CMB Temperature Anisotropy

Shant Baghram
23 June 2013

I. THE CMB POWER SPECTRUM

The Cosmic Microwave Background (CMB) radiation has an anisotropy of the order of $\mathcal{O}(10^{-5})$ in temperature. The statistics of this anisotropy is so wealthy in information that they were, are and will be many observations to detect this anisotropy originated on the last scattering surface of photons. The temperature anisotropy is a function of position $\mathbf{x}$, time t and the direction of the photons $\hat{n}$ reaching us:

$$\frac{\delta T}{T} = \frac{\delta T}{T}(\mathbf{x}, t, \hat{n}) \tag{1}$$

where T is the average density of CMB. We set $\mathbf{x} = \mathbf{x}_0$ and $t = t_0$ as in the present time and we mainly suppress this space and time dependence. On the other hand as the direction of the photons is defined in a 2D sphere $\hat{n}\epsilon S^2$, where we can expand the temperature anisotropy in terms of spherical harmonics $Y_{\ell m}$.

$$\frac{\delta T}{T}(\mathbf{x}_0, t_0, \hat{n}) = \sum_{\ell m} a_{\ell m}(\mathbf{x}_0) Y_{\ell m}(\hat{n}) \tag{2}$$

where the physics is imprinted in $a_{\ell m}$s and the $Y_{\ell m}$ are the basis of space where we expand the temperature anisotropy. In the case of statistical homogeneity, instead of computing the ensemble of realizations (where we can not calculate the ensemble average because we have only one observable Universe) we have an expectation value for the coefficients of $Y_{\ell m}$s by changing the direction we are looking to photons.

$$\langle a_{\ell m}.a^*_{\ell' m'} \rangle = \delta_{\ell\ell'}\delta_{mm'}C_\ell \tag{3}$$

where C_ℓ is the CMB anisotropy power spectrum, where now is just a function of moment ℓ. This moments is related to the angle (θ), between the two directions of observed photons as $\ell = \pi/\theta$ Now we can find the two-point function of the temperature anisotropy as:

$$\langle \frac{\delta T}{T}(\hat{n})\frac{\delta T}{T}(\hat{n}')\rangle_{\hat{n}.\hat{n}'\equiv\mu} = \sum_{\ell\ell' mm'} \langle a_{\ell m}.a^*_{\ell' m'}\rangle Y_{\ell m}(\hat{n}) Y^*_{\ell' m'}(\hat{n}') \tag{4}$$

Now by substituting Eq.(3) in equation above, we will have:

$$\langle \frac{\delta T}{T}(\hat{n})\frac{\delta T}{T}(\hat{n}')\rangle_{\hat{n}.\hat{n}'\equiv\mu} = \sum_{\ell} C_\ell \sum_{m=-\ell}^{m=+\ell} Y_{\ell m}(\hat{n}) Y^*_{\ell' m'}(\hat{n}') \tag{5}$$

Now we can use the theorem of spherical harmonics to convert their multiplication to Legendre function:

$$\sum_{m=-\ell}^{m=+\ell} Y_{\ell m}(\hat{n}) Y^*_{\ell' m'}(\hat{n}') = \frac{2\ell+1}{4\pi} P_\ell(\hat{n}.\hat{n}') \tag{6}$$

where P_ℓ is the Legendre function. Consequently the two point correlation function of temperature anisotropy is related to CMB power spectrum as:

$$\langle\frac{\delta T}{T}(\hat{n})\frac{\delta T}{T}(\hat{n}')\rangle_{\hat{n}.\hat{n}'\equiv\mu} = \frac{1}{4\pi}\sum_\ell(2\ell+1)C_\ell P_l(\mu) \tag{7}$$

The most dominant effect in CMB temperature anisotropy is the Sachs-Wolfe effect, where:

$$\frac{\Delta T}{T}(x_0,\hat{n},t_0) \simeq \frac{1}{3}\Psi(x_{dec},t_{dec}), \tag{8}$$

Ψ is the gravitational potential in time of decoupling. (The definition of gravitational potential and its gauge invariant version is discussed in perturbation theory) and the decoupling comoving distance is:

$$\mathbf{x}_{dec} = \mathbf{x}_0 + \hat{n}(t_0 - t_{dec}) \tag{9}$$

Now the Fourier transform of temperature anisotropy in terms of Fourier mode of gravitational potential is:

$$\frac{\Delta T}{T}(\mathbf{k},\hat{n},t_0) \simeq \frac{1}{3}\Psi(\mathbf{k},t_{dec})e^{i\mathbf{kn}(t_0-t_{dec})} \tag{10}$$

Now we are going to compute the the two point function of temperature anisotropy and translate it in Fourier space.

