Chapter 2

Bessel Functions

2.1 Bessel, Neumann, and Hankel Functions: $J_n(x)$, $N_n(x)$, $H_n^{(1)}(x)$, $H_n^{(2)}(x)$

Bessel functions are solutions of the following differential equation:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{2.1}$$

which is called the Bessel's differential equation. This is a second order differential equation and has two linearly independent solutions. *Any two* of the following functions are linearly independent solutions of (2.1)

$$J_\nu(x) \qquad N_\nu(x) \qquad H_\nu^{(1)}(x) \qquad H_\nu^{(2)}(x) \tag{2.2}$$

Thus, the general solution for (2.1) can be written as a linear combination of *any two* of the above functions. Usually the general solution is written as one of these forms:

$$AJ_\nu(x) + BN_\nu(x) \qquad\qquad AH_\nu^{(1)}(x) + BH_\nu^{(2)}(x) \tag{2.3}$$

When ν is not an integer ($\nu \neq n$) J_ν and $J_{-\nu}$ are also linearly independent solutions of (2.1), however, we usually never use $J_{-\nu}$. The Neumann function is related to J_ν and $J_{-\nu}$:

$$N_\nu(x) = \frac{\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \tag{2.4}$$

$$N_n(x) = \lim_{\nu\to n}\frac{\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \tag{2.5}$$

and Hankel functions of the first and second kind are related to Bessel and Neumann functions:

$$H_\nu^{(1)}(x) = J_\nu(x) + jN_\nu(x) \tag{2.6}$$

$$H_\nu^{(2)}(x) = J_\nu(x) - jN_\nu(x) \tag{2.7}$$

With a variable transformation $x = \kappa\rho$ equation (2.1) can be transformed into:

$$\rho^2y'' + \rho y' + (\kappa^2\rho^2 - \nu^2)y = 0 \tag{2.8}$$

whose independent solutions are $J_\nu(\kappa\rho)$ and $N_\nu(\kappa\rho)$. When $\nu = n$ is an integer J_n and J_{-n} are *not independent* anymore and we have:

$$\begin{array}{ll} J_{-n}(x) = (-1)^nJ_n(x) & N_{-n}(x) = (-1)^nN_n(x) \\ J_n(-x) = (-1)^nJ_n(x) & N_n(-x) = (-1)^n\left[2\,j\,J_n(x) + N_n(x)\right] \end{array} \tag{2.9}$$

Plots of the first three Bessel functions are shown in Fig. 2.1 and the first three Neumann functions are shown in Fig. 2.2

