به نام خدا

پاسخنامه تمرین سری ۷ نسبیت خاص :

سوال اول :

$$S = \int d^4x \, \mathcal{L}(x)$$

معادله حرکت : $\delta S = 0 \longrightarrow \dfrac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$

$$\mathcal{L} = -\frac{1}{2}\phi \Box \phi + \frac{1}{2} m \phi^2 - \frac{\lambda}{4!}\phi^4$$

$$\int (\phi \Box \phi)\, d^4x = \int d^4x \, \phi (\partial_\mu \partial^\mu)\phi = \int d^4x \left(\partial_\mu(\phi \partial^\mu \phi) - \partial_\mu \phi \partial^\mu \phi \right)$$

$$= \underbrace{\cancel{\int d^4x \, \partial_\mu (\phi \partial^\mu \phi)}}_{0} - \int d^4x \, (\partial_\mu \phi)(\partial^\mu \phi)$$

$$\longrightarrow \mathcal{L} = +\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2} m \phi^2 - \frac{\lambda}{4!}\phi^4$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = m\phi - \frac{\lambda}{3!}\phi^3$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu (\partial^\mu \phi) = \Box \phi \longrightarrow \boxed{\Box \phi = m\phi - \frac{\lambda}{3!}\phi^3}$$

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - A_\mu J^\mu \quad ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\text{E.M}: \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = m^2 A^\nu - J^\nu$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = -\frac{1}{4}\partial_\mu \left(\frac{\partial}{\partial(\partial_\mu A_\nu)} (F^{\alpha\beta} F_{\alpha\beta}) \right) = -\frac{2}{4}\partial_\mu \left(F^{\alpha\beta} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} \right)$$

$$= -\frac{1}{2}\partial_\mu \left(F^{\alpha\beta} \left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha \right) \right) = -\frac{1}{2}\partial_\mu \left(\underbrace{F^{\mu\nu} - F^{\nu\mu}}_{2F^{\mu\nu}} \right)$$

$$= -\partial_\mu F^{\mu\nu}$$

$$\longrightarrow \boxed{\partial_\mu F^{\mu\nu} + m^2 A^\nu - J^\nu = 0} \xrightarrow{\partial_\mu A^\mu = 0} \Box A^\nu + m^2 A^\nu = J^\nu$$

سوال دوم :

قانون پیوستگی در سیستم $\longrightarrow \partial_\mu J^\mu = 0 \quad : \quad J^\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha}$

$U(1) : \quad \phi \longrightarrow e^{i\alpha} \phi$

الف) ϕ : حقیقی $\qquad \mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m \phi^2$

$e^{i\alpha} = \pm 1 \longrightarrow$ تقارن گسسته

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \underbrace{\frac{\delta \phi}{\delta \alpha}}_{i\phi} = (\partial^\mu \phi)\, i\phi \longrightarrow \partial_\mu J^\mu \neq 0$$

ب) $\phi , \phi^* \qquad \mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m \phi^* \phi$

$$\frac{\delta \phi}{\delta \alpha} = i\phi \qquad \frac{\delta \phi^*}{\delta \alpha} = -i\phi^*$$

$$J^\mu = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}}_{\partial^\mu \phi^*} \frac{\delta \phi}{\delta \alpha} + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}}_{\partial^\mu \phi} \frac{\delta \phi^*}{\delta \alpha} = i\phi\, \partial^\mu \phi^* - i\phi^* \partial^\mu \phi$$

$$J^\mu = i(\phi\, \partial^\mu \phi^* - \phi^* \partial^\mu \phi)$$

$$\partial_\mu J^\mu = i\, \partial_\mu(\phi\, \partial^\mu \phi^*) - i\, \partial_\mu(\phi^* \partial^\mu \phi)$$
$$= i\, \cancel{\partial_\mu \phi\, \partial^\mu \phi^*} + i\phi \Box \phi^* - i\, \cancel{\partial_\mu \phi^* \partial^\mu \phi} - i\phi^* \Box \phi$$
$$= i(\phi \Box \phi^* - \phi^* \Box \phi)$$

معادله حرکت : $\quad \Box \phi = -m\phi \qquad \partial_\mu J^\mu = i(-m\phi\phi^* + m\phi\phi^*) = 0$

