1 Bessel, Neumann, and Hankel Functions: $J_\nu(x), \, N_\nu(x), \, H^{(1)}_\nu(x), \, H^{(2)}_\nu(x)$

Bessel functions are solutions of the following differential equation:

\[x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad (1.1)\]

Any two of the following functions are linearly independent solutions of (1.1)

\[J_\nu(x), \, N_\nu(x), \, H^{(1)}_\nu(x), \, H^{(2)}_\nu(x)\]

when $\nu$ is not an integer, $J_\nu(x)$ and $J_{-\nu}(x)$ are also linearly independent principal solutions of (1.1). The Neumann function $N_\nu(x)$ is related to $J_\nu$ and $J_{-\nu}$:

\[N_\nu(x) = \frac{\cos \nu \pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu \pi} \quad (1.2)\]

\[N_n(x) = \lim_{\nu \to n} \frac{\cos \nu \pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu \pi} \quad (1.3)\]

in some books Neumann functions are denoted by $Y_\nu(x)$ instead of $N_\nu(x)$. Hankel functions of the first and second kind are related to Bessel and Neumann functions:

\[H^{(1)}_\nu(z) \equiv J_\nu(z) + jN_\nu(z) = \frac{e^{-j\nu \pi} J_\nu(z) - J_{-\nu}(z)}{\sin \nu \pi} \quad (1.4)\]

\[H^{(2)}_\nu(z) \equiv J_\nu(z) - jN_\nu(z) = \frac{e^{j\nu \pi} J_\nu(z) - J_{-\nu}(z)}{j \sin \nu \pi} \quad (1.5)\]

With a variable transformation $x = \kappa \rho$ equation (1.1) can be transformed into:

\[\rho^2y'' + \rho y' + (\kappa^2 \rho^2 - \nu^2)y = 0 \quad (1.6)\]

When $\nu = n$ is an integer $J_n$ and $J_{-n}$ are not independent anymore and we have:

\[J_{-n}(x) = (-1)^n J_n(x) \quad N_{-n}(x) = (-1)^n N_n(x) \quad (1.7)\]

Plots of the first three Bessel and Neumann functions are shown in Fig. 1.1 and Fig. 1.2, respectively.

In general for arbitrary $\nu$ we have

\[J_\nu(e^{\pm j\pi} x) = e^{\pm j\nu \pi} J_\nu(x) \quad N_\nu(e^{\pm j\pi} x) = e^{\mp j\nu \pi} N_\nu(x) \pm 2j \cos \nu \pi J_\nu(x) \quad (1.8)\]

Figure 1.1: Bessel functions of the first kind

Figure 1.2: Bessel functions of the second kind
1.1 Asymptotic Approximations

1.1.1 Small Argument Limit \( |x| \to 0 \)

\[
J_0(x) \approx 1 - \frac{x^2}{4} \approx 1 \tag{1.9}
\]

\[
J_\nu(x) \approx \left( \frac{x}{2} \right)^\nu \frac{1}{\Gamma(\nu + 1)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{x}{2} \right)^n \tag{1.10}
\]

\[
N_\nu(x) \approx -\frac{1}{\pi} \left( \frac{2}{x} \right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{\pi n!} \left( \frac{2}{x} \right)^n \quad n \neq 0 \tag{1.11}
\]

\[
N_0(x) \approx \frac{2}{\pi} \ln \frac{x}{2} \quad \gamma = 1.78107241799 \ldots \text{ Euler's constant} \tag{1.12}
\]

1.1.2 Large Argument Limit \( |x| \gg |\nu|, -\pi < \arg(x) < \pi \)

\[
J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\nu \pi}{2} \right) \tag{1.13}
\]

\[
N_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{\nu \pi}{2} \right) \tag{1.14}
\]

1.1.3 Wronskian relations

The wronskian between two functions is defined by

\[
W\{f, g\} \triangleq f(x)g'(x) - f'(x)g(x) \tag{1.15}
\]

\[
W \{ J_\nu, N_\nu \} = J_{\nu+1}N_\nu - J_\nu N_{\nu+1} = \frac{2}{\pi x} \tag{1.16}
\]

which is independent of \( \nu \)

