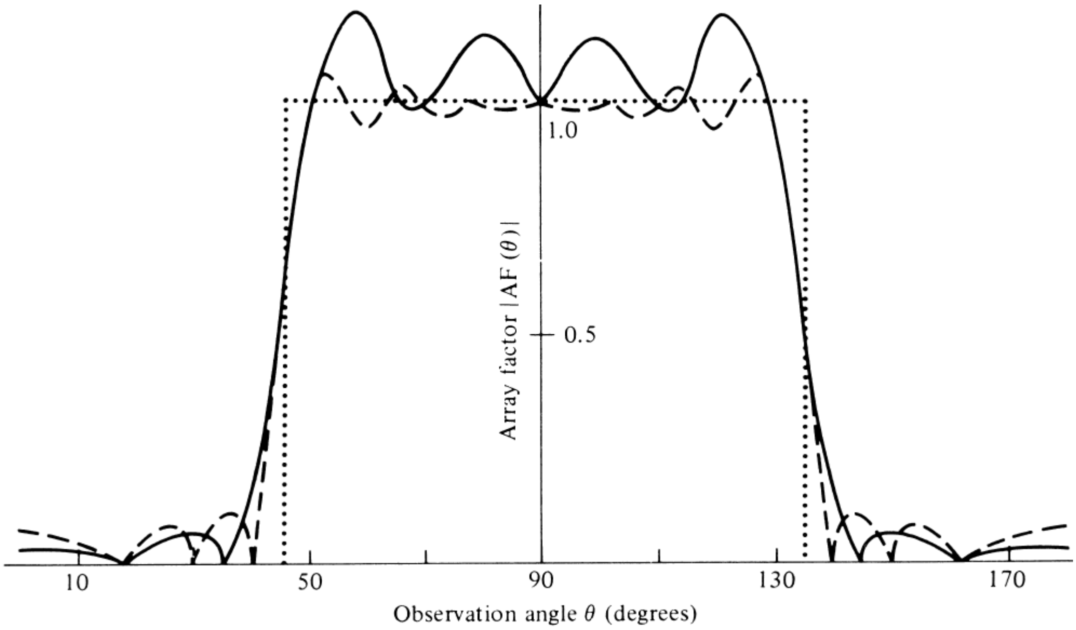
Fourier Transform Method (Linear Array)

