Tensor Analysis-I

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But one thing is certain, in all my life I have never labored nearly as hard, and
I have become imbued with great respect for mathematics, the subtler part of which I had in my simple-mindedness regarded as pure luxury until now. - Einstein

Vector Space

- A vector space (V, F) is defined as a set of vectors V and scalars F, that satisfy the familiar axioms of vector space structure.
- If any vector in the space can be expanded with a set $\{e_i\}_{i=1}^{n}$ of vectors, we say that this set is a basis if each e_i & $\mathrm{e_j}$ are mutually linearly independent. n if finite is unique and this unique number is denoted as the dimension of V i.e. $\dim(V) = n$.
- In the context of relativity as we will see, we will always work with finite dimensional vector spaces, so our mathematics would be rather simple and precise.

Vector Space

- For a basis $\{e_i\}_{i=1}^{n}$ we can expand each vector in unique way with certain coefficients, v^i:

$$v = \sum v^i e_i \equiv v^i e_i$$

The Dual Space

- A linear functional on V is a linear map $f: V \rightarrow F$. The set V^* of all linear functionals on V is called the dual space of V.
- One can easily realize that this space is also a vector space.
- It is customary to write $\langle v, F \rangle$ or $\langle f, v \rangle$ to denote $f(v)$. When written this way it is called the **natural pairing** or **dual pairing** between V and V^*. Elements of V^* are often called **covectors**.

The Dual Space

- If $\{e_i\}_{i=1}^n$ is a basis of *V*, there is a canonical **dual basis** or **cobasis** $\{\theta^j\}_{j=1}^n$ of V^*, defined by $\langle e_i, \theta^j \rangle = \delta_i^j$, where δ_i^j is the **Kronecker delta.**
- any element $f \in V^*$ can be expanded in terms of the dual basis,

$$f = f_i \theta^i$$

where $f_i \in F$ are scalars.

The Dual Space

- From the argument in the last page $\dim(V^*) = \dim(V)$ so these to spaces are isomorphic, but it seems that our isomorphy depends on our choice of basis $\{e_i\}_{i=1}^{n}$, for this reason we say that these spaces are not isomorphic in a natural way (there is an exception, that is the case where we have an innerproduct as will be defined later in these slides. In that case $Riesz\ Lemma$ guarantees the naturalness of this isomorphy.).
- On the other hand $V^{**}\ \&\ V$ are isomorphic naturally.

Inner product spaces

- So far everything we have said about vector spaces and linear maps works in essentially the same way regardless of the underlying field F, but there are are a few places where the underlying field matters and here we consider one of them.
- Although in SR we are concerned with real numbers, here we express the definitions for complex numbers to be more general.

Inner product spaces

- Thus, let F be a subfield of C, and let V be a vector space over F. A **sesquilinear form** on V is a map $g : V \times V \rightarrow F$ satisfying the following two properties.

1. **linear on the second entry**: $g(u, av + bw) = ag(u, v) + bg(u, w)$
2. **Hermitian**: $g(v, u) = \overline{g(u, v)}$

- Note that these two properties together imply that g is **antilinear** on the first entry.
- If F is a subfield of real numbers g would be bilinear and symmetric, so a real sesquilinear form is in fact a **symmetric bilinear form.**

Inner product spaces

- If the sesquilinear form g is

3. (3) **nondegenerate**, so that $g(u, v) = 0$ for all v implies $u = 0$, then it is called an **inner product**.

- A vector space equipped with an inner product is called an **inner product space**.

Inner product spaces

- By Hermiticity $g(u, u)$ is always a real number. We may thus distinguish some important subclasses of inner products. If $g(u, u) \geq 0$ (respectively, $g(u, u) \leq 0$) then g is **nonnegative definite** (respectively, **nonpositive definite**).
- A nonnegative definite (respectively, nonpositive definite) inner product satisfying the condition that $g(u, u) = 0$ implies $u = 0$ is **positive definite** (respectively, **negative definite**).

Inner product spaces

- A positive definite or negative definite inner product is always nondegenerate but the converse is not true.

Tensor product

- We define the tensor product space of two vector spaces, V, W by the space $V \otimes W = span\{e_i \otimes f_j\}_{\mathrm{ij}}$. And I hope that you are familiar with the definition of the tensor product of basis vectors in finite dimensions from linear algebra!
- We will use the term **tensor** to describe the general element of $V \otimes W$.

Dual representation of tensor product

- Every pair of vectors (v, w) where $v \in V, w \in W$ defines a bilinear map $V^* \times W^* \rightarrow F$, by setting

$$(v, w) : (\rho, \phi) \rightarrow \rho(v)\phi(w).$$

- Then one can define for any element in $\mathrm{A} \in V \otimes W$, a map $V^* \times W^* \rightarrow F$ by setting:

$$A = A^{ij} e_i \otimes f_j \rightarrow A(\rho, \phi) = A^{ij} \rho(e_i)\phi(f_j)$$

- One can also show that this map is one-to-one(exercise!). So this identification gives a dual representation of $V \otimes W$ as bilinear maps.

Multilinear maps and tensor spaces of type (r, s)

- From now on I work in real vector spaces, we refer to any multilinear map

$$T: \underbrace{V^* \times V^* \times \cdots \times V^*}_{r} \times \underbrace{V \times V \times \cdots \times V}_{s} \to R$$

as a **tensor of type** (*r, s*) on *V*. The integer *r* ≥ 0 is called the **contravariant degree** and *s* ≥ 0 the **covariant degree** of *T* . I will denote this space by $T^{(r,s)}$.