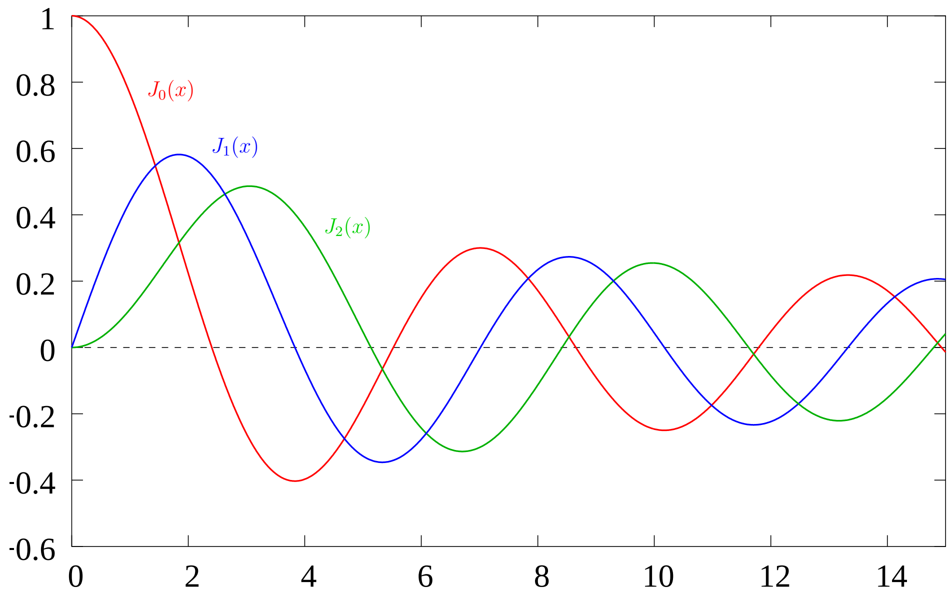

Figure 2.1: Bessel functions of the first kind

2.1.1 Small and Large Argument Approximations

Small Argument Limit $|x| \ll 1$

$$J_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n \tag{2.10}$$

$$N_n(x) \approx -\frac{(n-1)!}{\pi}\left(\frac{2}{x}\right)^n \qquad n \neq 0 \tag{2.11}$$

$$Y_0(x) \approx \frac{2}{\pi}\ln\frac{\gamma x}{2} \qquad \gamma = 1.78107241799\ldots \text{ Euler's constant} \tag{2.12}$$

Large Argument Limit $|x| \gg 1$

$$J_n(x) \approx \sqrt{\frac{2}{\pi\, x}}\cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) \tag{2.13}$$

$$N_n(x) \approx \sqrt{\frac{2}{\pi\, x}}\sin\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right) \tag{2.14}$$

2.1.2 Orthogonality Relationships and Fourier-Bessel Expansions

Bessel equation (2.8) can be written in the following form:

$$(\rho y')' + \left(\kappa^2\rho - \frac{n^2}{\rho}\right) y = 0 \tag{2.15}$$

If you compare (2.15) with (1.5), you can see that it is a Sturm-Liouville equation with $p(\rho) = \rho$, $q(\rho) = -\frac{n^2}{\rho}$, $w(\rho) = \rho$, and $\lambda = \kappa^2$. With appropriate boundary conditions on a finite interval such as $\rho \in [a, b]$ we can have a Sturm-Liouville eigenvalue problem (regular if $a > 0$ and irregular if $a = 0$). Therefore, all the properties of Sturm-Liouville eigenfunctions and eigenvalues will be applicable to this equation. Here we only consider two irregular cases [?].

First consider (2.15) on the interval $0 \leq \rho \leq b$ with boundary condition $y(b) = 0$. This is an irregular problem and we have to impose an extra condition at $\rho \to 0$ in order to have orthogonal eigenfunctions. This extra condition is that the solution and its derivative must be bounded as $\rho \to 0$. Since $N_n(\kappa\rho) \to -\infty$ as $\rho \to 0$, therefore, we can not have $N_n(\kappa\rho)$ in our solution. Consequently, eigenfunctions must be in the form of $J_n(\kappa\rho)$ and since $y(b) = 0$ we obtain the eigenvalues:

$$J_n(\kappa b) = 0 \Longrightarrow \kappa_m = \frac{\nu_{nm}}{b} \Longrightarrow \lambda_m = (\kappa_m)^2 = \left(\frac{\nu_{nm}}{b}\right)^2 \tag{2.16}$$

