, \mathbf{x}_j) - \frac{1}{m}\sum_{k=1}^{m} \langle \phi(\mathbf{x}_i), \phi(\mathbf{x}_k) \rangle - \frac{1}{m}\sum_{k=1}^{m} \langle \phi(\mathbf{x}_j), \phi(\mathbf{x}_k) \rangle + \left\| \mu_\phi \right\|^2 \\
    &= K(\mathbf{x}_i, \mathbf{x}_j) - \frac{1}{m}\sum_{k=1}^{m} K(\mathbf{x}_i, \mathbf{x}_k) - \frac{1}{m}\sum_{k=1}^{m} K(\mathbf{x}_j, \mathbf{x}_k) + \left\| \mu_\phi \right\|^2
    \end{aligned}$$
  - In other words, we can compute the centered kernel matrix using only the kernel function.

- **Normalizing in feature space:**
  - A common form of normalization is to ensure that points in feature space have unit length by replacing $\phi(\mathbf{x})$ with the corresponding unit vector $\phi_n(\mathbf{x}) = \frac{\phi(\mathbf{x})}{\|\phi(\mathbf{x})\|}$ .
  - The dot product in feature space then corresponds to the cosine of the angle between the two mapped points, because

$$\langle \phi_n(\mathbf{x}_i), \phi_n(\mathbf{x}_j) \rangle = \frac{\langle \phi(\mathbf{x}_i), \phi(\mathbf{x}_j) \rangle}{\|\phi(\mathbf{x}_i)\| . \|\phi(\mathbf{x}_j)\|} = \cos\theta.$$

  - If the mapped points are both centered and normalized, then a dot product corresponds to the correlation between the two points in feature space.
  - The normalized kernel function, $K_n$, can be computed using only the kernel function $K$, as

$$K_n(\mathbf{x}_i, \mathbf{x}_j) = \frac{\langle \phi(\mathbf{x}_i), \phi(\mathbf{x}_j) \rangle}{\|\phi(\mathbf{x}_i)\| . \|\phi(\mathbf{x}_j)\|} = \frac{K(\mathbf{x}_i, \mathbf{x}_j)}{\sqrt{K(\mathbf{x}_i, \mathbf{x}_i).K(\mathbf{x}_j, \mathbf{x}_j)}}$$

## Kernel-based algorithms

- The optimization problem for SVM is defined as

$$Minimize \frac{1}{2}\|\mathbf{w}\|^2 \qquad \text{subject to } y_k(\langle \mathbf{w}, \mathbf{x}_k\rangle + b) \geq 1 \text{ for all } k = 1, 2, \ldots, m$$

- In order to solve this constrained optimization problem, we use the Lagrangian function

$$L(\mathbf{w}, b, \alpha) = \frac{1}{2}\|\mathbf{w}\|^2 - \sum_{k=1}^{m} \alpha_k \left[y_k\left(\langle \mathbf{w}, \mathbf{x}_k\rangle + b\right) - 1\right]$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T$.

- Eliminating $\mathbf{w}$ and $b$ from $L(\mathbf{w}, b, a)$ using these conditions then gives the dual representation of the problem in which we maximize

$$\psi(\alpha) = \sum_{k=1}^{m} \alpha_k - \frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m} \alpha_k \alpha_j y_k y_j \langle \mathbf{x}_k, \mathbf{x}_j\rangle$$

- We need to maximize $\psi(\alpha)$ subject to constraints $\sum_{k=1}^{m} \alpha_k y_k = 0$ and $\alpha_k \geq 0$ $\forall k$.
- For optimal $\alpha_k$'s, we have $\alpha_k\left[1 - y_k\left(\langle \mathbf{w}, \mathbf{x}_k\rangle + b\right)\right] = 0$.
- To classify a data $\mathbf{x}$ using the trained model, we evaluate the following function

$$h(\mathbf{x}) = \text{sgn}\left(\sum_{k=1}^{m} \alpha_k y_k \langle \mathbf{x}_k, \mathbf{x}\rangle\right)$$

- This solution depends on the dot-product between two pints $\mathbf{x}_k$ and $\mathbf{x}$.

- By using kernel $K$, the dual representation of the problem in which we maximize

$$\psi(\alpha) = \sum_{k=1}^{m} \alpha_k - \frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m} \alpha_k \alpha_j y_k y_j K(\mathbf{x}_i, \mathbf{x}_j)$$

- To classify a data $x$ using the trained model, we evaluate the following function

$$h(\mathbf{x}) = \operatorname{sgn}\left(\sum_{k=1}^{m} \alpha_k y_k K(\mathbf{x}_k, \mathbf{x})\right)$$

- This solution depends on the dot-product between two pints $\mathbf{x}_k$ and $\mathbf{x}$.

