Theory of Slots in Rectangular Wave-Guides*

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A basic theory of slots in rectangular wave-guides is given. The analogy with a transmission line is developed and established, and detailed formulae for the reflection and transmission coefficients and for the "voltage amplitude" in the slot generated by a given incident wave are given. While the complete expressions for these quantities are quite complicated and involve the summation of infinite series, certain parts of the expressions are comparatively simple. In particular, the "resistance" or "conductance" of slots which are equivalent to series or shunt elements in a transmission line are given by fairly simple closed expressions. Guide-to-guide coupling by slots and slot arrays are also considered. A more detailed summary of the main results of the paper is given in Section 1.

1. INTRODUCTION. FUNDAMENTAL ASSUMPTIONS AND SUMMARY OF MAIN RESULTS

THE use of slots in wave-guides has proved a promising means of launching high frequency radiation. The subject has been extensively investigated by Watson.[1] In this paper, a fundamental theory of slots in wave-guides, based on the field equations, is attempted. Attention is confined, for the most part, to rectangular guides, but the principle of the method is general, and other shapes of guide could be considered if necessary.

The fundamental assumptions on which the theory is based are the following:

(1) The walls of the guide are perfectly conducting and of negligible thickness.

(2) The slot is narrow; to be more precise, we assume that

$$2 \log(\text{length of slot}/\text{width of slot}) \gg 1.$$

(3) In considering the field outside the guide, the penetration of the field into the region behind the face containing the slot is neglected. In other words, we treat the problem as if the guide-face containing the slot had an infinite perfectly conducting flange on it.

(4) The guide transmits only the H_{01}-wave, and the length of the slot is near that of the first "resonance" (i.e., near $\lambda/2$).

Of these assumptions, the third is probably the one most seriously at fault.[1a] Experiment indicates that, in some cases, the penetration of the field behind the face is by no means negligible. In the case of an *array* of slots, however, the assumption is probably a good one. The fourth assumption is not in any way essential to the development of the theory, but is made because it is probably satisfied in cases of practical importance. Much of the work is independent of this assumption.

In Section 2, as a preliminary, the problem of determining the field generated in a guide of arbitrary section by an assigned tangential electric field in the wall of the guide is solved. In Section 3, some general considerations relating to the slot problem are given, and the analogy of a slot to an antenna is pointed out. The contents of this section are probably not essentially new, but are given here for subsequent convenience.

In Section 4 the problem of a slot in a rectangular guide which transmits only the H_{01}-wave is considered. A quantity called the "voltage" at any point in the slot is introduced, which is the analog of the current in an antenna. This voltage varies approximately sinusoidally along the slot, as does the current in a half-wave antenna. Expressions for the amplitudes of the H_{01}-waves scattered in either direction, in terms of the voltage amplitude, are given, with particular cases, in Eqs. (25–31). It is then shown that, in a certain sense, the slot is equivalent to a network in a transmission line and can be characterized by the reflection and transmission

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[1] W. H. Watson, *The Physical Principles of Wave Guide Transmission* (Oxford University Press, London, 1947). The writer's work is there referred to, but the notation is different.

[1a] The referee has suggested that the finite conductivity of the walls might lead to large errors. A rough calculation made in the appendix indicates that this is not so, however.

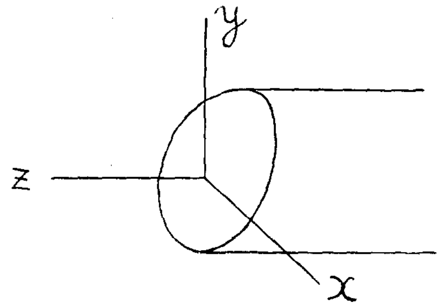

FIG. 1.

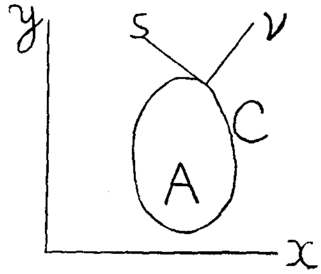

FIG. 2.

coefficients for waves incident from either direction. These coefficients satisfy two identical relations (Eq. (34)), one of which is required for the transmission line analogy to be complete and the other of which imposes a restriction on the type of network to which the slot is equivalent.

In Section 5 it is shown that the transmission coefficients can in part be calculated quite simply from energy considerations, and that this fact suffices to give expressions for the "conductance" or "resistance" of a slot when it is equivalent to a series or shunt element in a transmission line. Detailed formulae for the various relevant cases are given in Eqs. (45–48).

In Section 6, the complete solution of the problem of determining the voltage amplitude in a slot in terms of the amplitude of the incident wave is given (Eqs. (63–68)). This completes the discussion of the problem of a single slot in an infinite guide coupled to free space.

In Section 7, the question of guide-to-guide coupling is considered briefly, and it is shown that this is easily solved when once the solutions for each guide treated separately have been found.

In Section 8, an array of slots is considered, and a set of linear equations is developed for the determination of the voltage amplitudes in the various slots (Eqs. (71–73)). The coefficients in these equations are not hard to calculate when once the calculations for a single slot have been made. In Section 9, the case of a terminated guide is briefly considered.

Not much comparison of the theory with experiment seems to be possible at the present time and in the present state of the calculations. However, some of the formulae of Section 5 agree fairly well with experiment according to Watson (Chapter 6 of reference 1). Detailed calculations for a slot in the broad face of the guide have been in progress for some time, and it is hoped that these will be published soon.

I am greatly indebted to Professor W. H. Watson for many interesting and helpful discussions of the problems involved. I also wish to thank Mr. J. R. Pounder for assistance in the preparation of this paper.

2. THE FIELD GENERATED IN A WAVE-GUIDE OF ARBITRARY SECTION BY AN ASSIGNED TANGENTIAL ELECTRIC FIELD IN THE WALL OF THE GUIDE

As a preliminary, we consider the problem of finding the field generated in an infinite wave-guide by any assigned tangential electric field in the wall of the guide. In this section, we shall suppose that the cross section of the guide is arbitrary, since there is here no point in specializing for a rectangular guide. This problem has been solved by Bethe[2] by quite a different method. The method used here, however, offers certain advantages for the slot problem.

Take a system of right-handed axes with the z axis along the guide (Fig. 1), and let A denote the cross section of the guide and C the boundary of A (Fig. 2). Also let ν denote the outward normal to C and s the direction of the tangent, a rotation from ν to s being in the same sense as a rotation from x to y. A suffix ν or s will be used to denote the components of a vector in these directions.

The field within the guide can be expressed in terms of two functions ψ, Ψ, by means of the

[2] H. A. Bethe (not yet published).

formulae:

$$E_x=\frac{\partial^2\psi}{\partial z\partial x}+ik\frac{\partial\Psi}{\partial y},$$

$$E_y=\frac{\partial^2\psi}{\partial z\partial y}-ik\frac{\partial\Psi}{\partial x},$$

$$E_z=\frac{\partial^2\psi}{\partial z^2}+k^2\psi,$$

$$H_x=-ik\frac{\partial\psi}{\partial y}+\frac{\partial^2\Psi}{\partial z\partial x}, \qquad (1)$$

$$H_y=ik\frac{\partial\psi}{\partial x}+\frac{\partial^2\Psi}{\partial z\partial y},$$

$$H_z=\frac{\partial^2\Psi}{\partial z^2}+k^2\Psi,$$

where, in the usual notation, $k=\omega/c=2\pi/\lambda$ (λ=vacuum wave-length), and ψ, Ψ satisfy the wave equations

$$(\nabla^2+k^2)\psi=0, \quad (\nabla^2+k^2)\Psi=0. \qquad (2)$$

We may conveniently refer to ψ, Ψ, as the *generating functions* for E- and H-waves, respectively.

It is assumed that all complex field quantities vary with the time according to the factor $e^{-i\omega t}$, this factor being omitted, and that the actual physical field quantities are the real parts of the corresponding complex expressions.