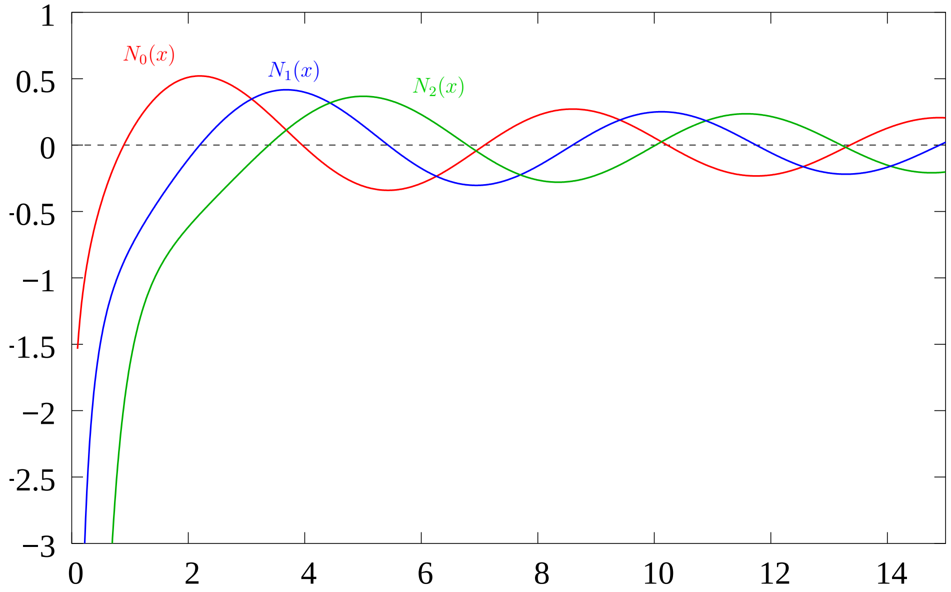

Figure 2.2: Bessel functions of the second kind

in which ν_{nm} is the m^{th} root of the Bessel function $J_n(x) = 0$, *i.e.* $J_n(\nu_{nm}) = 0$. These eigenvalues are all real and have all the properties that we explained for Sturm-Liouville problem. We have the following orthogonality property over the interval $[0, b]$ with respect to weight function $w(\rho) = \rho$:

$$\int_0^b J_n\left(\frac{\nu_{nm}}{b}\rho\right) J_n\left(\frac{\nu_{nk}}{b}\rho\right)\rho\, d\rho = \begin{cases} 0, & m \neq k \\ \dfrac{b^2}{2}\left[J_{n+1}(\nu_{nm})\right]^2, & m = k \end{cases} \tag{2.17}$$

Any piecewise continuous function $f(\rho)$ for which $f(b) = 0$ and is defined on the interval $[0, b]$ can be expanded in a series of above eigenfunctions:

$$f(\rho) = \sum_{m=1}^{\infty} A_m J_n\left(\frac{\nu_{nm}}{b}\rho\right) \tag{2.18}$$

in which the coefficients A_m are obtained by using the orthogonality property (2.17)

$$A_m = \frac{2}{b^2\left[J_{n+1}(\nu_{nm})\right]^2} \int_0^b f(\rho) J_n\left(\frac{\nu_{nm}}{b}\rho\right)\rho\, d\rho \tag{2.19}$$

Expression (2.18) is called the Fourier-Bessel series expansion of $f(\rho)$.

As another case, consider (2.15) on the interval $0 \leq \rho \leq b$ with boundary condition $y'(b) = 0$. This is an irregular problem and we have to impose an extra condition at $\rho \to 0$ in order to have orthogonal eigenfunctions. This extra condition is that the solution and its derivative must be bounded as $\rho \to 0$. Since $N_n(\kappa\rho) \to \infty$ as $\rho \to 0$, therefore, we can not have $N_n(\kappa\rho)$ in our solution. Consequently, eigenfunctions must be in the form of $J_n(\kappa\rho)$ and since $y'(b) = 0$ we obtain the eigenvalues:

$$J_n'(\kappa b) = \left.\frac{dJ_n(\kappa\rho)}{d\rho}\right|_{\rho=b} = 0 \Longrightarrow \kappa_m = \frac{\nu_{nm}'}{b} \Longrightarrow \lambda_m = (\kappa_m)^2 = \left(\frac{\nu_{nm}'}{b}\right)^2 \tag{2.20}$$

in which ν_{nm}' is the m^{th} root of the derivative of Bessel function $J_n'(x) = 0$, *i.e.* $J_n'(\nu_{nm}') = 0$. We have the following orthogonality property over the interval $[0, b]$ with respect to weight function $w(\rho) = \rho$:

$$\int_0^b J_n\left(\frac{\nu_{nm}'}{b}\rho\right) J_n\left(\frac{\nu_{nk}'}{b}\rho\right)\rho\, d\rho = \begin{cases} 0, & m \neq k \\ \dfrac{b^2}{2}\left(1 - \dfrac{n^2}{\nu_{nm}'^2}\right)\left[J_n(\nu_{nm}')\right]^2, & m = k \end{cases} \tag{2.21}$$

