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Robust Control over Uncertain Communication Channels

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Abstract— **In this paper, the mathematical framework for studying robust control over uncertain communication channels is introduced. The theory is developed by generalizing the classical information theoretic measures of information and the fundamental theory of Shannon to the robust analog, which are subject to uncertainty in the source and the communication channel. By invoking this mathematical framework, necessary conditions are presented for robust stabilizability and observability of fully-observed, finite dimensional, discrete-time invariant, noiseless uncertain linear systems over uncertain communication channels.**

I. INTRODUCTION

One of the issues that has begun to emerge in a number of applications, such as sensor networking, large scale teleoperation, and etc., is how to control systems by communicating information reliably, through limited capacity channels, when the subsystems are subject to uncertainty. Typical examples are applications in which a single dynamical system sends feedback information to a distant controller via a communication link with finite capacity. In the absence of uncertainty in the controlled system and the communication channel, important results are derived in [1]-[8]. Specially, the aim of these articles is to find necessary and sufficient conditions for stabilizability, when there are channel capacity and power constraints. For finite dimensional discrete-time invariant linear systems, it is shown that the transmission data rate (or channel capacity) required to stabilize a controlled system must be at least equal to the sum of the logarithms of the magnitude of the unstable open-loop eigenvalues.

The objective of this paper is to address similar questions, when there is uncertainty in the controlled system and communication link. In particular, to find necessary conditions on the channel capacity which ensure robust observability and stabilizability. The necessary steps in realizing such a study consists of the followings.

1. Give precise definitions of entropy, channel capacity, and rate distortion, when the communication blocks are subject to uncertainty.
2. Extend the fundamental theory of Shannon to communication systems subject to uncertainty.
3. Derive necessary and sufficient conditions on the communication blocks which are subject to uncertainty in order to ensure robust observability and stabilizability of the controlled system.

Clearly, the first two questions (above) are addressed by generalizing information theory to robust information theory which consists of the robust version of the classical source coding, channel coding, and rate distortion to their robust analog, which are subject to uncertainty. Then we show that the robust channel capacity of a communication link must be at least equal to the robust entropy of the source in order to ensure reliable communication. Subsequently, we find necessary condition for robust observability and stabilizability of uncertain plants over uncertain communication channels.

In Section II, the precise notion of the robust communication system, and the corresponding information theoretic measures, which are necessary to analyze such systems are introduced. One of the fundamental results which is required to address issue 3 above is the derivation of a lower bound for robust rate distortion which is found in Section III. Furthermore, in this section a robust version of the information transmission theorem is introduced. This theorem provides an upper bound for robust rate distortion in terms of the robust channel capacity. In Section IV, a necessary condition for robust observability and stabilizability is derived for fully observed, finite dimensional, discrete-time invariant, noiseless uncertain linear systems over uncertain channels.

II. ROBUST COMMUNICATION SYSTEMS

A. Communication System

Let $(\Omega, \mathcal{F}(\Omega))$ denote a measurable space in which $\mathcal{F}(\Omega)$ is the σ-field generated by Ω, and let $\mathcal{M}_1(\Omega)$ be the set of countable additive probability measure on $(\Omega, \mathcal{F}(\Omega))$. In general Ω is assumed to be the space of function $\Phi(t) \in \Omega$, where the argument t takes values within specified interval or is identified with discrete series of values, so that the variable $\Phi(t)$ may be treated as a continuous or discrete-time random process.

Consider the communication diagram given in Fig. 1. Here $(\mathcal{X}, \mathcal{F}(\mathcal{X}))$ is the source measurable space, and $(\widetilde{\mathcal{X}}, \mathcal{F}(\widetilde{\mathcal{X}}))$ is the source reproduction measurable space. The channel input and output measurable spaces are $(\mathcal{Z}, \mathcal{F}(\mathcal{Z}))$ and $(\widetilde{\mathcal{Z}}, \mathcal{F}(\widetilde{\mathcal{Z}}))$, respectively.

Information Source An information source is often specified by the probability measure $P_X : \mathcal{F}(\mathcal{X}) \rightarrow [0,1]$ induced by the source on $(\mathcal{X}, \mathcal{F}(\mathcal{X}))$(e.g., $P_X \in \mathcal{M}_1(\mathcal{X})$). In general, the source is uncertain, that is, P_X is unknown but belongs to the uncertainty set $P_X \in \mathcal{M}_{SU} \subset \mathcal{M}_1(\mathcal{X})$.