Let S denote the surface of the guide. Then we regard E_z, E_s as being assigned functions of position on S, it being assumed in the first place that these functions are continuous and possess continuous derivatives. We also assume that E_z, E_s tend to zero sufficiently rapidly as $z\rightarrow\pm\infty$. The boundary conditions of the problem are then that E_z, E_s are assigned on S, and that at $z=\pm\infty$ only outgoing and damped waves are present.

Now, inside the guide, E_z satisfies a wave equation similar to (2). Hence, by a well-known formula in the theory of Green's functions

$$E_z(P)=\int_S\frac{\partial G_1(P,P')}{\partial\nu'}E_z(P')dS', \qquad (3)$$

where $P(x, y, z)$ denotes the point at which E_z is estimated and $P'(x', y', z')$ a point in the domain of integration S. The Green's function G_1, for any two points P, P' within S, may be defined as follows:

(1) G_1, regarded as a function of P, satisfies the wave equation $(\nabla^2+k^2)G_1=0$ everywhere inside S except at P'.

(2) $G_1=0$ when P is on S.

(3) As $P\rightarrow P'$, G_1 becomes infinite like $1/4\pi r$ where r denotes the distance from P to P'.

Further, from the Maxwell equations, we easily find that, on S,

$$\frac{\partial H_z}{\partial\nu}=\frac{i}{k}\left[\left(\frac{\partial^2}{\partial z^2}+k^2\right)E_s-\frac{\partial^2E_z}{\partial z\partial s}\right].$$

Hence, similarly to (3),

$$H_z(P)=\frac{i}{k}\int_S G_2(P,P')\left[\left(\frac{\partial^2}{\partial z'^2}+k^2\right)E_s(P')-\frac{\partial^2}{\partial z'\partial s'}E_z(P')\right]dS', \qquad (4)$$

where the Green's function G_2 satisfies conditions which are the same as for G_1, except that condition (2) is replaced by $\partial G_2/\partial\nu=0$ on S, and that, in condition (3), $1/4\pi r$ is to be replaced by $-1/4\pi r$.

By repeated integration by parts with respect to z' or s', and by using the conditions that E_z, E_s vanish at $z=\pm\infty$, we can replace (4) by

$$H_z(P)=\frac{i}{k}\int_S\left[\left(\frac{\partial^2}{\partial z'^2}+k^2\right)G_2(P,P')\cdot E_s(P')-\frac{\partial^2G_2(P,P')}{\partial z'\partial s'}E_z(P')\right]dS'. \qquad (5)$$

Equations (3) and (5) now give the field components E_z, H_z everywhere in the guide if G_1, G_2 can be found. Further, the rates of change of E_z, E_s on S can now be made as large as desired, and (3) and (5) will also apply to the case of an aperture in a perfectly conducting guide, when there will, in general, be discontinuities in E_z, E_s at the edge of the aperture.

We shall now outline a method for finding the Green's function G_1, as follows: consider the problem of finding a function $f(x, y, z)$ which satisfies the equation

$$(\nabla^2+k^2)f=\phi(x, y, z), \qquad (6)$$

with the boundary conditions that $f=0$ on S, and that f gives outgoing or damped waves at $z=\pm\infty$. It can then be seen without difficulty that by making

$$\phi(x, y, z)\rightarrow -\delta(x-x')\delta(y-y')\delta(z-z'), \qquad (7)$$

where δ denotes Dirac's "delta-function," we obtain the required function G_1.

We suppose f expanded in terms of the functions for E-waves,

$$f=\sum_n a_n(z)\psi_n(x, y), \qquad (8)$$

where ψ_n, μ_n $(n=1, 2, \cdots)$ are the normalized eigenfunctions and eigenvalues of the problem:

$$\left.\begin{aligned} &\frac{\partial^2\psi_n}{\partial x^2}+\frac{\partial^2\psi_n}{\partial y^2}+\mu_n{}^2\psi_n=0 \text{ in } A,\\ &\psi_n=0 \text{ on } C,\\ &\int_A \psi_n{}^2 dxdy=1. \end{aligned}\right\} \qquad (9)$$

Substituting (8) in (6), using (9) and the orthogonality relations of the ψ_n, we find

$$(d^2a_n/dz^2)+u_n{}^2a_n = \int_A \phi(x, y, z)\psi_n(x, y)dxdy, \qquad (10)$$

where

$$u_n=(k^2-\mu_n{}^2)^{\frac{1}{2}}, \qquad (11)$$

the positive square root being taken in (11) if u_n is real and the positive-imaginary root if u_n is pure imaginary. We now solve (10) by the method of variation of constants, using the boundary conditions that $a_n(z)\sim e^{iu_nz}$ as $z\rightarrow+\infty$ and $a_n(z)\sim e^{-iu_nz}$ as $z\rightarrow-\infty$ to determine the arbitrary constants. Applying (7), we then finally obtain

$$G_1(P, P') = -\sum_n \frac{1}{2iu_n}\psi_n(x, y)\psi_n(x', y')e^{iu_n|z-z'|}. \qquad (12)$$

The function G_2 is found in a similar manner to be[3]

$$G_2(P, P') = \sum_n \frac{1}{2iU_n}\Psi_n(x, y)\Psi_n(x', y')e^{iU_n|z-z'|}, \qquad (13)$$

where, analogously to (9), (11), we define Ψ_n, M_n, U_n, the corresponding quantities for H-waves, by:

$$\left.\begin{aligned} &\frac{\partial^2\Psi_n}{\partial x^2}+\frac{\partial^2\Psi_n}{\partial y^2}+M_n{}^2\Psi_n=0 \text{ in } A,\\ &\frac{\partial\Psi_n}{\partial\nu}=0 \text{ on } C,\\ &\int_A \Psi_n{}^2 dxdy=1, \end{aligned}\right\} \qquad (14)$$

$$U_n=(k^2-M_n{}^2)^{\frac{1}{2}}, \qquad (15)$$

the same convention as to the sign of the square root in (15) being used as in (11).

Substituting (12), (13), in (3), (5), we have E_z, H_z at any point in the guide. If we now use (2), and the third and sixth of (1), we obtain (no additive functions evidently being necessary):

$$\psi(P) = -\sum_n \frac{1}{2iu_n\mu_n{}^2}\psi_n(x, y) \times\int_S \frac{\partial\psi_n(x', y')}{\partial\nu'}e^{iu_n|z-z'|}E_z(P')dS', \qquad (16)$$

$$\Psi(P) = \sum_n \frac{1}{2kU_n}\Psi_n(x, y) \times\int_S \Psi_n(x', y')e^{iU_n|z-z'|}E_s(P')dS' \pm\frac{i}{k}\sum_n \frac{1}{2M_n{}^2}\Psi_n(x, y) \times\int_S \frac{\partial\Psi_n(x', y')}{\partial s'}e^{iU_n|z-z'|}E_z(P')dS', \qquad (17)$$

the $+$ or $-$ sign being taken in (17) according as $z>z'$ or $z<z'$.

[3] We omit a term in G_2 which is proportional to $e^{ik|z-z'|}$, since it is evident from (5) that this does not contribute to the field.

Equations (16), (17) are our final results and give the generating functions for E- and H-waves in terms of the assigned values of E_z, E_s on the wall of the guide and the functions defining the free propagation of E- and H-waves in the guide. In the case of a perfectly conducting guide with an aperture, the integrations need, of course, only be extended over the aperture.

3. THE SLOT PROBLEM IN GENERAL AND THE ANTENNA ANALOGY

To solve the slot problem, it is necessary to find fields inside and outside the guide such that the tangential components of **E** and **H** are continuous as we cross the slot, there being a given incident wave in the guide.

Consider for a minute the case where, instead of the wave-guide, we have an infinite perfectly conducting plane with a narrow slot in it and a wave incident on it from one side. From the symmetry (or near-symmetry) of Maxwell's equations in **E** and **H**, it is evident that this problem is essentially the same as that of an antenna in the form of a narrow strip of metal scattering an incident wave, only with the **E**- and **H**-vectors interchanged.[4] Using this analogy, we can see that in the slot problem the only tangential components of **E** and **H** in the slot which are of importance are the component of **H** along the slot and the component of **E** across the slot. Further, near resonance (length of slot approximately equal to an odd number of half-wave-lengths), the "voltage," or integral of **E** across the slot, which corresponds to the current in the antenna problem, is approximately sinusoidal with the free-space wave-length, vanishing at the ends of the slot. The complex amplitude of the voltage will depend on the exact length and width of the slot and the data for the incident wave. Off resonance, the voltage is comparatively small.