Any piecewise continuous function $f(\rho)$ for which $f'(b) = 0$ and is defined on the interval $[0, b]$ can be expanded in a series of above eigenfunctions:

$$f(\rho) = \sum_{m=1}^{\infty} A_m J_n\left(\frac{\nu'_{nm}}{b}\rho\right) \tag{2.22}$$

in which the coefficients A_m are obtained by using the orthogonality property (2.21)

$$A_m = \frac{2}{b^2\left(1 - \frac{n^2}{\nu'^2_{nm}}\right)[J_n(\nu'_{nm})]^2} \int_0^b f(\rho) J_n\left(\frac{\nu'_{nm}}{b}\rho\right)\rho\, d\rho \tag{2.23}$$

If the interval is $[a, b]$ and $a > 0$, then the general form of eigenfunctions would be $A_m J_n(\kappa_m \rho) + B_m N_n(\kappa_m \rho)$. The boundary conditions at $\rho = a$ and $\rho = b$ will determine the eigenvalues κ_m and we have similar orthogonality property between the eigenfunctions as well. However, we will not study this case further.

2.1.3 Recursion Relationships and Wronskians

Consider $Z_n(x)$ to be either of $J_n(x)$ or $N_n(x)$ or any linear combination of these two functions. Then, the following recursive formulas are applicable (n can be any number):

$$Z_{n-1}(x) + Z_{n+1}(x) = \frac{2n}{x} Z_n(x) \tag{2.24}$$

$$Z_{n-1}(x) - Z_{n+1}(x) = 2Z'_n(x) \tag{2.25}$$

$$Z'_n(x) + \frac{n}{x} Z_n(x) = Z_{n-1}(x) \tag{2.26}$$

$$Z'_n(x) - \frac{n}{x} Z_n(x) = -Z_{n+1}(x) \tag{2.27}$$

$$(x^n Z_n(x))' = x^n Z_{n-1}(x) \tag{2.28}$$

$$\left(x^{-n} Z_n(x)\right)' = -x^{-n} Z_{n+1}(x) \tag{2.29}$$

in particular $J'_0(x) = -J_1(x)$. Equation (2.28) and (2.29) are very useful when integrating over Bessel functions.

$$J_n(x)Y_{n+1}(x) - J_{n+1}(x)N_n(x) = J'_n(x)N_n(x) - J_n(x)Y'_n(x) = -\frac{2}{\pi x} \tag{2.30}$$

2.1.4 Series and Integral Relationships

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{+\infty} J_n(x)\, t^n \tag{2.31}$$

$$e^{ix\cos\phi} = \sum_{n=-\infty}^{+\infty} i^n J_n(x) e^{in\phi} \tag{2.32}$$

$$\int Z_n(\alpha x) Z_n(\beta x) x\, dx = x\, \frac{\beta Z_n(\alpha x) Z_{n-1}(\beta x) - \alpha Z_{n-1}(\alpha x) Z_n(\beta x)}{\alpha^2 - \beta^2} \tag{2.33}$$

$$\int Z_n^2(\alpha x) x\, dx = \frac{x^2}{2}\left[Z_n^2(\alpha x) - Z_{n-1}(\alpha x) Z_{n+1}(\alpha x)\right] \tag{2.34}$$

$$\int x^{n+1} Z_n(x)\, dx = x^{n+1} Z_{n+1}(x) \tag{2.35}$$

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\beta - in\beta}\, d\beta = \frac{1}{\pi}\int_0^{\pi} \cos(n\beta - x\sin\beta)\, d\beta \tag{2.36}$$

$Z_n(x)$ can be any of $J_n(x)$ or $N_n(x)$ or $AJ_n(x) + BN_n(x)$