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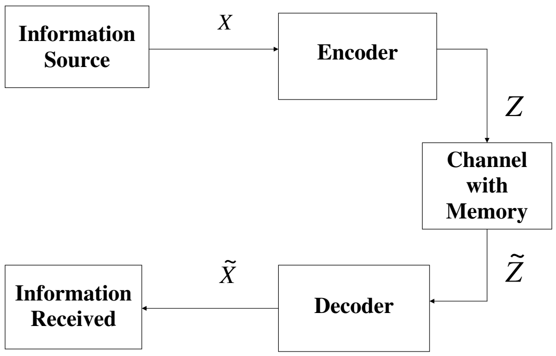

Fig. 1. Block diagram of communication system

Communication Channel. A Communication channel is a probabilistic mapping

$$P_{\widetilde{Z}|Z}(B,z) \stackrel{\triangle}{=} \Pr(\widetilde{Z} \in B|Z=z), B \in \mathcal{F}(\widetilde{\mathcal{Z}}), z \in \mathcal{Z},$$
$$P_{\widetilde{Z}|Z} : \mathcal{F}(\widetilde{Z}) \times \mathcal{Z} \rightarrow [0,1], \quad (1)$$

which satisfies the following conditions:
1)For every $B \in \mathcal{F}(\widetilde{\mathcal{Z}})$, the function $P_{\widetilde{Z}|Z}(B,.)$ is an $\mathcal{F}(\mathcal{Z})$-measurable function.
2) For every $z \in \mathcal{Z}$, the set function $P_{\widetilde{Z}|Z}(.,z)$ is a probability measure on $\mathcal{F}(\widetilde{\mathcal{Z}})$.
A mapping which satisfies 1, and 2 is called a stochastic kernel, and clearly, $P_{\widetilde{Z}|Z}(.,z) \in \mathcal{M}_1(\widetilde{\mathcal{Z}}),\ \forall z \in \mathcal{Z}$.
Since the channel in general is uncertain, the stochastic kernels $P_{\widetilde{Z}|Z}$ belongs to the uncertainty set $P_{\widetilde{Z}|Z} \in \mathcal{M}_{CU} \subset \mathcal{M}_1(\widetilde{\mathcal{Z}})$. Moreover, the channel could be with memory (e.g., the output of the channel is dependent on the past outputs or inputs to the channel, and/ or the channel has feedback).
Encoder. An encoder is a stochastic kernel

$$P_{Z|X}(A,x) = \Pr(Z \in A|X=x), A \in \mathcal{F}(\mathcal{Z}), x \in \mathcal{X}. \quad (2)$$

Decoder. A decoder is a stochastic kernel

$$P_{\widetilde{X}|\widetilde{Z}}(C,\widetilde{z}) = \Pr(\widetilde{X} \in C|\widetilde{Z}=\widetilde{z}), C \in \mathcal{F}(\widetilde{\mathcal{X}}), \widetilde{z} \in \widetilde{\mathcal{Z}}. \quad (3)$$

Deterministic encoders and decoders correspond to delta measures and hence they follow from (2) and (3). The definition of encoder, channel and decoder as stochastic kernel implies that $\mathcal{X} \rightarrow \mathcal{Z} \rightarrow \widetilde{\mathcal{Z}} \rightarrow \widetilde{\mathcal{X}}$ forms a Markov chain.
The above construction implies that the probability measure induced by the input of the channel on $(\mathcal{Z}, \mathcal{F}(\mathcal{Z}))$ can be defined through the Radon-Nikodym derivative

$$P_Z(A) = \int P_{Z|X}(A,x)dP_X(x), \forall A \in \mathcal{F}(\mathcal{Z}), x \in \mathcal{X},$$
$$\text{(or in compact notation): } P_Z = P_{Z|X} \otimes P_X. \quad (4)$$

Often, it is necessary to impose certain limitation on the input to the channel (such as average channel input power). These kinds of limitation are introduced by assuming that the probability measure corresponding to the channel input measurable space $(\mathcal{Z}, \mathcal{F}(\mathcal{Z}))$ belongs to a smaller class $P_Z \in \mathcal{M}_{CI} \subset \mathcal{M}_1(\mathcal{Z})$. Finally, let $P_{\widetilde{X}}$ denote the reproduction source probability measure.

B. Robust Information Theoretic Measures

In this section, we introduce robust entropy of the source, robust channel capacity and the robust rate distortion. We also extend the fundamental theory of Shannon to the robust analog which are subject to uncertainty in communication blocks.
The robust definition of information theoretic measures are given using relative entropy between two measures which is defined below.

Definition 2.1: (Relative Entropy) The relative entropy of two probability measures π and ν on $(\Omega, \mathcal{F}(\Omega))$ is defined by

$$H(\pi|\nu) \stackrel{\triangle}{=} \quad (5)$$
$$\begin{cases} \int \log \frac{d\pi(x)}{d\nu(x)} d\pi(x) & if \quad \pi << \nu, \ \log \frac{d\pi(x)}{d\nu(x)} \in L^1(\pi) \\ \infty & if \quad otherwise \end{cases}$$

where " $<<$ " denotes absolute continuity of measures.
Next, by invoking the relative entropy, the mutual information is defined as follow.