A consideration of the way in which this result is obtained will now show that it will still hold approximately for a slot in a wave-guide coupled to free space, or for a slot in guide-to-guide coupling: near resonance the voltage is approximately sinusoidal and vanishes at the ends of the slot, the only exceptions being in the immediate neighborhood of a sharp bend, as when a slot extends around an edge in a rectangular guide. In accordance with the fourth of our fundamental assumptions (Section 1), we shall consider only the case where we are near the first resonance (length of slot $\sim\lambda/2$).

It follows from what has been said above and from the third of our fundamental assumptions that the radiation pattern outside the guide in front of the face containing the slot should be the same as for a half-wave dipole, with the **E**- and **H**-vectors interchanged. Experiments of Watson and collaborators[1] indicate that this conclusion is not well satisfied in some cases. The explanation is that the third assumption referred to above is at fault.

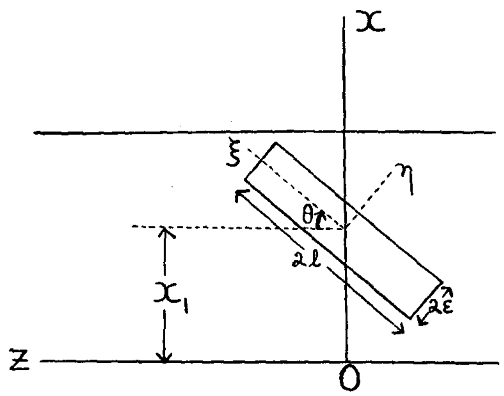

FIG. 3.

4. A SLOT IN A RECTANGULAR GUIDE AND THE TRANSMISSION LINE ANALOGY

We shall now consider the special case of a rectangular guide which transmits only the H_{01}-wave, one of whose faces contains a slot of length $2l$ and width 2ϵ (Fig. 3). We shall now specify our axes of Section 2 more precisely by taking either direction along one of the edges of the guide lying in the face containing the slot as the z axis, with the x axis through the center of the slot and the y axis along the normal to the guide-face drawn *into* the guide, so that Fig. 3 views the guide-face containing the slot from the *outside*. We shall then specify the position of the slot by x_1, the distance of its center from the z axis, and the angle θ which the direction of the slot makes with the z axis, θ being positive

[4] See Babinet's Principle as formulated by H. G. Booker (not yet published) and E. T. Copson, Proc. Roy. Soc. A186, 100 (1946).

if measured in the same sense as a rotation from Oz to Ox (Fig. 3).

We also take axes (ξ, η) through the center of the slot as shown, the ξ axis being along the slot, and a rotation from the ξ axis to the η axis being in the same sense as a rotation from Oz to Ox. We then define the "voltage" as

$$V(\xi)=\int E_\eta(\xi, \eta)d\eta \qquad (18)$$

taken across the slot.

Let a, $b(a>b)$ be the dimensions of the guide (Fig. 4). Suppose now that we have an incident H_{01}-wave traveling in the positive z-direction in an infinite guide. If the slot is in the broad face of the guide, then, with the axes of Fig. 3, the generating function for the incident wave is

$$\Psi^{(0)}=A\cos(\pi x/a)e^{iUz}, \qquad (19)$$

where

$$U=(k^2-\pi^2/a^2)^{\frac{1}{2}}=2\pi/\lambda_{\text{guide}}, \qquad (20)$$

and A is an arbitrary constant, which we shall term the *amplitude of the wave* (at the center of the slot). The slot will scatter H_{01}-waves (as well as damped waves) backwards $(z<0)$ and forwards $(z>0)$. Let these waves have amplitudes B, C, respectively, at the center of the slot, so that they are given by generating functions

$$\begin{aligned}\Psi^{(-)}&=B\cos(\pi x/a)e^{-iUz},\\ \Psi^{(+)}&=C\cos(\pi x/a)e^{iUz}.\end{aligned} \qquad (21)$$

We shall then define the reflection and transmission coefficients α, β by

$$\alpha=B/A, \quad \beta=1+C/A. \qquad (22)$$

Similarly, if we have an incident H_{01}-wave of amplitude A' traveling in the *negative* z-direction, and waves scattered backwards $(z>0)$ and forwards $(z<0)$ of amplitudes B', C', respectively, we define the reflection and transmission coefficients α', β' by

$$\alpha'=B'/A', \quad \beta'=1+C'/A'. \qquad (23)$$

If the slot is in the narrow face of the guide, and we keep the axes of Fig. 2, we must write y in place of x in (19) and (21); otherwise the same definitions (22), (23) hold for the reflection and transmission coefficients.

According to the previous section (and as

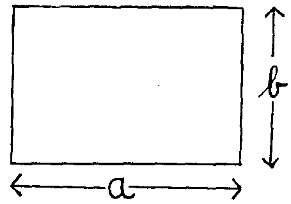

FIG. 4.

will be verified in detail in Section 6), the voltage in the slot, defined by (18), will be approximately sinusoidal, and vanishing at the ends of the slot, for the case under consideration. Suppose, then, that for an incident H_{01}-wave of amplitude A traveling in the positive direction the voltage is[5]

$$V=P\cos k\xi, \qquad (24)$$

where the constant P may be termed the *voltage amplitude*. Similarly, let the voltage amplitude be P' when an incident wave of amplitude A' is traveling in the negative z-direction.

Formula (17) now gives the amplitudes B, C, of the scattered waves, as defined by (21), which are generated by the voltage (24) in the slot (or the corresponding amplitudes B', C' when the voltage amplitude is P'). Only a single term of the infinite series in (17) is, of course, needed for this, the other terms giving damped waves. In making the calculation we can neglect the width of the slot when once the voltage has been introduced by (18). Using the eigenfunction and eigenvalue for the H_{01}-wave, and making use of (1), it is found that the amplitudes of the scattered waves are, in all cases, given by:

$$\begin{aligned}B/P=C'/P'=\zeta/\pi^2kUb,\\ C/P=B'/P'=\zeta^*/\pi^2kUb,\end{aligned} \qquad (25)$$

where the star denotes the conjugate complex, and ζ is a dimensionless quantity given by

$$\zeta=\int_{-\lambda/4}^{\lambda/4} f(\xi)\cos k\xi d\xi, \qquad (26)$$

where $f(\xi)/a$ denotes the component of $\mathbf{H}$ along the slot at the point ξ in an H_{01}-wave of *unit* amplitude traveling in the positive z-direction.[6]

[5] This voltage does not vanish at the ends unless the length of the slot is *exactly* $\lambda/2$. It is permissible to suppose this, however, as far as this part of the calculation is concerned (see Section 6).

[6] That ζ must be expressible in this manner becomes clear when the problem is treated by Bethe's method (reference 2).