1.2 Integral Representations

When \( n \) is an integer:

\[
J_\alpha(x) = \frac{e^{-j\alpha(x+\frac{x}{2})}}{2\pi} \int_0^{2\pi} e^{jx \cos(\phi - \alpha)} e^{j\alpha \phi} d\phi = \frac{1}{2\pi} \int_0^{2\pi} e^{jx \sin \phi} e^{j\alpha \phi} d\phi \tag{1.17}
\]

\[
J_\alpha(x) = \frac{e^{-j\alpha(x+\frac{x}{2})}}{2\pi} \int_{-\pi}^{\pi} e^{jx \cos(\phi - \alpha)} e^{j\alpha \phi} d\phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{jx \sin \phi} e^{j\alpha \phi} d\phi \tag{1.18}
\]

\[
J_n(x) = \frac{e^{-jx \alpha \frac{x}{2}}}{\pi} \int_0^{\pi} \cos(n\phi) e^{jx \cos \phi} d\phi \tag{1.19}
\]

\[
J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin \phi - n\phi) d\phi \tag{1.20}
\]

\[
J_0(x) = \frac{2}{\pi} \int_0^{\pi} \cos(x \sin \phi) d\alpha = 2 \int_0^{\pi} \cos(x \cos \phi) d\alpha \tag{1.21}
\]

\[
J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin \phi) d\alpha \tag{1.22}
\]

In general for arbitrary \( \nu \)

\[
J_\nu(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin \phi - \nu \phi) d\phi - \frac{\sin \nu \pi}{\pi} \int_0^\infty e^{-x \sinh t - \nu t} dt \quad \Re\{x\} > 0 \tag{1.23}
\]

\[
J_\nu(x) = \frac{1}{\pi} \int_0^\infty \sin \left( x \cosh t - \frac{\nu \pi}{2} \right) \cosh \nu t dt \tag{1.24}
\]

\[
N_\nu(x) = \frac{1}{\pi} \int_0^{\pi} \sin(x \sin \phi - \nu \phi) d\phi - \frac{1}{\pi} \int_0^\infty \left( e^{\nu t} + e^{-\nu t} \cos \nu \pi \right) e^{-x \sinh t} dt \quad \Re\{x\} > 0 \tag{1.25}
\]
1.3 Orthogonality Relationships and Fourier-Bessel Series

Bessel equation (1.6) can be written in the following form:

\[-(\rho y')' + \frac{\nu^2}{\rho} y - \kappa^2 \rho y = 0 \implies L[y] = \kappa^2 y \quad L = -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{\nu^2}{\rho^2} \tag{1.26}\]

This is a Sturm-Liouville equation with \( p(\rho) = \rho, \; q(\rho) = \frac{\nu^2}{\rho}, \; w(\rho) = \rho, \) and \( \lambda = \kappa^2. \) With appropriate boundary conditions on a finite interval such as \( [a, b] \) we can have a Sturm-Liouville eigenvalue problem (regular if \( a > 0 \) and irregular if \( a = 0 \)). Here \( n \) can be any real and non-negative number.

CASE I: Consider (1.26) on the interval \( 0 \leq \rho \leq b \) with boundary condition \( y(b) = 0. \) At \( \rho \to 0 \) we require the solution to be bounded. Thus, we can not have \( N_n(\kappa \rho) \) and eigenfunctions must be in the form of \( J_n(\kappa \rho). \)

Eigenvalues are:

\[ J_n(\kappa b) = 0 \implies \kappa_m = \frac{\nu_{nm}}{b} \implies \lambda_m = (\kappa_m)^2 = \left( \frac{\nu_{nm}}{b} \right)^2 \quad (1.27)\]

in which \( \nu_{nm} \) is the \( m^{th} \) root of the Bessel function \( J_n(x) = 0, \) i.e. \( J_n(\nu_{nm}) = 0. \) The following orthogonality property exists:

\[ \int_0^b J_n \left( \frac{\nu_{nm}}{b} \rho \right) J_m \left( \frac{\nu_{nk}}{b} \rho \right) \rho \, d\rho = \begin{cases} 0, & m \neq k \\ \frac{b^2}{2} |J_{n+1}(\nu_{nm})|^2, & m = k \end{cases} \quad (1.28)\]

For any piecewise continuous function \( f(\rho) \) we have:

\[ f(\rho) \sim \sum_{m=1}^{\infty} F_m J_n \left( \frac{\nu_{nm}}{b} \rho \right) \quad (1.29)\]

in which the coefficients \( F_m \) are obtained by using the orthogonality property (1.28)

\[ F_m = \frac{2}{b^2 |J_{n+1}(\nu_{nm})|^2} \int_0^b f(\rho) J_n \left( \frac{\nu_{nm}}{b} \rho \right) \rho \, d\rho \quad (1.30)\]