Definition 2.2: (Mutual Information) The mutual information between two random variable M and N is defined as the relative entropy between the joint probability measure $P_{M,N}$ and the product of marginal $P_M \otimes P_N$ via

$$I(M;N) \stackrel{\triangle}{=} H(P_{M,N}|P_M \otimes P_N). \quad (6)$$

Next we define the concept of robust entropy and subsequently robust entropy rate for a family of sources. The entropy of the source represents the amount of information generated by the source symbols. The robust definition of entropy first is appeared in [9].

Definition 2.3: (Robust Entropy and Robust Entropy Rate) Let X be a random variable (or random process) with probability measure $P_X \in \mathcal{M}_1(\mathcal{X})$ representing the uncertain source outcomes in which $P_X \in \mathcal{M}_{SU}$. Let $Q_X \in \mathcal{M}_1(\mathcal{X})$ be a fixed measure. Then the robust entropy of X with respect to Q_X is defined by

$$H_{robust}(P_X^*) = \sup_{P_X \in \mathcal{M}_{SU}} -H(P_X|Q_X), \quad (7)$$

where $P_X^* = argsup_{P_X \in \mathcal{M}_{SU}} - H(P_X|Q_X)$.
Subsequently, if $X = (x_0, ..., x_{T-1})$ represents a sequence with length T of source symbols produced by an uncertain source, the robust entropy rate is defined by

$$\mathcal{H}_{robust}(\mathcal{X}) = \lim_{T \rightarrow \infty} \frac{1}{T} H_{robust}(P_X^*), \quad (8)$$

if the limit exist.

Remark 2.4: When Q_X is the Lebesgue measure and there is no uncertainty in source distribution, (e.g., $\mathcal{M}_{SU} = \{M_X\}$, where M_X is the nominal source distribution), the robust entropy and the robust entropy rate is reduced to the classical Shannon entropy and Shannon entropy rate.

The importance of the robust definition of entropy can be understood in terms of the so called robust Shannon first coding theorem [10]. This theorem states the following. Sourcewords of blocklength T produced by a discrete memoryless source (e.g., a finite alphabet source with i.i.d. outcomes) with unknown source distribution P_X, $P_X \in \mathcal{M}_{SU} = \mathcal{M}_{SUR} \triangleq \{P_X \in \mathcal{M}_1(\mathcal{X}); H(P_X|M_X) \leq R_c\}$ (where M_X is nominal source distribution under which the source letters are i.i.d., and $R_c \geq 0$ controls the size of uncertainty), can be encoded into codewords of blocklength r from a coding alphabet of size k, with decoding probability p_e arbitrary small for T-sufficiently large, regardless of the true source distribution, if $\sup_{P_X \in \mathcal{M}_{SUR}} H(P_X) \leq \frac{r}{T} \log k$.

Theorem 2.5: [9] If $\mathcal{M}_{SU} = \mathcal{M}_{SUR} = \{P_X \in \mathcal{M}_1(\mathcal{X}), H(P_X|M_X) \leq R_c\}$, where M_X is the nominal source distribution and $M_X << Q_X$

$$\begin{aligned} H_{robust}(P_X^{*,s^*}) &= H_{robust}(R_c) \\ &= min_{s\geq 0}[sR_c + (1+s) \\ &\quad . \ln \int (\frac{dM_X}{dQ_X})^{-\frac{1}{1+s}} dM_X], \text{ nats}, \\ \frac{dP_X^{*,s}}{dM_X} &= \frac{(\frac{dM_X}{dQ_X})^{-\frac{1}{1+s}}}{\int (\frac{dM_X}{dQ_X})^{-\frac{1}{1+s}} dM_X}, \end{aligned} \tag{9}$$

where the minimizing $s^* \geq 0$ in (9) is the unique solution of $H(P_X^{*,s}|M_X) = R_c$.