With this interpretation of $f(\xi)$ the results (25), (26) hold whether the slot is in the broad or the narrow face of the guide. Explicitly, we have, for the slot in the broad face,

$$f(\xi)=e^{iU\xi\cos\theta}\left[\frac{\pi^2}{a}\cos\theta\cos\frac{\pi}{a}(x_1+\xi\sin\theta)\right.$$
$$\left.-i\pi U\sin\theta\sin\frac{\pi}{a}(x_1+\xi\sin\theta)\right],$$

and for the slot in the narrow face,

$$f(\xi)=(\pi^2/a)\cos\theta e^{iU\xi\cos\theta}.$$

We thus find, for the slot in the broad face:

$$\zeta=\frac{\pi^2}{ka}\cos\theta\left[\cos\frac{\pi x_1}{a}I(\theta)-i\sin\frac{\pi x_1}{a}J(\theta)\right]$$
$$+\frac{\pi U}{k}\sin\theta\left[\cos\frac{\pi x_1}{a}J(\theta)-i\sin\frac{\pi x_1}{a}I(\theta)\right], \quad (27)$$

where

$$I(\theta)=\frac{\cos(p\pi/2)}{1-p^2}+\frac{\cos(q\pi/2)}{1-q^2},$$
$$J(\theta)=\frac{\cos(p\pi/2)}{1-p^2}-\frac{\cos(q\pi/2)}{1-q^2},$$
$$p=(U/k)\cos\theta-(\pi/ka)\sin\theta,$$
$$q=(U/k)\cos\theta+(\pi/ka)\sin\theta.$$

As particular cases of this,[7] we have for the *centered inclined slot* $(x_1=a/2)$:

$$\zeta=(i\pi/k)[U\sin\theta I(\theta)+(\pi/a)\cos\theta J(\theta)]; \quad (28)$$

for the *longitudinal slot* $(\theta=0)$:

$$\zeta=2ka\cos(\pi U/2k)\cos(\pi x_1/a); \quad (29)$$

and for the *transverse slot* $(\theta=\pi/2)$:

$$\zeta=-(2\pi ik/U)\cos(\pi^2/2ka)\sin(\pi x_1/a). \quad (30)$$

For the slot in the narrow face we find

$$\zeta=\frac{2\pi^2k}{a(k^2-U^2\cos^2\theta)}\cos\theta\cos(Ul\cos\theta), \quad (31)$$

the result being in this case independent of x_1.

[7] The terminology here used is that of Watson (reference 1).

Formula (25) thus gives the amplitudes of the scattered waves in terms of the voltage amplitudes P or P'. To find the reflection and transmission coefficients (22), (23), we must find the voltage amplitude in terms of the amplitude of the incident wave, i.e., we must find P/A, P'/A'. This is a complicated problem which will be taken up in Section 6. We shall, however, anticipate the results of that section by quoting the formulae for the voltage amplitudes:

$$P/A=\zeta/Ka,\quad P'/A'=\zeta^*/Ka, \quad (32)$$

where ζ denotes the same quantity as defined in (26) above, and K is another (complex) dimensionless constant whose complete expression is given in Section 6. These formulae hold whether the slot is in the broad or the narrow face.

From Eqs. (22), (23), (25), and (32), we now have for the reflection and transmission coefficients:

$$\alpha=\gamma\zeta^2/K,\quad \alpha'=\gamma\zeta^{*2}/K,$$
$$\beta=\beta'=1+\gamma|\zeta|^2/K, \quad (33)$$

where

$$\gamma=1/\pi^2kUab. \quad (33')$$

We thus see that the four coefficients α, β, α', β' are not independent. They are subject to the *two* relations:

$$\beta=\beta',\quad \alpha\alpha'=(1-\beta)^2. \quad (34)$$

The first of the relations (34) shows that, as far as H_{01}-waves are concerned, the slot is completely analogous to a network connecting two portions of an infinite transmission line, if we define the reflection and transmission coefficients by means (say) of the voltage waves. The equality of the transmission coefficients for waves incident from either direction in a transmission line is, in fact, easily shown to hold under all circumstances by means of ordinary line theory.[8] The second of the relations (34) shows, however, that the equivalent network for a slot is not of the most general type: there is an identical relation connecting the three quantities which characterize the network.

It is now clear that the two coefficients α, β

[8] This was pointed out to the writer by J. R. Pounder. That *some* identical relation must exist between the four coefficients α, α', β, β' for a transmission line follows from the fact that a network can be characterized by only three independent constants.

(and the associated coefficients α', β' deduced from them by (34)) characterize the slot completely as far as standing-wave-ratio measurements are concerned; and that such quantities as the impedance presented by a slot with a given termination (the terminal impedance being defined by means of a reflection coefficient at the termination), or the equivalent network of a slot, can be worked out in terms of these two coefficients by using ordinary transmission line theory. The only proviso is that the place at which the measurements are made, and the termination, must be at a sufficient distance from the slot for the damped waves to be negligible.

5. CALCULATION OF THE REAL PART OF K. RESISTANCE OR CONDUCTANCE OF SERIES AND SHUNT SLOTS

Although the complete calculation of the constant K in (32) is a complicated matter, which will be considered in the next section, the *real* part of K can be calculated quite simply from energy considerations, as follows.[9] Using the notation of the previous section, and considering the case where the incident wave is traveling in the positive z-direction, the mean flow of energy down the guide (in the z-direction) at $z=-\infty$ is easily calculated to be (in Gaussian units)

$$(\pi\omega bU/16a)(|A|^2-|B|^2). \tag{35}$$

Similarly, the mean flow of energy down the guide at $z=+\infty$ is found to be

$$(\pi\omega bU/16a)[|A|^2+|C|^2+2\mathcal{R}(AC^*)], \tag{36}$$

where $\mathcal{R}$ means "real part of." The difference between (35) and (36) must be equal to the mean flow of energy out of the slot. Using our third fundamental assumption (Section 1), this can be calculated in the same way as the flux of energy from a half-wave dipole:[4] it is equal to

$$(73/480\pi^2)c|P|^2. \tag{37}$$

Equating (37) to the difference between (35) and (36), and dividing by $|C|^2$, we obtain, on using (25),

$$\mathcal{R}\left(\frac{1}{\beta-1}\right)=-1-\frac{73a}{60\pi^3kUb}\left|\frac{P}{C}\right|^2.$$

Hence, from (33), (33′), and (25),

$$\mathcal{R}(K)=-(73/60\pi)-\gamma|\zeta|^2, \tag{38}$$

$$\mathcal{R}\left(\frac{1}{\beta-1}\right)=-1-\frac{73}{60\pi\gamma|\zeta|^2}. \tag{39}$$

The results (38), (39) hold, of course, whether the slot is in the broad or the narrow face.

If the equivalent network of the slot can be replaced by simple series or shunt elements, the result (39) suffices to calculate the resistance or conductance of these elements. In terms of the reflection and transmission coefficients, the conditions for the network to be equivalent to a series element are, from ordinary transmission line theory,

$$\alpha=\alpha'=1-\beta, \tag{40}$$

and the admittance of the series element is then

$$Y=(1/2\alpha)-\tfrac{1}{2}. \tag{41}$$

The conditions for the network to be equivalent to a shunt element are

$$\alpha=\alpha'=\beta-1, \tag{42}$$

and the impedance of the shunt element is then

$$Z=-(1/2\alpha)-\tfrac{1}{2}. \tag{43}$$

Reference to (33) then shows that the slot is equivalent to a series element in a transmission line if ζ is a pure imaginary, and to a shunt element if ζ is real. From (27) and (31) we then see that the transverse slot and the centered inclined slot in the broad face are equivalent to series elements, while the longitudinal slot in the broad face and the slot in any position in the narrow face are equivalent to shunt elements.

From (39)–(43) we now have for the "conductance" of the "series" slot or the "resistance" of the "shunt" slot the same expression, namely,

$$G_{\text{series}}=R_{\text{shunt}}=73/120\pi\gamma|\zeta|^2. \tag{44}$$

From (28)–(31), we therefore have:

Transverse slot in broad face (series):

$$\frac{1}{G}=\frac{480\pi^3}{73}\cdot\frac{\gamma k^2}{U^2}\cdot\cos^2\frac{\pi^2}{2ka}\cdot\sin^2\frac{\pi x_1}{a}. \tag{45}$$

[9] We could, of course, obtain the real part from the complete expression for K to be given in Section 6; but the present method is much simpler and forms a useful check on the work.

Centered inclined slot in broad face (series):

$$\frac{1}{G}=\frac{120\pi^3}{73}\cdot\frac{\gamma}{k^2a^2}[Ua\sin\theta I(\theta)+\pi\cos\theta J(\theta)]^2. \quad (46)$$

Longitudinal slot in broad face (shunt):

$$\frac{1}{R}=\frac{480\pi}{73}\cdot\gamma k^2a^2\cdot\cos^2\frac{\pi U}{2k}\cdot\cos^2\frac{\pi x_1}{a}. \quad (47)$$

Slot in narrow face (shunt):

$$\frac{1}{R}=\frac{480\pi^5}{73}\cdot\frac{\gamma}{k^2a^2}\cdot\left[\frac{\cos\theta\cos(Ul\cos\theta)}{1-(U/l)^2\cos^2\theta}\right]^2. \quad (48)$$

In the Eqs. (45)–(48), the half-length of the slot, l, can be put exactly equal to $\lambda/4$, but they should hold approximately for any length of slot close to $\lambda/2$, since the real part of K does not vary rapidly with l. If we define "resonance" by the condition: reactance $=0$, i.e., imaginary part of $K=0$, then actually *at* resonance (i.e., length of slot so that reactance $=0$), we can of course put $R=1/G$. Off resonance, however, we cannot do this, since the imaginary part of K, and hence the reactance or susceptance, varies rapidly with l.