Expression (1.29) is called the **Fourier-Bessel Series** expansion of \( f(\rho). \) Note that the series always converges to zero at \( \rho = b. \)

CASE II: Consider (1.26) on the interval \( 0 \leq \rho \leq b \) with boundary condition \( y'(b) = 0. \) At \( \rho \to 0 \) we require the solution to be bounded. Again eigenfunctions must be in the form of \( J_n(\kappa \rho) \) and since \( y'(b) = 0 \) we obtain the eigenvalues:

\[ J_n'(\kappa b) = \left. \frac{dJ_n(\kappa \rho)}{d\rho} \right|_{\rho=b} = 0 \implies \kappa_m = \frac{\nu_{nm}'}{b} \implies \lambda_m = (\kappa_m)^2 = \left( \frac{\nu_{nm}'}{b} \right)^2 \quad (1.31)\]

in which \( \nu_{nm}' \) is the \( m^{th} \) root of the derivative of Bessel function \( J_n'(x) = 0, \) i.e. \( J_n'(\nu_{nm}') = 0. \) The following orthogonality property holds:

\[ \int_0^b J_n \left( \frac{\nu_{nm}'}{b} \rho \right) J_m \left( \frac{\nu_{nk}'}{b} \rho \right) \rho \, d\rho = \begin{cases} 0, & m \neq k \\ \frac{b^2}{2} \left( 1 - \frac{n^2}{\nu_{nm}^2} \right) |J_n(\nu_{nm})|^2, & m = k \end{cases} \quad (1.32)\]

For any piecewise continuous function \( f(\rho) \) we can write:

\[ f(\rho) \sim \sum_{m=1}^{\infty} F_m J_n \left( \frac{\nu_{nm}'}{b} \rho \right) \quad (1.33)\]

in which the coefficients \( F_m \) are obtained by using the orthogonality property (1.32)

\[ F_m = \frac{2}{b^2 \left( 1 - \frac{n^2}{\nu_{nm}^2} \right) |J_n(\nu_{nm})|^2} \int_0^b f(\rho) J_n \left( \frac{\nu_{nm}'}{b} \rho \right) \rho \, d\rho \quad (1.34)\]
Again this is called Fourier-Bessel expansion of \( f(\rho) \). Note that the derivative of the series always converges to zero at \( \rho = b \).

If the interval is \([a, b]\) and \( a > 0\), then the SLP is regular and the general form of eigenfunctions would be \( A_m J_n(\kappa_m \rho) + B_m N_n(\kappa_m \rho) \). The boundary conditions at \( \rho = a \) and \( \rho = b \) will determine the eigenvalues \( \kappa_m \) and we have similar orthogonality property between the eigenfunctions as well.

### 1.4 Recursion Relationships

Consider \( Z_\nu(x) \) to be \( J_\nu(x) \) or \( N_\nu(x) \) or \( H_\nu^{(1)}(x) \) or \( H_\nu^{(2)}(x) \) or any linear combination of these functions. Then, the following recursive formulas are applicable (\( \nu \) can be any number):

\[
\begin{align*}
Z_{\nu-1}(x) + Z_{\nu+1}(x) &= \frac{2\nu}{x} Z_\nu(x) \\
Z_{\nu-1}(x) - Z_{\nu+1}(x) &= 2Z'_{\nu}(x) \\
Z'_\nu(x) + \frac{\nu}{x} Z_\nu(x) &= Z_{\nu-1}(x) \\
Z'_\nu(x) - \frac{\nu}{x} Z_\nu(x) &= -Z_{\nu+1}(x) \\
[x^\nu Z_\nu(x)]' &= x^\nu Z_{\nu-1}(x) \\
[x^{-\nu} Z_\nu(x)]' &= -x^{-\nu} Z_{\nu+1}(x)
\end{align*}
\]

in particular \( Z'_0(x) = -Z_1(x) \). Equation (1.39) and (1.40) are very useful when integrating over Bessel functions.