Remark 2.6: If $X = \{x_0, ..., x_{T-1}\}$ is a sequence with length T of symbols produced by an uncertain source with distribution P_X, $\mathcal{M}_{SU} = \mathcal{M}_{SUR}$, M_X (consequently P_X)$<<$ Q_X,and Q_X is the Lebesgue measure (consequently, $\mu_X(x) = \frac{dM_X}{dQ_X}$ is the probability density function, P.D.F., and also $p_X(x) = \frac{dP_X}{dQ_X}$ is P.D.F.) from Theorem 2.5 follows that

$$\begin{aligned} &H_{robust}(TR_c) = \\ &min_{s\geq 0}[sTR_c + (1+s) \ln \int \mu_X(x)^{\frac{s}{1+s}} dx] \end{aligned} \tag{10}$$

and

$$p_X^{*,s}(x) = \frac{\mu_X(x)^{\frac{s}{1+s}}}{\int \mu_X(x)^{\frac{s}{1+s}} dx}, \quad p_X^{*,s}(x) = \frac{dP_X^{*,s}}{dQ_X} \text{ (P.D.F.)}, \tag{11}$$

where the minimizing $s^* \geq 0$ in (10) and (11) is the unique solution of $H(P_X^{*,s^*}|M_X) = TR_c$.

Corollary 2.7: [9] Under assumption of Remark 2.6 if the nominal source distribution M_X has corresponding Td-dimensional Gaussian density with mean m and variance Γ_X, $\forall R_c \in [0, \infty)$

$$\begin{aligned} \frac{1}{T} H_{robust}(TR_c) &= \frac{d}{2} \ln(\frac{1+s}{s}) + \frac{d}{2} \ln(2\pi e) \\ &\quad + \frac{1}{2T} \ln \det \Gamma_X, \end{aligned} \tag{12}$$

where $s > 0$ is the unique solution of the following nonlinear equation

$$R_c = -\frac{d}{2} \ln(\frac{1+s}{s}) + \frac{d}{2s}. \tag{13}$$

Corollary 2.8: Under assumption of Remark 2.6 when $\mu_X(x)$ and $p_X(x)$ correspond to probability mass function (P.M.F.), that is, $\mu_X(x) = \sum_i \mu_X(x_i)\delta(x_i)$ and $p_X(x) = \sum_i p_X(x_i)\delta(x_i)$, where $\delta(.)$ is delta measure, (10) and (11) are reduced to

$$\begin{aligned} \frac{1}{T} H_{robust}(TR_c) &= min_{s\geq 0}[sR_c + \frac{1+s}{T} \\ &\quad . \ln \sum_i \mu_X(x_i)^{\frac{s}{1+s}}], \end{aligned} \tag{14}$$

$$p_X^{*,s}(x_i) = \frac{\mu_X(x_i)^{\frac{s}{1+s}}}{\sum_i \mu_X(x_i)^{\frac{s}{1+s}}}, \tag{15}$$

where the minimizing $s^* \geq 0$ in (14) and (15) is the unique solution of $H(P_X^{*,s^*}|M_X) = TR_c$.

Next, we define robust channel capacity for communication channels.

Definition 2.9: (Robust Channel Capacity) In many practical applications the communication channel $P_{\widetilde{Z}|Z} : \mathcal{F}(\widetilde{\mathcal{Z}}) \times \mathcal{Z} \to [0,1]$ belongs to the set $P_{\widetilde{Z}|Z} \in \mathcal{M}_{CU} \subset \mathcal{M}_1(\widetilde{\mathcal{Z}})$, called the channel uncertainty set. For these channels, the robust channel capacity is defined by

$$C_{robust} = \sup_{P_Z \in \mathcal{M}_{CI}} \inf_{P_{\widetilde{Z}|Z} \in \mathcal{M}_{CU}} I(Z; \widetilde{Z}). \tag{16}$$

When the channel input and output measurable spaces correspond to the sequences

$$\begin{aligned} &(\mathcal{Z}, \mathcal{F}(\mathcal{Z})) = (\mathcal{Z}_{0,n-1}, \mathcal{F}_{0,n-1}^{\mathcal{Z}}) \\ &\triangleq \times_{k=0}^{n-1}(\mathcal{Z}_k, \mathcal{F}(\mathcal{Z}_k)), \quad n = 1,2,...,\infty. \\ &(\widetilde{\mathcal{Z}}, \mathcal{F}(\widetilde{\mathcal{Z}})) = (\widetilde{\mathcal{Z}}_{0,n-1}, \mathcal{F}_{0,n-1}^{\widetilde{\mathcal{Z}}}) \\ &\triangleq \times_{k=0}^{n-1}(\widetilde{\mathcal{Z}}_k, \mathcal{F}(\widetilde{\mathcal{Z}}_k)), \quad n = 1,2,...,\infty, \end{aligned} \tag{17}$$

where $(\mathcal{Z}_k, \mathcal{F}(\mathcal{Z}_k))$ and $(\widetilde{\mathcal{Z}}_k, \mathcal{F}(\widetilde{\mathcal{Z}}_k))$ are exemplars of measurable space $(\mathcal{Z}_I, \mathcal{F}(\mathcal{Z}_I))$ and $(\widetilde{\mathcal{Z}}_O, \mathcal{F}(\widetilde{\mathcal{Z}}_O))$ (which are the channel input and output measurable alphabet sets respectively), and $Z = (z_0, z_1, ..., z_{n-1})$ is an element in $\mathcal{Z}_{0,n-1}$, and similarly for an element in $\widetilde{Z}_{0,n-1}$, we have the following definition for robust channel capacity.