6. CALCULATION OF VOLTAGE AMPLITUDE

We proceed now to the more difficult problem of calculating the voltage amplitude when the incident H_{01}-wave is given.

Referring to Fig. 3, our problem is, in accordance with what was said in Section 3, to find the field component E_η in the slot (or the voltage V) in order that H_ξ may be continuous as we go from inside to outside the slot. We also impose the conditions[10]

$$\left.\begin{aligned} E_\eta&=0 \quad\text{when}\quad \xi=\pm l,\\ \text{i.e.,}\qquad V(\xi)&=0 \quad\text{when}\quad \xi=\pm l.\end{aligned}\right\} \quad (49)$$

From (1), (16), (17), we have for the component H_ξ in the slot of the field generated inside the guide by the field E_η:

$$\begin{aligned} H_\xi(\xi,\eta)&=(H_z\cos\theta+H_x\sin\theta)_{y=0}\\ &=\frac{\cos^2\theta}{2k}\sum_n\frac{M_n{}^2}{U_n}\Psi_n\\ &\times\int_S e^{iU_n|z-z'|}\Psi_n{}'E_\eta{}'d\xi'd\eta'\\ &\mp\frac{i}{2k}\cos\theta\sin\theta\sum_n\Psi_n\\ &\times\int_S e^{iU_n|z-z'|}\frac{\partial\Psi_n{}'}{\partial x'}E_\eta{}'d\xi'd\eta'\\ &+\frac{k}{2}\sin^2\theta\sum_n\frac{1}{u_n\mu_n{}^2}\frac{\partial\psi_n}{\partial y}\\ &\times\int_S e^{iu_n|z-z'|}\frac{\partial\psi_n{}'}{\partial y'}E_\eta{}'d\xi'd\eta'\\ &\pm\frac{i}{2k}\cos\theta\sin\theta\sum_n\frac{\partial\Psi_n}{\partial x}\\ &\times\int_S e^{iU_n|z-z'|}\Psi_n{}'E_\eta{}'d\xi'd\eta'\\ &+\frac{\sin^2\theta}{2k}\sum_n\frac{U_n}{M_n{}^2}\frac{\partial\Psi_n}{\partial x}\\ &\times\int_S e^{iU_n|z-z'|}\frac{\partial\Psi_n{}'}{\partial x'}E_\eta{}'d\xi'd\eta', \end{aligned} \quad (50)$$

where S now denotes the area of the slot. In (50) we have written for brevity

$$\psi_n=\psi_n(x,y),\quad \Psi_n=\Psi_n(x,y),\quad \psi_n{}'=\psi_n(x',y'),$$
$$\Psi_n{}'=\Psi_n(x',y'),\quad E_\eta{}'=E_\eta(\xi',\eta'),$$

and we are to put, after differentiation, $y=y'=0$. The functions ψ_n, Ψ_n and constants μ_n, M_n are the normalized eigenfunctions and eigenvalues for the E- and H-waves in the rectangular guide, as defined by (9) and (14). All these are, of course, *damped* waves except the H_{01}-wave.

Now we have (Fig. 3),

$$\frac{\partial}{\partial\xi}=\cos\theta\frac{\partial}{\partial z}+\sin\theta\frac{\partial}{\partial x},$$

$$\frac{\partial^2}{\partial\xi^2}=\cos^2\theta\frac{\partial^2}{\partial z^2}+\sin^2\theta\frac{\partial^2}{\partial x^2}+2\cos\theta\sin\theta\frac{\partial^2}{\partial z\partial x},$$

[10] These are special cases of the condition that in an aperture in a perfectly conducting screen the component of **E** parallel to the edge of the aperture tends to zero as we tend to the edge from inside the aperture.

so that we can rewrite (50) as follows:

$$H_\xi(\xi,\eta)=\frac{i}{k}\left(\frac{\partial^2}{\partial\xi^2}+k^2\right)\times\int_S G_2(\xi,\eta;\,\xi',\eta')E_\eta'd\xi'd\eta'$$
$$+\frac{\sin^2\theta}{2k}\sum_n\int_S\left(\frac{U_n}{M_n^2}\frac{\partial\Psi_n}{\partial x}\frac{\partial\Psi_n'}{\partial x'}-\frac{1}{U_n}\frac{\partial^2\Psi_n}{\partial x^2}\Psi_n'\right)e^{iU_n|z-z'|}E_\eta'd\xi'd\eta'$$
$$+\frac{k}{2}\sin^2\theta\sum_n\int_S\left(\frac{1}{u_n\mu_n^2}\frac{\partial\psi_n}{\partial y}\frac{\partial\psi_n'}{\partial y'}e^{iu_n|z-z'|}-\frac{1}{U_n}\Psi_n\Psi_n'e^{iU_n|z-z'|}\right)E_\eta'd\xi'd\eta'$$
$$\mp\frac{i}{2k}\cos\theta\sin\theta\sum_n\int_S\left(\Psi_n\frac{\partial\Psi_n'}{\partial x'}+\Psi_n'\frac{\partial\Psi_n}{\partial x}\right)\times e^{iU_n|z-z'|}E_\eta'd\xi'd\eta',\quad(51)$$

where G_2 is the Green's function given by (13). In (51) we are to put $y=y'=0$ after differentiation, and $G_2(\xi,\eta;\,\xi',\eta')$ means that we are to do the same with G_2.

The integral occurring in the first term in (51) tends to infinity as $\epsilon\to 0$ on account of the singularity in the Green's function G_2, but the remaining three terms remain finite as $\epsilon\to 0$.[11] Making $\epsilon\to 0$ in the last three terms of (51), therefore, and introducing the voltage $V(\xi)$ defined by (18), we can, with an error of the order of ϵ/l, replace (51) by

$$H_\xi(\xi,\eta)=\frac{i}{k}\left(\frac{\partial^2}{\partial\xi^2}+k^2\right)\times\int_S G_2(\xi,\eta;\,\xi',\eta')E_\eta(\xi',\eta')d\xi'd\eta'+\int_{-l}^{l}F(\xi,\xi')V(\xi')d\xi',\quad(52)$$

[11] Each of the three terms actually consists of the sum of two parts, the infinities of which cancel. The considerations of Section 3 show that the only singularity which should occur in H_ξ as $\epsilon\to 0$ is correctly given by the first term of (51).

where

$$F(\xi,\xi')=\frac{\sin^2\theta}{2k}\sum_n\left(\frac{U_n}{M_n^2}\frac{\partial\Psi_n}{\partial x}\frac{\partial\Psi_n'}{\partial x'}-\frac{1}{U_n}\frac{\partial^2\Psi_n}{\partial x^2}\Psi_n'\right)e^{iU_n|z-z'|}$$
$$+\frac{k}{2}\sin^2\theta\sum_n\left(\frac{1}{u_n\mu_n^2}\frac{\partial\psi_n}{\partial y}\frac{\partial\psi_n'}{\partial y'}e^{iu_n|z-z'|}-\frac{1}{U_n}\Psi_n\Psi_n'e^{iU_n|z-z'|}\right)\mp\frac{i}{2k}\cos\theta\sin\theta$$
$$\times\sum_n\left(\Psi_n\frac{\partial\Psi_n'}{\partial x'}+\Psi_n'\frac{\partial\Psi_n}{\partial x}\right)e^{iU_n|z-z'|},\quad(53)$$

and in (53) we are to put, after differentiation,

$$x=x_1+\xi\sin\theta,\quad z=\xi\cos\theta,\quad x'=x_1+\xi'\sin\theta,\quad z'=\xi'\cos\theta,\quad y=y'=0.\quad(54)$$