### 1.5 Series and Integral Relationships

\[
e^{-jk\rho \cos \phi} = \sum_{n=-\infty}^{+\infty} (-j)^n J_n(k\rho) e^{in\phi} \quad e^{jk\rho \cos \phi} = \sum_{n=-\infty}^{+\infty} j^n J_n(k\rho) e^{in\phi}
\]

\[
e^{-jk\rho \sin \phi} = \sum_{n=-\infty}^{+\infty} (-1)^n J_n(k\rho) e^{in\phi} \quad e^{jk\rho \sin \phi} = \sum_{n=-\infty}^{+\infty} J_n(k\rho) e^{in\phi}
\]

In the following expressions \( Z_n(x) \) and \( B_n(x) \) can be any of \( J_n(x) \), \( N_n(x) \), \( H_n^{(1)}(x) \), \( H_n^{(2)}(x) \) or linear combinations of them. \( m, n, \alpha, \beta \) are arbitrary real numbers.

\[
\int Z_n(\alpha x) B_n(\beta x) x \, dx = x \frac{\beta Z_n(\alpha x) B_n(\beta x) - \alpha Z_n(\alpha x) B_n(\beta x)}{\alpha^2 - \beta^2}
\]

\[
= x \frac{\alpha Z_n(\alpha x) B_n(\beta x) - \beta Z_n(\alpha x) B_n(\beta x)}{\alpha^2 - \beta^2}
\]

\[
\int Z_n^2(\alpha x) x \, dx = \frac{x^2}{2} [Z_n^2(\alpha x) - Z_n(\alpha x) Z_{n+1}(\alpha x)]
\]

\[
\int x^{n+1} Z_n(x) \, dx = x^{n+1} Z_{n+1}(x)
\]

\[
\int x^{-n+1} Z_n(x) \, dx = -x^{-n+1} Z_{n+1}(x)
\]

\[
\int \frac{1}{x} Z_n(\alpha x) B_m(\alpha x) \, dx = \alpha x \frac{Z_n(\alpha x) B_{m+1}(\alpha x) - Z_{n+1}(\alpha x) B_m(\alpha x)}{n^2 - m^2} + \frac{Z_n(\alpha x) B_m(\alpha x)}{n + m}
\]

\[
\int Z_1(x) \, dx = -Z_0(x)
\]

\[
\int x Z_0(x) \, dx = x Z_1(x)
\]

### 2 Modified Bessel Functions \( I_\nu(x) \) and \( K_\nu(x) \)

Modified Bessel functions are solutions of the following differential equation:

\[
x^2 y'' + xy' - (x^2 + \nu^2)y = 0
\]
which is called the modified Bessel’s differential equation. The general solution of (2.1) can be written as a linear combination of the modified Bessel functions of the first and second kind:

$$AI_\nu(x) + BK_\nu(x)$$

When $\nu$ is not an integer ($\nu \neq n$) $I_\nu$ and $I_{-\nu}$ are linearly independent (principal) solutions of (2.1), however, we usually use $I_\nu(x)$ and $K_\nu(x)$ (also called Kelvin function) which is related to

$$K_\nu(x) = \frac{\pi}{2\sin \nu \pi} [I_{-\nu}(x) - I_\nu(x)]$$ (2.2)

With a variable transformation $x = \kappa \rho$ equation (2.1) can be transformed into:

$$\rho^2 y'' + \rho y' - (\kappa^2 \rho^2 + \nu^2)y = 0$$ (2.3)

whose independent solutions are $I_\nu(\kappa \rho)$ and $K_\nu(\kappa \rho)$. When $\nu$ is an integer $I_n$ and $I_{-n}$ are not independent anymore and we have $I_{-n}(x) = I_n(x)$. Furthermore, for arbitrary $\nu$ we always have:

$$I_\nu(e^{\pm j\pi} x) = e^{\pm j\nu \pi} I_\nu(x) \quad K_\nu(e^{\pm j\pi} x) = \mp j\nu \pi K_\nu(x)$$ (2.4)

Plots of the first three modified Bessel functions of the first and second kind are shown in Fig. 2.1 and Fig. 2.2, respectively.

2.1 Small and Large Argument Approximations

2.1.1 Small Argument Limit $|x| \to 0$

$$I_0 \approx 1 + \frac{x^2}{4} \approx 1$$ (2.5)

$$I_\nu(x) \approx \frac{(x/2)^\nu}{\Gamma(\nu + 1)} \to \frac{1}{n!} \left(\frac{x}{2}\right)^n$$ (2.6)

$$K_\nu(x) \approx \frac{\Gamma(\nu)}{2} \left(\frac{2}{x}\right)^\nu \to \frac{(n-1)!}{2} \left(\frac{2}{x}\right)^n \quad \nu \neq 0$$ (2.7)

$$K_0(x) \approx -\ln \frac{\gamma x}{2} \quad \gamma = 1.78107241799 \ldots \text{ Euler’s constant}$$ (2.8)

2.1.2 Large Argument Limit $|x| \to \infty$

$$I_\nu(x) \approx \frac{1}{\sqrt{2\pi x}} e^x$$ (2.9)

$$K_\nu(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}$$ (2.10)