Definition 2.10: (Robust Channel Capacity for Sequences) When the channel is unknown but belongs to the uncertainty set $P_{\widetilde{Z}|Z} \in \mathcal{M}_{CU} \subset \mathcal{M}_1(\widetilde{\mathcal{Z}}_{0,n-1})$, the robust channel capacity is defined by

$$\begin{aligned} C_{robust} &= \lim_{n\to\infty} \frac{1}{n} C_{n,robust}, \\ C_{n,robust} &= \sup_{P_Z \in \mathcal{M}_{CI}} \inf_{P_{\widetilde{Z}|Z} \in \mathcal{M}_{CU}} I(Z; \widetilde{Z}). \end{aligned} \tag{18}$$

The above definitions for robust channel capacity are so called information definitions for channel capacity. The importance of these definitions can be understood in terms of the robust Shannon second coding theorem [11] which relates the information channel capacity to the maximum transmission data rate for reliable transmission known as operational channel capacity. This theorem states that for uncertain additive white Gaussian channels a transmission data rate is achievable (e.g., there exits a sequence of $(2^{nR}, n)$ code with maximum probability of decoding error $\lambda^{(n)} \to 0$, uniformly over all uncertain channel models) if and only if $R \leq C_{robust}$.

Next we proceed by defining the robust rate distortion. This is a measure of the minimum rate under which an end to end transmission with distortion up to distortion level D is possible. This definition first appeared in [12].

Definition 2.11: (Robust Rate Distortion) Let $\mathcal{M}_{DC} = \{Q_{\widetilde{X}|X}; \int_{\mathcal{X}\times\widetilde{\mathcal{X}}} \rho(x,\widetilde{x}) dQ_{\widetilde{X}|X} dP_X \leq D\}$ be the set of distortion constraints, in which $D \geq 0$ is the distortion level and $\rho : \mathcal{X} \times \widetilde{\mathcal{X}} \to [0,\infty)$ is the distortion measure. The robust rate distortion is defined by

$$\begin{aligned} &R_{robust}(D) = \inf_{Q_{\widetilde{X}|X} \in \mathcal{M}_{DC}} \sup_{P_X \in \mathcal{M}_{SU}} I(X;\widetilde{X}) \\ =& \inf_{Q_{\widetilde{X}|X} \in \mathcal{M}_{DC}} \sup_{P_X \in \mathcal{M}_{SU}} H(P_X \otimes Q_{\widetilde{X}|X} | P_X \otimes P_{\widetilde{X}}). \end{aligned} \tag{19}$$

When the source and reproduction measurable spaces correspond to sequences

$$\begin{aligned} &(\mathcal{X}, \mathcal{F}(\mathcal{X})) = (\mathcal{X}_{0,T-1}, \mathcal{F}^{\mathcal{X}}_{0,T-1}) \\ &\triangleq \times_{k=0}^{T-1} (\mathcal{X}_k, \mathcal{F}(\mathcal{X}_k)), \qquad T = 1,2,...,\infty. \\ &(\widetilde{\mathcal{X}}, \mathcal{F}(\widetilde{\mathcal{X}})) = (\widetilde{\mathcal{X}}_{0,T-1}, \mathcal{F}^{\widetilde{\mathcal{X}}}_{0,T-1}) \\ &\triangleq \times_{k=0}^{T-1} (\widetilde{\mathcal{X}}_k, \mathcal{F}(\widetilde{\mathcal{X}}_k)), \qquad T = 1,2,...,\infty, \end{aligned} \tag{20}$$

where $(\mathcal{X}_k, \mathcal{F}(\mathcal{X}_k))$ and $(\widetilde{\mathcal{X}}_k, \mathcal{F}(\widetilde{\mathcal{X}}_k))$ are exemplars of measurable space $(\mathcal{X}_S, \mathcal{F}(\mathcal{X}_S))$ and $(\widetilde{\mathcal{X}}_R, \mathcal{F}(\widetilde{\mathcal{X}}_R))$ (which are the source and reproduction measurable alphabet sets respectively), and $X = (x_0, x_1, ..., x_{T-1})$ is an element in $\mathcal{X}_{0,T-1}$, and similarly for an element in $\widetilde{\mathcal{X}}_{0,T-1}$, we have the following definition for robust rate distortion.