We now consider the field generated *outside* the guide. With our third fundamental assumption (Section 1), this is simply the field due to E_η across the slot regarded as being in an infinite conducting plane. This problem can be solved by introducing the "associate Hertz vector," or it can be inferred by analogy with the corresponding problem of a thin antenna in the form of a flat strip having a given current distribution.[4] We omit the details and give only the final result:

$$H_\xi(\xi,\eta)=\frac{i}{2\pi k}\left(\frac{\partial^2}{\partial\xi^2}+k^2\right)\times\int_S\frac{e^{ikr}}{r}E_\eta(\xi',\eta')d\xi'd\eta',\quad(55)$$

where

$$r=[(\xi-\xi')^2+(\eta-\eta')^2]^{\frac{1}{2}}.$$

Equations (52) and (55) now give the component H_ξ in the slot of the fields generated inside and outside the guide, respectively, by E_η. According to the continuity condition mentioned at the beginning of this section, the difference between (55) and (52) must be equal to the component H_ξ of the incident field. We denote this latter component by $Af(\xi)/a$, where A is the amplitude of the incident wave at the center of the slot, so that $f(\xi)$ is identical with the function introduced in (26). The continuity condition then gives

$$\frac{i}{k}\left(\frac{d^2}{d\xi^2}+k^2\right)\int_S\left[G_2(\xi,\eta;\xi',\eta')-\frac{1}{2\pi}\frac{e^{ikr}}{r}\right]E_\eta(\xi',\eta')d\xi'd\eta'$$

$$=-\int_{-l}^{l}F(\xi,\xi')V(\xi')d\xi'-Af(\xi)/a. \quad (56)$$

We have written an ordinary, instead of a partial, derivative on the left-hand side of (56), since the right-hand side is a function of ξ only, and the same must therefore be true of the left-hand side. (This is consistent with the assumption —see Section 3—that we can neglect the component of $\mathbf{H}$ across the slot.)

Regarding (56) as a differential equation for the left-hand side, and solving by the method of variation of constants, the general solution can be written

$$\frac{i}{k}\int_S\left[G_2(\xi,\eta;\xi',\eta')-\frac{1}{2\pi}\frac{e^{ikr}}{r}\right]E_\eta(\xi',\eta')d\xi'd\eta'$$

$$=\frac{i}{k}\cos k\xi\int_{-l}^{\xi}d\xi\int_{-l}^{l}d\xi'\cdot\sin k\xi F(\xi,\xi')V(\xi')$$

$$-\frac{i}{k}\sin k\xi\int_{-l}^{\xi}d\xi\int_{-l}^{l}d\xi'\cdot\cos k\xi F(\xi,\xi')V(\xi')$$

$$+\frac{A}{ka}\cos k\xi\int_{-l}^{\xi}\sin k\xi f(\xi)d\xi$$

$$-\frac{A}{ka}\sin k\xi\int_{-l}^{\xi}\cos k\xi f(\xi)d\xi$$

$$+C_1\cos k\xi+C_2\sin k\xi, \quad (57)$$

where C_1, C_2 are arbitrary constants. Equation (57) is now an integral equation for the determination of E_η, or $V(\xi)$, the constants C_1, C_2 being determined by the boundary conditions (49). We shall adopt a method of solution analogous to one used by Hallén[12] in connection with a similar integral equation occurring in an antenna problem. We rewrite the integral on the left-hand side of (57) as

$$\int_S\left(G_2-\frac{1}{2\pi}\frac{e^{ikr}}{r}\right)E_\eta(\xi,\eta')d\xi'd\eta'$$

$$+\int_S\left(G_2-\frac{1}{2\pi}\frac{e^{ikr}}{r}\right)$$

$$\times[E_\eta(\xi',\eta')-E_\eta(\xi,\eta')]d\xi'd\eta'. \quad (58)$$

The contribution to the first integral in (58) from a strip of width $d\eta'$ parallel to the η axis and at a distance η' from it is

$$E_\eta(\xi,\eta')d\eta'\int_{-l}^{l}\left(G_2-\frac{1}{2\pi}\frac{e^{ikr}}{r}\right)d\xi'. \quad (59)$$

Now from the definition of G_2 it follows that, allowing for the "image effect" when the point (x, y, z) is near a wall of the guide, $G_2\sim -1/2\pi r$ when r is small. Hence, since the major contribution to the integral in (59) comes from points where r is small, the value of the integral is approximately

$$-(1/4\pi)\int_{-l}^{l}d\xi'/r.$$

This latter integral has, with an error of order ϵ/l, the value $2\log(2l/|\eta-\eta'|)$ except when ξ is near $\pm l$. But since E_η vanishes when $\xi=\pm l$, this exception is unimportant. We may, therefore, with an error of order ϵ/l, replace the first integral in (58) by

$$-\frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\log\frac{2l}{|\eta-\eta'|}E_\eta(\xi,\eta')d\eta'. \quad (60)$$

The second integral in (58), on the other hand, remains finite as $\epsilon\rightarrow 0$, and may (with an error of order ϵ/l) be replaced by

$$\int_{-l}^{l}\left[G_2(\xi,\xi')-\frac{1}{2\pi}\frac{e^{ik|\xi-\xi'|}}{|\xi-\xi'|}\right][V(\xi')-V(\xi)]d\xi',$$

where $G_2(\xi,\xi')$ means that we are to make $\epsilon\rightarrow 0$ in $G_2(\xi,\eta;\xi',\eta')$.

We now see that (with our approximations) the integral in (60) must be independent of η, which shows that E_η must vary across the slot in the same way that the electrostatic charge density varies across an infinite conducting strip of the

[12] E. Hallén, Nova Acta Reg. Soc. Upsaliensis 11, No. 4 (1938).

same width. We can therefore write

$$\int_{-\epsilon}^{\epsilon} \log\frac{2l}{|\eta-\eta'|} E_\eta(\xi,\eta')d\eta' = \log\frac{2l}{\bar{\epsilon}} V(\xi), \quad (61)$$

where the constant $\bar{\epsilon}$, defined by (61), can be calculated from the known solution of the electrostatic problem of the infinite conducting strip. Obviously $\bar{\epsilon}$ is of the order of ϵ and it really makes no difference, to the order to which we are working, whether we take $\bar{\epsilon}=\epsilon$ or use the correct value. As however, the calculation is easily made, we shall use the correct value; it proves to be $\bar{\epsilon}=\epsilon/2$. Collecting our results, we now see that the integral equation (57) can be replaced by

$$-\frac{i}{2\pi k}\log\frac{4l}{\epsilon}V(\xi) = \frac{A}{k}\cos k\xi \int_{-l}^{\xi} \sin k\xi f(\xi)d\xi$$

$$-\frac{A}{k}\sin k\xi \int_{-l}^{\xi} \cos k\xi f(\xi)d\xi$$

$$+C_1\cos k\xi + C_2 \sin k\xi + \Lambda(\xi), \quad (62)$$

where

$$\Lambda(\xi) = -\frac{i}{k}\int_{-l}^{l} G_2(\xi,\xi')[V(\xi')-V(\xi)]d\xi'$$

$$+\frac{i}{2\pi k}\int_{-l}^{l}\frac{e^{ik|\xi-\xi'|}}{|\xi-\xi'|}[V(\xi')-V(\xi)]d\xi'$$

$$+\frac{i}{k}\cos k\xi\int_{-l}^{\xi}d\xi\int_{-l}^{l}d\xi' \sin k\xi F(\xi,\xi')V(\xi')$$

$$-\frac{i}{k}\sin k\xi\int_{-l}^{\xi}d\xi\int_{-l}^{l}d\xi' \cos k\xi F(\xi,\xi')V(\xi').$$