**Theorem (Rademacher complexity of kernel-based hypotheses)**

*Let $K : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ be a PDS kernel and let $\phi : \mathcal{X} \mapsto \mathbb{H}$ be a feature mapping associated to $K$. Let also $S \subseteq \{\mathbf{x} \mid \mathbf{K}(\mathbf{x}, \mathbf{x}) \leq r^2\}$ be a sample of size $m$ and let $H = \{\mathbf{x} \mapsto \langle \mathbf{w}, \phi(\mathbf{x}) \rangle \mid \|\mathbf{w}\|_{\mathbb{H}} \leq \Lambda\}$ for some $\Lambda \geq 0$. Then*

$$\hat{\mathcal{R}}_S(H) \leq \frac{\Lambda\sqrt{\operatorname{Tr}[\mathbf{K}]}}{m} \leq \sqrt{\frac{r^2\Lambda^2}{m}}.$$

**Proof.**

$$\begin{aligned}
\hat{\mathcal{R}}_S(H) &= \frac{1}{m} \mathop{\mathbb{E}}_{\sigma}\left[\sup_{\|\mathbf{w}\| \leq \Lambda} \sum_{i=1}^{m} \sigma_i \langle \mathbf{w}, \phi(\mathbf{x}_i) \rangle\right] = \frac{1}{m} \mathop{\mathbb{E}}_{\sigma}\left[\sup_{\|\mathbf{w}\| \leq \Lambda} \left\langle \mathbf{w}, \sum_{i=1}^{m} \sigma_i \phi(\mathbf{x}_i) \right\rangle\right] \\
&\leq \frac{\Lambda}{m} \mathop{\mathbb{E}}_{\sigma}\left[\left\|\sum_{i=1}^{m} \sigma_i \phi(\mathbf{x}_i)\right\|_{\mathbb{H}}\right] \leq \frac{\Lambda}{m}\sqrt{\mathop{\mathbb{E}}_{\sigma}\left[\left\|\sum_{i=1}^{m} \sigma_i \phi(\mathbf{x}_i)\right\|_{\mathbb{H}}^2\right]} = \frac{\Lambda}{m}\sqrt{\mathop{\mathbb{E}}_{\sigma}\left[\sum_{i,j=1}^{m} \sigma_i \sigma_j \langle \phi(\mathbf{x}_i), \phi(\mathbf{x}_j) \rangle\right]} \\
&\leq \frac{\Lambda}{m}\sqrt{\mathop{\mathbb{E}}_{\sigma}\left[\sum_{i=1}^{m} \|\phi(\mathbf{x}_i)\|^2\right]} = \frac{\Lambda}{m}\sqrt{\mathop{\mathbb{E}}_{\sigma}\left[\sum_{i=1}^{m} \mathbf{K}(\mathbf{x}_i, \mathbf{x}_i)\right]} \\
&\leq \frac{\Lambda\sqrt{\operatorname{Tr}[\mathbf{K}]}}{m} = \sqrt{\frac{r^2\Lambda^2}{m}}
\end{aligned}$$

□

**Theorem (Margin bounds for kernel-based hypotheses)**

*Let* $\mathbf{K} : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}$ *be a PDS kernel with* $r^2 = \sup_{\mathbf{x} \in \mathcal{X}} \mathbf{K}(\mathbf{x}, \mathbf{x})$. *Let* $\phi : \mathcal{X} \mapsto \mathbb{H}$ *be a feature mapping associated to* $\mathbf{K}$ *and let* $H = \{x \mapsto \langle \mathbf{w}, \phi(\mathbf{x}) \rangle \mid \|\mathbf{w}\|_{\mathbb{H}} \leq \Lambda\}$ *for some* $\Lambda \geq 0$. *Fix* $\rho > 0$. *Then for any* $\delta > 0$, *each of the following statements holds with probability at least* $(1 - \delta)$ *for any* $h \in H$:

$$\mathbf{R}(h) \leq \hat{\mathbf{R}}_{S,\rho}(h) + 2\sqrt{\frac{r^2\Lambda^2/\rho^2}{m}} + \sqrt{\frac{\log(1/\delta)}{2m}}$$

$$\mathbf{R}(h) \leq \hat{\mathbf{R}}_{S,\rho}(h) + 2\sqrt{\frac{\mathrm{Tr}\,[\mathbf{K}]\Lambda^2/\rho^2}{m}} + 3\sqrt{\frac{\log(2/\delta)}{2m}}$$

1. Chapter 16 of Shai Shalev-Shwartz and Shai Ben-David Book[1]
2. Chapter 6 of Mehryar Mohri and Afshin Rostamizadeh and Ameet Talwalkar Book[2].

Mohri, Mehryar, Afshin Rostamizadeh, and Ameet Talwalkar (2018). *Foundations of Machine Learning*. Second Edition. MIT Press.

Shalev-Shwartz, Shai and Shai Ben-David (2014). *Understanding machine learning: From theory to algorithms*. Cambridge University Press.

Questions?