Definition 2.12: (Robust Rate Distortion for Sequences) When the true probability measure of the source sequences belongs to the uncertainty set $P_X \in \mathcal{M}_{SU} \subset \mathcal{M}_1(\mathcal{X}_{0,T-1})$, the robust rate distortion is defined by

$$\begin{aligned} &R_{robust}(D) = \lim_{T\to\infty} \frac{1}{T} R_{T,robust}(D), \; R_{T,robust} = \\ &\inf_{Q_{\widetilde{X}|X} \in \mathcal{M}_{DC}} \sup_{P_X \in \mathcal{M}_{SU}} H(P_X \otimes Q_{\widetilde{X}|X} | P_X \otimes P_{\widetilde{X}}). \end{aligned} \tag{21}$$

Theorem 2.13: (Robust Rate Distortion) [12] Suppose $e^{s\rho} \in L_1(\widetilde{\mathcal{X}}, \mathcal{F}(\widetilde{\mathcal{X}}), P_{\widetilde{X}})$, $\forall s \in \Re$. Then the solution to the problem (19) with relative entropy constraint (e.g., $\mathcal{M}_{SU} = \mathcal{M}_{SUR} = \{P_X \in \mathcal{M}_1(\mathcal{X}); H(P_X|M_X) \leq R_c\}$, M_X is the nominal source distribution) is given by

$$\begin{aligned} R(D) = sD &+ \lambda R_c \\ &+ \lambda \log \int_{\mathcal{X}} \Big(\int_{\widetilde{\mathcal{X}}} e^{s\rho(x,\widetilde{x})} dP_{\widetilde{X}} \Big)^{-\frac{1}{\lambda}} dP_X, \end{aligned} \tag{22}$$

where $s \leq 0$ and $\lambda > 0$ are Lagrange multipliers.
Moreover the infimum is attained at

$$dP^*_X = \frac{e^{\frac{l(x)}{\lambda}} dM_X}{\int_{\mathcal{X}} e^{\frac{l(x)}{\lambda}} dM_X}, \tag{23}$$

$$l(x) = \int_{\widetilde{\mathcal{X}}} \log\Big(e^{-s\rho(x,\widetilde{x})} \frac{dQ^*_{\widetilde{X}|X}}{dP_{\widetilde{X}}} \Big) dQ^*_{\widetilde{X}|X}, \tag{24}$$

and the supremum is attained at

$$dQ^*_{\widetilde{X}|X} = \frac{e^{s\rho(x,\widetilde{x})} dP_{\widetilde{X}}}{\int_{\widetilde{\mathcal{X}}} e^{s\rho(x,\widetilde{x})} dP_{\widetilde{X}}}. \tag{25}$$

The importance of the robust rate distortion can be understood in terms of the robust Shannon third coding theorem [12]. This theorem considers an uncertain discrete memoryless source $(\mathcal{X}, \mathcal{F}(\mathcal{X}), P_X)$, where the source distribution P_X belongs to the relative entropy uncertainty set $P_X \in \mathcal{M}_{SUR}$ (in this theorem, it is assumed that under the nominal source distribution, the source letters are i.i.d.). This theorem states that there exits a D-admissible code of blocklength T for T sufficiently large, regardless of the true source distribution if and only if $R_{robust}(D) < R$. That is, $R_{robust}(D)$ is the operational rate distortion.

III. Lower Bound for Robust Rate Distortion, Robust Information Transmission Theorem

In this section a lower bound in terms of robust entropy is obtained for robust rate distortion. Also a robust extension of information transmission theorem is presented. This theorem provides a necessary condition for end to end transmission with average distortion up to distortion level D, when the source and communication channel are subject to uncertainty.

Since the explicit expression for robust rate distortion is difficult to obtain, it is desirable to have a lower bound which is easily computed. This lower bound will be used in the next section to relate robust channel capacity to the parameters of the plant for stabilizability and observability purpose.

Lemma 3.1: (Lower Bound for Robust Rate Distortion) Assume the true unknown source distribution is absolutely continuous with respect to the Lebesgue measure and $P_X \in \mathcal{M}_{SU}$. Let also $\rho(x,\widetilde{x}) = \rho(x - \widetilde{x})$. Then a lower bound for $R_{robust}(D)$ is given by

$$\begin{aligned} &R_{robust}(D) \geq \\ &\quad \sup_{P_X \in \mathcal{M}_{SU}} -H(P_X|Q_X) - \max_{G \in G_D} -H(G|Q_X), \end{aligned} \tag{26}$$

where $Q_X \in \mathcal{M}_1(\mathcal{X})$ is the Lebesgue measure, $G << Q_X$ (e.g., $g(x) = \frac{dG}{dQ_X}$ is P.D.F.), and G_D is defined by

$$G_D \;=\; \{G \in \mathcal{M}_1(\mathcal{X}); g = \frac{dG}{dQ_X}, \int \rho(x) g(x) dx \leq D\}. \tag{27}$$