We can now solve (62) by successive approximations, regarding $\log 4l/\epsilon$ as being large, in accordance with our second fundamental assumption. For a first approximation, we neglect $\Lambda(\xi)$ in (62); for a second approximation, we substitute the first approximation for $V(\xi)$ in $\Lambda(\xi)$, and so on. We then find $V(\xi)$ as a series in the small quantity $(\log 4l/\epsilon)^{-1}$—or more accurately in $(2\log 4l/\epsilon)^{-1}$. The arbitrary constants C_1, C_2 are then determined by using the boundary conditions (49). We are here only concerned with the case $l\sim\lambda/4$, i.e. $kl\sim\pi/2$. We then find that we must proceed to at least the *second* approximation to find the constant C_1, but that having found it the term proportional to C_1 in the first approximation is the major term. Confining ourselves to this approximation, the solution is as follows:

$$V(\xi) = P\cos k\xi,$$

where

$$P/A = \zeta/Ka, \quad (63)$$

ζ being given by (26), and where

$$K = -\frac{i}{\pi}\log\frac{4l}{\epsilon}\cos kl$$

$$-i\int_{-l}^{l}[G_2(l,\xi')+G_2(-l,\xi')]\cos k\xi' d\xi'$$

$$+\frac{i}{\pi}\int_{-l}^{l}\frac{e^{ik(l-\xi')}}{l-\xi'}\cos k\xi' d\xi'$$

$$-\iint_{-l}^{l} F(\xi,\xi')\cos k\xi \cos k\xi' d\xi d\xi'. \quad (64)$$

In all terms in K except the first, we are to put $l=\lambda/4$, $kl=\pi/2$.

If the incident wave is traveling in the negative z-direction, and is of amplitude A', we find similarly that the corresponding voltage amplitude P' is given by

$$P'/A' = \zeta^*/Ka. \quad (65)$$

The results (63) and (65) are those which have already been used in Section 4.

If we write

$$K = K_1 + iK_2,$$

then K_1 is a comparatively simple expression which has already been given in (38). For K_2 we can write, to the order to which we are working,

$$K_2 = (\delta L/\lambda)\log(2\lambda/w) + K_2', \quad (66)$$

where $w=2\epsilon$ is the width of the slot, $\delta L = 2l-\lambda/2$ is the *excess* of the length of the slot over a half-wave-length, and K_2' denotes the imaginary part of the expression for K in (64) omitting the first term, so that K_2' does not depend (to our order of approximation) on the length or width of the slot. From (13) and (53), it will be seen that K_2' is given by quite a complicated expression in the form of a doubly-infinite series.

If the slot is in the broad face of the guide, we have explicitly from (13) and (53), on using the eigenfunctions and eigenvalues for the rectangular guide:[13]

$$G_2(\xi,\xi')=\frac{2}{iab}\sum_{n,m}\frac{\delta_{nm}}{u_{nm}}\cos\frac{n\pi x}{a}\times\cos\frac{n\pi x'}{a}e^{iu_{nm}|\xi-\xi'|\cos\theta},\quad(67)$$

$$F(\xi,\xi')=\frac{2\sin^2\theta}{kab}\sum_{n,m}\delta_{nm}\left(\frac{n\pi}{a}\right)^2\times\left[\frac{u_{nm}}{\mu_{nm}^2}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}+\frac{1}{u_{nm}}\cos\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}\right]e^{iu_{nm}|\xi-\xi'|\cos\theta}$$
$$+\frac{2k\sin^2\theta}{ab}\sum_{n,m}\delta_{nm}\left[\frac{(m\pi/b)^2}{u_{nm}\mu_{nm}^2}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}-\frac{1}{u_{nm}}\cos\frac{n\pi x}{a}\cos\frac{n\pi x'}{a}\right]e^{iu_{nm}|\xi-\xi'|\cos\theta},\quad(68)$$

where

$$x=x_1+\xi\sin\theta,\qquad x'=x_1+\xi'\sin\theta,$$
$$\mu_{nm}^2=(n\pi/a)^2+(m\pi/b)^2,\quad u_{nm}=(k^2-\mu_{nm}^2)^{\frac{1}{2}},$$
$$\delta_{nm}=1,\text{ unless } n \text{ or } m=0,$$
$$=\tfrac{1}{2},\qquad \text{if } n \text{ or } m=0.$$

The summations in (67) and (68) are taken with respect to n and m from 0 to ∞ with the exception of the values $n=m=0$. The positive, or positive-imaginary, square root for u_{nm} is to be taken.

If the slot is in the narrow face of the guide, we need merely interchange a and b in (67) and (68), and, of course, use the appropriate expression for ζ (i.e., (31) instead of (27)).

7. GUIDE-TO-GUIDE COUPLING

Consider now the case where the slot couples two infinite guides—guide 1, where we have an incident (exciting) wave, and guide 2, where waves are generated. Suppose that Fig. 3 gives the orientation of the slot relative to the standard axes for guide 1. Then the orientation of the slot relative to the standard axes for guide 2 will be given by an exactly similar figure (with, of course, different parameters x_1, θ in general). The only difference is that, if we keep the direction of the ξ axis the same in both guides and preserve the relative orientations of the (ξ,η) axes, then the η axes are in opposite directions for the two guides, so that the voltage is the *negative* of that defined by (18) when considered from the point of view of guide 2. This is because the normal to the guide-face drawn *into* the guide has opposite directions for the two guides.

We can then carry through an exactly similar analysis to that used in Section 6, the only difference being that instead of the solution *outside* the guide there used, we must use the solution in guide 2. Obviously the voltage is still approximately sinusoidal and we find for the amplitude P, if the incident wave is of amplitude A and traveling in the positive z-direction,

$$P/A=\zeta_1/K_{12}a_1,\quad(69)$$

where the subscripts 1 in ζ, a refer to guide 1, and where

$$K_{12}=-\frac{i}{\pi}\log\frac{4l}{\epsilon}\cos kl-i\int_{-l}^{l}[G_2^{(1)}(l,\xi')+G_2^{(1)}(-l,\xi')]\cos k\xi'd\xi'-i\int_{-l}^{l}[G_2^{(2)}(l,\xi')+G_2^{(2)}(-l,\xi')]\cos k\xi'd\xi'-\int\int_{-l}^{l}F^{(1)}(\xi,\xi')\cos k\xi\cos k\xi'd\xi d\xi'-\int\int_{-l}^{l}F^{(2)}(\xi,\xi')\cos k\xi\cos k\xi'd\xi d\xi',\quad(70)$$

the superscripts 1, 2 in the functions G_2, F in (70) referring to guides 1, 2, respectively. If the incident wave is traveling in the negative z-direction, we must write ζ_1^* in place of ζ_1 in (69).

Energy considerations similar to those used in Section 5 show that

$$\mathcal{R}(K_{12})=-\gamma_1|\zeta_1|^2-\gamma_2|\zeta_2|^2,$$

[13] The term proportional to $\cos\theta\sin\theta$ in (53) has been omitted from $F(\xi,\xi')$, since it can be seen that this vanishes on integration.

where γ_1, γ_2 are the constants γ defined by (33) for the two guides.

Knowing the voltage amplitude, the reflection and transmission coefficients for guide 1 and the field generated in guide 2 are, of course, easily calculated. It will be seen that if the calculations have been made for the slot in each guide separately, the calculations for the guide-to-guide coupling are readily performed. In fact, reference to (64) and (70) shows that most of the terms in K_{12} are simply the sum of the corresponding terms for the K's of the two guides. It will also be observed that our third fundamental assumption (Section 1) is not used in this case, so that the results should be more accurate.

8. SLOT ARRAYS

We now consider the case of an array of N slots numbered 1, 2, $\cdots$, N in the same face of an infinite guide, and coupled to empty space. It is evident that if the lengths of all the slots are approximately $\lambda/2$ the voltage in each slot is still approximately sinusoidal and the problem is to calculate the amplitudes $P_1, P_2, \cdots, P_N$ in terms of the amplitude of the incident wave. Having done this, we can find the radiation pattern outside the guide, and the reflection and transmission coefficients, etc., inside the guide.