Multilinear maps and tensor spaces of type (r, s)

- If T is a tensor of type (r, s) and S is a tensor of type (p, q) then define $T \otimes S$, called their **tensor product**, to be the tensor of type $(r + p, s + q)$ defined by,

$$(T \otimes S)\left(w^1, \dots, w^r, \rho^1, \dots, \rho^p, u_1, \dots, u_s, v_1, \dots, v_q\right)$$
$$= T(w^1, \dots, w^r, u_1, \dots, u_s)S\left(\rho^1, \dots, \rho^p, v_1, \dots, v_q\right)$$

- This product generalizes the definition of tensor products of vectors and convectors that we introduced earlier.

Multilinear maps and tensor spaces of type (r, s)

- You are advised to show as an exercise that tensor products,

$$e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_s}, \qquad i_k, j_l = 1, \ldots, n$$

  form a basis for $T^{(r,s)}$.
- then one can write:

$$\forall T \in T^{(r,s)}: \; T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_s}$$

$$T^{i_1 \ldots i_r}_{j_1 \ldots j_s} = T(\theta^{i_1}, \ldots, \theta^{i_r}, e_{j_1}, \ldots, e_{j_s})$$

Multilinear maps and tensor spaces of type (r, s)

- Let $\{e_i\}$ and $\{e'_j\}$ be two bases of V related by

$$e_i = A^j_i e'_j, \qquad e'_j = (A^{-1})^k_j e_k$$

- Then one can see that the dual basis transforms by

$$\theta'^j = A^j_k \theta^k$$

Multilinear maps and tensor spaces of type (r, s)

- Then for a general tensor we have:

$$T = T^{i_1 \dots i_r}_{j_1 \dots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \theta^{j_1} \otimes \cdots \otimes \theta^{j_s} = T'^{i'_1 \dots i'_r}_{j'_1 \dots j'_s} e'_{i'_1} \otimes \cdots \otimes e'_{i'_r} \otimes \theta'^{j'_1} \otimes \cdots \otimes \theta'^{j'_s}$$

- Then we can see (exercise):

$$T'^{i'_1 \dots i'_r}_{j'_1 \dots j'_s} = A^{i'_1}_{i_1} \dots A^{i'_r}_{i_r} (A^{-1})^{j_1}_{j'_1} \dots (A^{-1})^{j_s}_{j'_s} T^{i_1 \dots i_r}_{j_1 \dots j_s}$$

Index Notation, a physicist's way of looking at tensors!

- Physicists simply define tensors as objects that transform the way we explained in the last slide. This definition is practical but imprecise and one should always keep in mind what one really means by a tensor.

Operations on tensors

- **Contraction**: The process of tensor product creates tensors of higher degree from those of lower degrees, We now describe an operation that lowers the degree of tensor.
- Firstly, consider a mixed tensor $T = T^i_j \, e_i \otimes \varepsilon^j$ of type (1, 1). Its contraction is defined to be a scalar denoted $C^1_1 T$, given by

$$C^1_1 T = T\,(\varepsilon^i, e_i) = T\,(\varepsilon^1, e_1) + \cdots + T\,(\varepsilon^n, e_n).$$

Operations on tensors

- Although a basis of V and its dual basis have been used in this definition, it is independent of the choice of basis

$$T\left(\epsilon'^{i'}, e'_{i'}\right) = A^{i'}_{i} A^{-1k}_{i'} T\left(\epsilon^{i}, e_{k}\right) = T(\epsilon^{i}, e_{i})$$

- More generally, for a tensor T of type (r,s) with both r > 0 and s > 0 one can define its (p, q)-contraction (1 ≤ p ≤ r, 1 ≤ q ≤ s) to be the tensor $C^p_q T$ of type (r − 1,s − 1) defined by

$$\left(C^p_q T\right)(w^1, \ldots, w^{r-1}, v_1, \ldots, v_{s-1})$$
$$= \sum_{k=1}^{n} T\left(w^{1,\ldots,} w^{p-1}, \epsilon^k, w^{p+1}, \ldots, w^{r-1}, v_1, \ldots, v_{q-1}, e_k, v_{q+1}, \ldots, v_{s-1}\right).$$

Operations on tensors

- One can then easily find the contracted components of a tensor,

$$\left(C_q^p T\right)^{i_1 \dots i_{r-1}}_{j_1 \dots j_{s-1}} = T^{i_1 \dots i_{p-1} k \, i_{p+1} \dots i_{r-1}}_{j_1 \dots j_{q-1} k \, j_{q+1} \dots j_{s-1}}$$

Operations on tensors

- **Raising and lowering indices:** Finally we define one last notion which is widely used in relativity. Let V be a real inner product space with metric tensor $g = g_{ij}\, \varepsilon^i \otimes \varepsilon^j$, then one can see:

$$u.v = g_{ij}u^i u^j = C_1^1 C_2^2 g \otimes u \otimes v$$
$$u_i = g_{ij}u^j = C_2^1(g \otimes u)$$
$$w^i = g^{ij}w_j = C_1^2(g^{-1} \otimes w)$$