Moreover, if ρ is such that $\int e^{s\rho(x)}dx < \infty$ ($s < 0$), the maximum over $G \in G_D$ is attained at distribution with corresponding P.D.F $g(x)$, satisfying the following two conditions

$$g(x) = \frac{e^{s\rho(x)}}{\int e^{s\rho(x)}dx}$$
$$\int \rho(x)g(x)dx = D. \tag{28}$$

Proof: The lower bound (26) follows by generalizing the Shannon lower bound [13] to the case when the source distribution is uncertain. The detail of the proof can be found in [14].

Next, by invoking data processing inequality we derive a robust analog of the information transmission theorem. This theorem provides a necessary condition for end to end transmission up to a distortion level D, (e.g. $E\rho(X, \widetilde{X}) \leq D$), when there is uncertainty on the source as well as communication channel. In the next section, this theorem will be used to relate the robust channel capacity required for stabilizability and observability purpose to the parameters of plant.

Theorem 3.2: (Robust Information Transmission Theorem) A necessary condition for reproducing the source output X up to distortion level D by $\widetilde{X}$ at the output of the decoder, when there is uncertainty on the source and communication channel is

$$\begin{aligned} C_{robust} &= \sup_{P_Z \in \mathcal{M}_{CI}} \inf_{P_{\widetilde{Z}|Z} \in \mathcal{M}_{CU}} I(Z; \widetilde{Z}) \\ &\geq \inf_{Q_{\widetilde{X}|X} \in \mathcal{M}_{DC}} \sup_{P_X \in \mathcal{M}_{SU}} I(X; \widetilde{X}) \\ &= R_{robust}(D). \end{aligned} \tag{29}$$

Proof: [14].

In Theorem 3.2 source, reproduction, channel input and channel output spaces can be replaced by spaces correspond to sequences. The next Corollary shows that Theorem 3.2 is applicable to sequence of source and channel symbols with length T and n ($T \leq n$), while the feedback encoder and feedback decoder are employed (See Fig. 2).

Corollary 3.3: (Robust Information Transmission Theorem for Sequences) [14] A necessary condition for reproducing the source output $X = (x_0, x_1, x_2, ..., x_{T-1})$ up to distortion level D by $\widetilde{X} = (\widetilde{x}_0, \widetilde{x}_1, \widetilde{x}_2, ..., \widetilde{x}_{T-1})$ at the output of the decoder for n-times channel use ($T \leq n$), when there is uncertainty on the source and communication channel is

$$C_{n,robust} \geq R_{T,robust}(D). \tag{30}$$

Remark 3.4: Corollary 3.3 extends the result given in [8] to the case when source and communication link are subject to uncertainty.

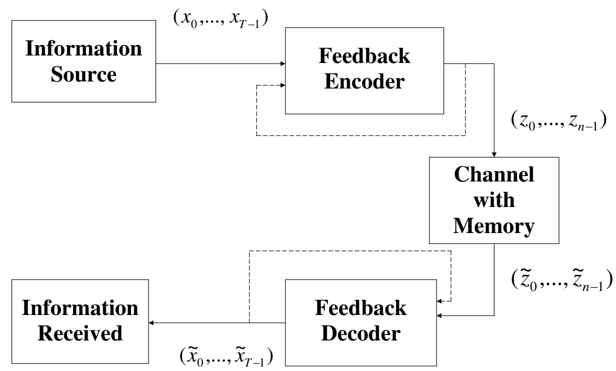

Fig. 2. Block diagram of communication system

IV. Necessary Condition for Observability and Stabilizability

In this section, we are concerned with the family of uncertain systems described by the filter probability space $(\Omega, \mathcal{F}(\Omega), \{\mathcal{F}_t\}_{t\geq 0}, \{P_X^{\Delta A}; \Delta A \in S \subset \Re^{n\times n}\})$, in which $P_X^{\Delta A}$, $\Delta A \in S$, induced by the uncertain system

$$X_{t+1} = (A + \Delta A)X_t + BU_t, \ t \geq 0, \ X_0 \in \Re^n, \tag{31}$$

where $\{X_t\}$ is $\Re^n$ valued state process, $\{U_t\}$ is a $\Re^m$-valued control process, $A \in \Re^{n\times n}$, and $B \in \Re^{n\times m}$. It is assumed that the initial position, X_0, is distributed according to the probability density p_{X_0} which has finite entropy $H(p_{X_0})$. Moreover, system (31) is assumed to be detectable and stabilizable for each $\Delta A \in S$.