Considering, say, the jth slot, we proceed as before, but now in finding the component of $\mathbf{H}$ along the slot we must include, in addition, the contributions from the voltages in all the other slots, when reckoning the field either inside or outside the guide. We must also allow for the fact that the incident wave has a different phase at the center of each slot. The calculations go very much the same as before, finding a first approximation, and then a second approximation, to the voltage in each slot. We omit the details and quote the final result: the voltage amplitudes are determined by the set of linear equations:

$$\sum_{j'=1}^{N} p_{jj'}P_{j'}=(A/a)\zeta_j e^{iUz_j}, \quad j'=1,\cdots,N, \quad (71)$$

where it has been assumed that the incident wave is traveling in the positive z-direction and has amplitude A at some *arbitrary* point in the guide, z_j being the distance down the guide from this point, measured in the positive z-direction, of the center of the jth slot, and where

$$p_{jj}=K_j, \quad (72)$$

$$p_{jj'}=\int_{-l}^{l}\left[F_{jj'}^{(\mathrm{out})}(\xi)-F_{jj'}^{(\mathrm{in})}(\xi)\right]\cos k\xi d\xi, \quad j\neq j'. \quad (73)$$

In (71), (72), ζ_j, K_j mean the constants ζ, K, already defined, calculated for the jth slot, while in (73), $F_{jj'}^{(\mathrm{out})}(\xi)$ denotes the component of $\mathbf{H}$ along the jth slot due to a sinusoidal voltage of *unit* amplitude in the j'th slot, calculated for *outside* the guide, and $F_{jj'}^{(\mathrm{in})}(\xi)$ has a similar meaning when calculated for *inside* the guide. $F_{jj'}^{(\mathrm{in})}$ can be found from formulae already given; it would probably be sufficient to confine ourselves to the contribution from the H_{01}-wave generated by the jth slot, so that $F_{jj'}^{(\mathrm{in})}(\xi)$ would be quite a simple expression. $F_{jj'}^{(\mathrm{out})}$ can be found by the method outlined in Section 6 for finding the field outside the guide, and again yields a simple closed expression.

There is thus no particular difficulty about calculating the coefficients $p_{jj'}(j\neq j')$, while the coefficients p_{jj} are given by the calculations for a *single* slot. The coefficients $p_{jj'}(j\neq j')$ express, of course, the interaction between the slots. For distant slots, $F_{jj'}^{(\mathrm{out})}$ decreases as $(z_j-z_{j'})^{-1}$, while $F_{jj'}^{(\mathrm{in})}$ remains constant. While the calculation of all the coefficients $p_{jj'}$ and the solution of the set of equations (71) would in general present a formidable task, it would appear that in the arrays of practical importance considerable simplifications are possible. The K_j might be determined experimentally.

If the incident wave is traveling in the negative z-direction we need merely write $\zeta_j^* e^{-iUz_j}$ in place of ζe^{iUz_j} in (71).

9. THE CASE OF A TERMINATED GUIDE

So far we have dealt entirely with infinite guides. In practice, however, the guide may have an unmatched termination, so that we must take account of the termination (a guide with a matched termination can, of course, be treated as an infinite guide). We shall indicate briefly here how a termination of arbitrary impedance can be dealt with, and it will appear that this case can be deduced very simply from that of the infinite guide already dealt with.

Suppose we take the z axis pointing towards the termination, and that we have available solutions for the infinite guide for incident waves traveling in either direction. Assuming that all slots are at a sufficient distance from the termination (and in the negative z-direction from it) for all waves but the H_{01}-wave to be damped out at the termination, the infinite-guide-solution for a wave incident in the positive z-direction will, in the neighborhood of the termination, be an H_{01}-wave whose generating function is, say, $\chi_1^{(+)}$. Similarly, the infinite-guide-solution for a wave incident in the negative z-direction will, in the neighborhood of the termination, be a superposition of H_{01}-waves traveling in both directions, so that its generating function will be of the form $\chi_2^{(-)}+\alpha\chi_2^{(+)}$, where α is a constant of the nature of a reflection coefficient (the superscripts +, − refer to waves traveling in the positive and negative z-directions, respectively).

We now consider the field given by

$$\chi_1^{(+)}+\lambda[\chi_2^{(-)}+\alpha\chi_2^{(+)}],$$

where λ is an arbitrary constant. By proper choice of λ we can now make the ratio of the amplitudes of the incident to the reflected waves at the termination anything we please, and hence can give the termination an arbitrary "impedance." Thus a linear combination of the two solutions for the infinite guide, for incident waves traveling in opposite directions, gives the solution for the semi-infinite guide with an arbitrary termination. In particular, the voltage amplitude in any slot will be

$$P_1+\lambda P_2$$

where P_1, P_2, are the voltage amplitudes for the two solutions for the infinite guide.

APPENDIX

On the Correction Due to Imperfect Conductivity of the Guide Walls

To take account completely of the finite conductivity of the walls of the guide would be quite possible, but would lead to elaborate computations. The following very rough calculation, in which we consider the problem of Section 5, indicates, however, that the error caused by the assumption of perfect conductivity is not likely to be great.

The chief losses caused by the imperfect conductivity of the walls will occur near the slot, where the fields are large. In this region we can estimate the tangential components of the magnetic field by analogy with the problem of the half-wave antenna (see Section 3). We thus find that, near the slot, the major component of $\mathbf{H}$ is that perpendicular to the length of the slot, and is of amount

$$H=\frac{P}{2\pi\eta}\left[(l-\xi)\frac{e^{ikr_2}}{r_2}-(l+\xi)\frac{e^{ikr_1}}{r_1}\right],$$

where (ξ,η) are the coordinates of a point in the guide face containing the slot relative to the axes of Fig. 3, r_1, r_2 are the distances of the point from the ends of the slot, and the notation is otherwise that used before.[14] Except near the ends of the slot, this can be replaced approximately, if η is small, by

$$H=\frac{P}{2\pi\eta}[e^{ik(l-\xi)}-e^{ik(l+\xi)}]. \tag{A1}$$

If, as we may assume, the depth of penetration is small compared with the thickness of the wall, we have approximately for the tangential *electric* field when imperfect conductivity is allowed for,[15]

$$\mathbf{E}=(1-i)\left(\frac{f}{4\sigma}\right)^{\frac{1}{2}}(\mathbf{H}\times\mathbf{n}), \tag{A2}$$

where $\mathbf{n}$ is a unit vector drawn into the guide wall, f is the frequency, and σ the conductivity of the guide wall.

Using (A1), (A2), we thus have for the mean flux of energy into the guide wall, per unit area,

$$\frac{c}{8\pi}\Re(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{n}=\frac{c}{8\pi}\left(\frac{f}{4\sigma}\right)^{\frac{1}{2}}|H|^2 =\frac{c}{16\pi^3}\left(\frac{f}{4\sigma}\right)^{\frac{1}{2}}\frac{|P|^2}{\eta^2}(1-\cos 2k\xi).$$

The total mean flux of energy into the guide walls is thus of the order

$$2\cdot\frac{c}{16\pi^3}\left(\frac{f}{4\sigma}\right)^{\frac{1}{2}}|P|^2\cdot l'\int_\epsilon\frac{d\eta}{\eta^2}\sim\frac{c}{8\pi^3}\left(\frac{f}{4\sigma}\right)^{\frac{1}{2}}|P|^2\frac{l'}{\epsilon}, \tag{A3}$$

where l' is a length of the order of the length of the slot, and 2ϵ is, as before, the width of the slot.

Comparing (A3) with the expression (37) for the mean flux of energy out of the slot, we see that the error involved in assuming perfect conductivity of the walls is of the order

$$\frac{1}{\pi}\left(\frac{f}{4\sigma}\right)^{\frac{1}{2}}\frac{l'}{\epsilon}.$$

For Copper and a wave-length of the order of 1 cm., $f:4\sigma$ is of the order 10^{-6}. A reasonable value of l'/ϵ would be about 30. The above error is then about 10^{-2} or one percent.

[14] See, for instance, Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 457, formula 76, on interchanging **E** and **H** and allowing for the difference in units. Note that Stratton's I_0 corresponds to $(c/2\pi)P$ in the present case. We have disregarded an unimportant phase factor.

[15] See, for instance, reference 14, p. 534, formula 47, allowing for the difference in units.