$$\begin{aligned}\langle\ & \frac{\delta T}{T}(\mathbf{x}_0,\hat{n},t_0)\frac{\delta T}{T}(\mathbf{x}_0,\hat{n}',t_0)\rangle_{\hat{n}.\hat{n}'\equiv\mu} = \frac{1}{(2\pi)^6}\int d^3kd^3k'e^{i\mathbf{x}_0(\mathbf{k}-\mathbf{k}')}\langle\frac{\delta T}{T}(\mathbf{k},\hat{n},t_0)\frac{\delta T^*}{T}(\mathbf{k},\hat{n}',t_0)\rangle \\ = & \frac{1}{9(2\pi)^6}\int d^3kd^3k'e^{i\mathbf{x}_0(\mathbf{k}-\mathbf{k}')}\langle\Psi(\mathbf{k})\Psi^*(\mathbf{k}')\rangle e^{i\mathbf{kn}(t_0-t_{dec})}e^{-i\mathbf{k}'\mathbf{n}'(t_0-t_{dec})} \\ = & \frac{1}{9(2\pi)^6}\int d^3kd^3k'e^{i\mathbf{x}_0(\mathbf{k}-\mathbf{k}')}\langle\Psi(\mathbf{k})\Psi^*(\mathbf{k}')\rangle\sum_{\ell\ell'=0}^{\infty}(2\ell+1)(2\ell'+1) \\ \times\ & i^{\ell-\ell'}j_\ell(k(t_0-t_{dec}))j_{l'}(k'(t_0-t_{dec}))P_\ell(\hat{k}.\hat{n})P_{\ell'}(\hat{k}'.\hat{n}')\end{aligned} \tag{11}$$

where we have used the relation of plane wave with Legendre polynomial as:

$$e^{i\mathbf{kn}(t_0-t_{dec})} = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell j_\ell(k(t_0-t_{dec}))P_\ell(\hat{k}.\hat{n}) \tag{12}$$

where $k\equiv|\mathbf{k}|$ and $\hat{k}\equiv\frac{\mathbf{k}}{k}$ Now we know that the gravitational potential two point function in Fourier space is related to the matter power spectrum. as:

$$\langle\Psi(\mathbf{k})\Psi^*(\mathbf{k}')\rangle \equiv (2\pi)^3\delta^{3D}(\mathbf{k}-\mathbf{k}')P_\Psi(\mathbf{k}) \tag{13}$$

By the Dirac delta function appeared in the definition of power spectrum the integral in Eq. (11), can be calculated as:

$$\begin{aligned}\langle\frac{\delta T}{T}(\mathbf{x}_0,\hat{n},t_0)\frac{\delta T}{T}(\mathbf{x}_0,\hat{n}',t_0)\rangle &= \frac{1}{9(2\pi)^3}\int d^3kP_\Psi(\mathbf{k})\sum_{\ell\ell'=0}^{\infty}(2\ell+1)(2\ell'+1)i^{\ell-\ell'} \\ &\times\ j_\ell(k(t_0-t_{dec}))j_{l'}(k(t_0-t_{dec}))P_\ell(\hat{k}.\hat{n})P_{\ell'}(\hat{k}.\hat{n}')\end{aligned} \tag{14}$$

Now we can expand the Legendre polynomials in terms of spherical harmonics:

$$P_\ell(\hat{k}.\hat{n}) = \frac{4\pi}{2\ell+1}\sum_m Y^*_{\ell m}(\hat{k})Y_{\ell m}(\hat{n}) \tag{15}$$
$$P_{\ell'}(\hat{k}.\hat{n}') = \frac{4\pi}{2\ell'+1}\sum_{m'} Y^*_{\ell' m'}(\hat{k})Y_{\ell' m'}(\hat{n}')$$

Now as the integral over momentum is $\int d^3k \equiv \int k^2 dk d\Omega_{\hat{k}}$ and the orthogonality condition:

$$\int d\Omega_{\hat{k}} Y^*_{\ell m}(\hat{k})Y_{\ell' m'}(\hat{k}') = \delta_{\ell\ell'}\delta_{mm'} \tag{16}$$

and also the relation between the spherical harmonics and Legendre function is:

$$\sum_m Y^*_{\ell m}(\hat{n})Y_{\ell' m'}(\hat{n}') = \frac{2\ell+1}{4\pi}P_\ell(\mu) \tag{17}$$

where $\mu = \hat{n}.\hat{n}'$. Finally using the orthogonality relation and the exchange between the spherical harmonics with Legendre polynomials the two point function of temperature anisotropy becomes:

$$\langle\frac{\delta T}{T}(\mathbf{x}_0, \hat{n}, t_0)\frac{\delta T}{T}(\mathbf{x}_0, \hat{n}', t_0)\rangle_{\hat{n}.\hat{n}'\equiv\mu} \tag{18}$$
$$\simeq \sum_\ell \frac{2\ell+1}{4\pi}P_\ell(\mu)\frac{2}{\pi}\int\frac{dk}{k}\frac{1}{9}P_\psi(k)k^3 j^2_\ell(k(t_0 - t_dec))$$

Now by combing the Eqs.(3,18) we will find the CMB anisotropy power spectrum as:

$$C_\ell \simeq \frac{2}{9\pi}\int_0^\infty \frac{dk}{k}P_\Psi(\mathbf{k})k^3 j^2_\ell(k(t_0 - t_{dec})) \tag{19}$$

The equation above shows that how we can relate the CMB anisotropy power spectrum to the gravitational potential power spectrum. On the other hand the oscillatory behavior of CMB power spectrum is induced via the square of the bessel function.