$\Box \phi^* = -m\phi^*$

$\partial_\mu J^\mu = 0$ ✓

سوال سوم:

$$\begin{cases} E = -\nabla\phi - \dfrac{\partial A}{\partial t} \\ B = \nabla\times A \end{cases} \qquad \begin{cases} \nabla\cdot E = \rho \\ \nabla\times B = J + \dfrac{\partial E}{\partial t} \end{cases}$$

الف/ $\nabla\cdot E = \rho \longrightarrow \nabla\cdot E = \nabla\cdot\left(-\nabla\phi - \dfrac{\partial A}{\partial t}\right) = -\nabla^2\phi - \dfrac{\partial}{\partial t}(\nabla\cdot A) = \dfrac{\partial^2\phi}{\partial t^2} - \nabla^2\phi$

$= \Box\phi \longrightarrow \Box\phi = \rho$

$$\nabla\times B - \frac{\partial E}{\partial t} = \nabla\times(\nabla\times A) - \frac{\partial}{\partial t}\left(-\nabla\phi - \frac{\partial A}{\partial t}\right) = \nabla(\cancel{\nabla\cdot A}) - \nabla^2 A + \nabla\Big(\underbrace{\cancel{\frac{\partial\phi}{\partial t}}}_{-\nabla\cdot A}\Big) + \frac{\partial^2 A}{\partial t^2}$$

$$= \frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \Box A \longrightarrow \Box A = J$$

$$\begin{cases} \Box\phi = \rho \\ \Box A = J \end{cases} \longrightarrow \Box A^\mu = J^\mu \longrightarrow \partial_\alpha\partial^\alpha A^\mu = J^\mu$$

ب/ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\partial_\nu F^{\nu\mu} = \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) = \Box A^\mu - \partial^\mu(\cancel{\partial_\nu A^\nu}) = \Box A^\mu$$

$$\Box A^\mu = J^\mu \longrightarrow \partial_\alpha F^{\alpha\mu} = J^\mu$$

ج/ $\mathcal{L}_{em} = -\frac{1}{4} F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu$

معادله حرکت: $\dfrac{\partial\mathcal{L}_{em}}{\partial A_\nu} - \partial_\mu\left(\dfrac{\partial\mathcal{L}_{em}}{\partial(\partial_\mu A_\nu)}\right) = 0 \xrightarrow[m=0]{\text{اگر در لاگرانژی سوال اول}} \Box A^\nu = J^\nu$

$$\partial_\mu F^{\mu\nu} = J^\nu$$

$F^{\mu\nu}F_{\mu\nu}$:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \qquad F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$= 2(-E_1^2 - E_2^2 - E_3^2 + B_1^2 + B_2^2 + B_3^2) = 2(|\vec{B}|^2 - |\vec{E}|^2)$$

$$F^{\mu\nu}F_{\mu\nu} = 2(B^2 - E^2)$$

$$J^\mu = 0 \longrightarrow \mathcal{L}_{em} = -\frac{1}{4} F^{\mu\nu}F_{\mu\nu}$$

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\nu)} \partial_0 A_\nu - \mathcal{L} \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$$

$$\mathcal{H} = -F^{0\nu}(\partial_0 A_\nu) - \mathcal{L} = -F^{0\nu}(\partial_0 A_\nu) + \frac{1}{4}\underbrace{F^{\mu\nu}F_{\mu\nu}}_{2(B^2-E^2)}$$

$$F^{0\nu} = (0, -E_1, -E_2, -E_3)$$

$$\partial_0 A_\nu = (\dot{\phi}, -\partial_0 A_i) \longrightarrow F^{0\nu}(\partial_0 A_\nu) = \dot{\phi}(0) + E_i \partial_0 A_i = E_i \dot{A}_i$$

$$\vec{E} = -\nabla\phi - \frac{\partial A}{\partial t} \longrightarrow E_i = -\partial_i \phi - \dot{A}_i \longrightarrow \dot{A}_i = -\partial_i\phi - E_i$$

$$\longrightarrow F^{0\nu}(\partial_0 A_\nu) = E_i(-\partial_i\phi - E_i) = -E^2 - E_i\partial_i\phi$$

$$\mathcal{H} = -F^{0\nu}(\partial_0 A_\nu) + \frac{1}{2}(B^2 - E^2) = E^2 + E_i\partial_i\phi + \frac{1}{2}(B^2 - E^2)$$

$$= \frac{1}{2}(B^2 + E^2) + E_i\partial_i\phi$$

$$\boxed{\mathcal{H}_{em} = \frac{1}{2}(B^2 + E^2) + E_i\partial_i\phi}$$

Scanned by CamScanner