In (31), ΔA is an unknown matrix, which belongs to the uncertainty set S. Let $P_X^{\Delta A}$ be the probability measure induced by (31), when $\Delta A \in S$. Then $P_X^{\Delta A}$ is a function of ΔA and it is induced by $\{X_t\}_{t\geq 0}$. Since ΔA belongs to the uncertainty set S, $P_X^{\Delta A}$ belongs to the uncertainty set $\mathcal{M}_{SU} = \{P_X^{\Delta A}; \Delta A \in S, P_X^{\Delta A} << Lebesgue \ \ measure\}$. The nominal model system is described by

$$\begin{aligned} &(\Omega, \mathcal{F}(\Omega), \{\mathcal{F}_t\}_{t\geq 0}, P_X) : X_{t+1} = AX_t + BU_t, \\ &t \geq 0, \quad X_0 \sim p_{X_0}, \ X_0 \in \Re^n. \end{aligned} \tag{32}$$

The family of systems (31) is cascaded with an uncertain communication channel (See Fig. 3) and the objective is to find necessary condition for almost surely uniform asymptotically observability and stabilizability, which are defined as follows.

Definition 4.1: Define the error by $E_t \triangleq X_t - \widetilde{X}_t$, where $\widetilde{X}_t$ is the output of decoder. The family of systems (31), is almost surely uniform asymptotically observable over uncertain communication channel, if there exists an encoder and decoder such that

$$\sup_{P_X^{\Delta A} \in \mathcal{M}_{SU}} \Pr(\lim_{t\to\infty} ||E_t||_2 \neq 0) = 0. \tag{33}$$

The family of systems (31) is almost surely uniform asymptotically stabilizable over uncertain communication channel

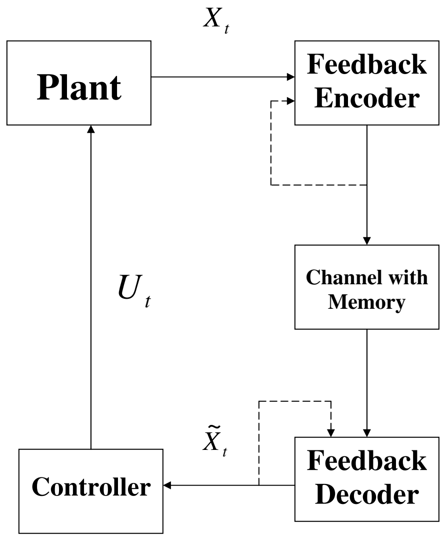

Fig. 3. Networked control system

if there exist an encoder, decoder and controller such that

$$\sup_{P_X^{\Delta A} \in \mathcal{M}_{SU}} \Pr(\lim_{t\to\infty} ||X_t||_2 \neq 0) = 0. \tag{34}$$

Proposition 4.2: Given the family of systems (31), a necessary condition on robust channel capacity ($C_{robust} = \lim_{n\to\infty} \frac{1}{n} C_{n,robust}$) for almost surely uniform asymptotically observability and stabilizability is

$$C_{robust} \geq \sum_{i=1}^{n} \max\{0, \log|\lambda_i(A + \Delta A_{max})|\}. \tag{35}$$

where $\Delta A_{max} \in S$ is chosen to maximize

$$|\det(A + \Delta A)| = \prod_{i=1}^{n} |\lambda_i(A + \Delta A)| \tag{36}$$

Proof: The proof of observability is given by assuming the existence of encoder/decoder pair under which the observability condition in the sense of Definition 4.1 is obtained. This assumption implies that a robust rate distortion with arbitrary small distortion value ϵ is obtained which from robust information transmission theorem (Theorem 3.2 and Corollary 3.3) requires that the robust channel capacity is lower bounded by robust rate distortion and, consequently from Lemma 3.1, it follows that it is lower bounded by the robust entropy of the source. Therefore, by calculating the robust entropy of the source (plant), the lower bound (35) is obtained for the robust channel capacity.

The proof of stabilizability is given by assuming existence of controller, encoder, and decoder under which the stabilizability condition in the sense of Definition 4.1 is obtained. Under this assumption, it is shown that a robust rate distortion with arbitrary small distortion value ϵ is obtained, consequently as it was shown in the proof of uniform observability, a necessary condition for obtaining such a rate distortion is given by (35). Details of the proof can be found in [14].

Remark 4.3: In proving the necessary condition shown previously, we did not need to explicitly describe the encoder, decoder and channel, consequently the condition holds independently of the choice of these subsystems.

Remark 4.4: Proposition 4.2 extends the result derived in [8] under Propositions 3.2 and 3.3 to the case when the controlled system and communication links are subject to uncertainty.

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