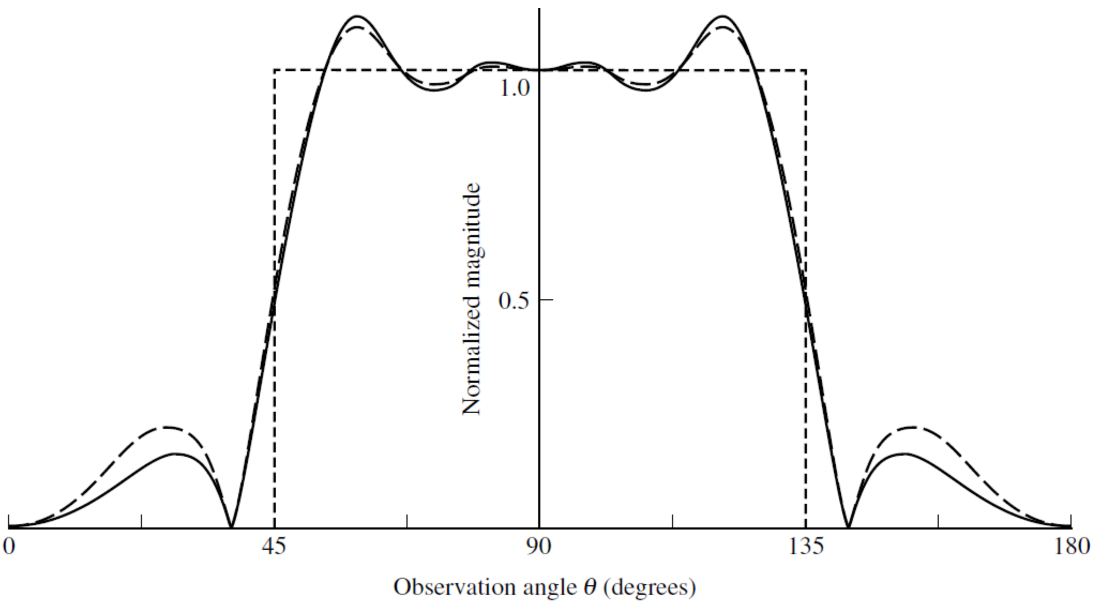


Desired pattern

Linear array ($N = 11, d = \lambda/2$) SLL = -24.29 dB

Linear array ($N = 21, d = \lambda/2$) SLL = -19.33 dB

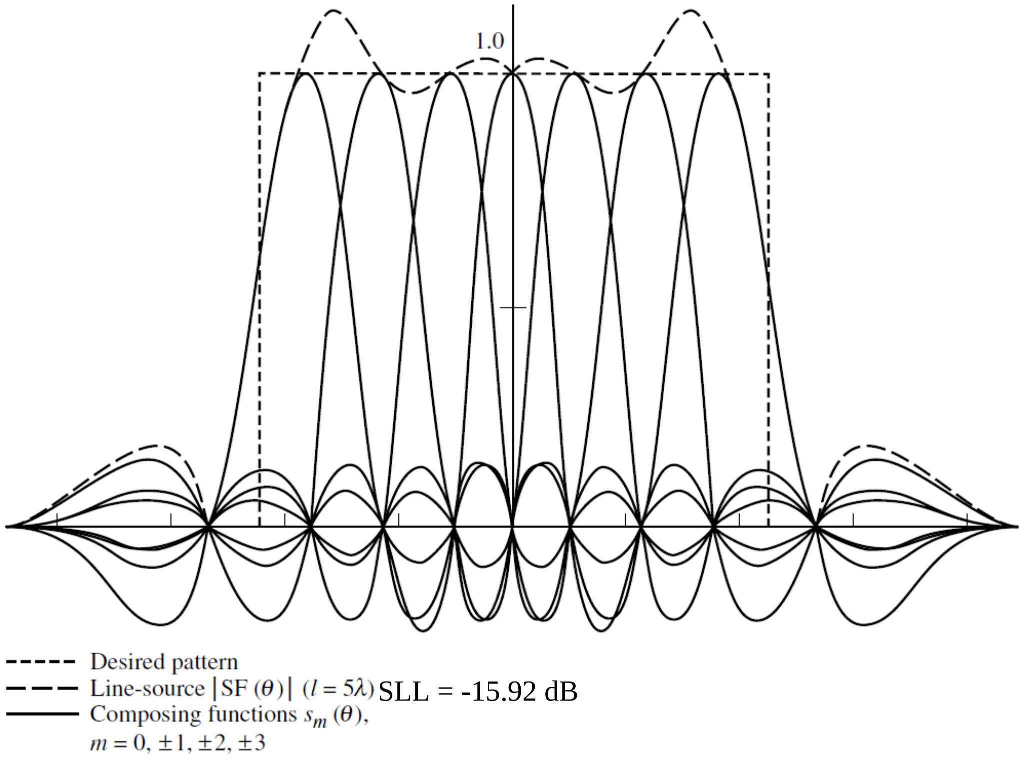
Woodward-Lawson Method (Line Source)



Desired pattern

Line-source $|SF(\theta)|$ ($l = 5\lambda$) SLL = -15.92 dB

Linear array $|AF(\theta)|$ ($N = 10, d = \lambda/2$) SLL = -13.10 dB



Chebyshev Polynomials

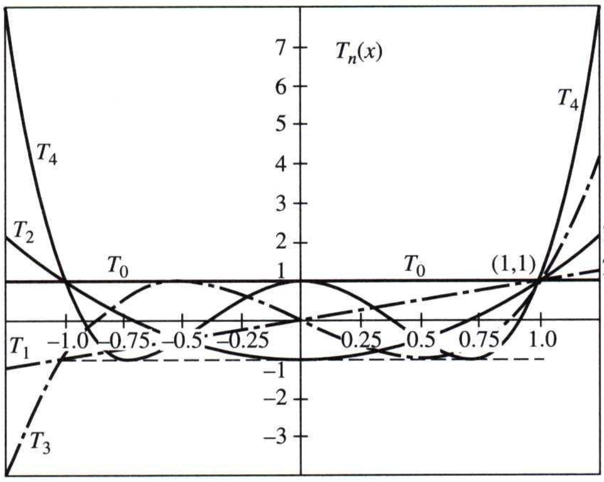


Figure 8-5 Chebyshev polynomials $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$, and $T_4(x)$.

$$z = \cos u$$

$$m = 0 \quad \cos(mu) = 1 = T_0(z)$$
$$m = 1 \quad \cos(mu) = z = T_1(z)$$
$$m = 2 \quad \cos(mu) = 2z^2 - 1 = T_2(z)$$
$$m = 3 \quad \cos(mu) = 4z^3 - 3z = T_3(z)$$
$$m = 4 \quad \cos(mu) = 8z^4 - 8z^2 + 1 = T_4(z)$$
$$m = 5 \quad \cos(mu) = 16z^5 - 20z^3 + 5z = T_5(z)$$
$$m = 6 \quad \cos(mu) = 32z^6 - 48z^4 + 18z^2 - 1 = T_6(z)$$
$$m = 7 \quad \cos(mu) = 64z^7 - 112z^5 + 56z^3 - 7z = T_7(z)$$
$$m = 8 \quad \cos(mu) = 128z^8 - 256z^6 + 160z^4 - 32z^2 + 1 = T_8(z)$$
$$m = 9 \quad \cos(mu) = 256z^9 - 576z^7 + 432z^5 - 120z^3 + 9z = T_9(z)$$

$$T_1(q\cos\theta) = q\cos\theta$$

$$T_2(q\cos\theta) = q^2\cos 2\theta + (q^2 - 1)$$

$$T_3(q\cos\theta) = q^3\cos 3\theta + (3q^3 - 3q)\cos\theta$$

$$T_4(q\cos\theta) = q^4\cos 4\theta + (4q^4 - 4q^2)\cos 2\theta + (3q^4 - 4q^2 + 1)$$

$$T_5(q\cos\theta) = q^5\cos 5\theta + (5q^5 - 5q^3)\cos 3\theta + (10q^5 - 15q^3 + 5q)\cos\theta$$

$$T_6(q\cos\theta) = q^6\cos 6\theta + (6q^6 - 6q^4)\cos 4\theta + (15q^6 - 24q^4 + 9q^2)\cos 2\theta + (10q^6 - 18q^4 + 9q^2 - 1)$$

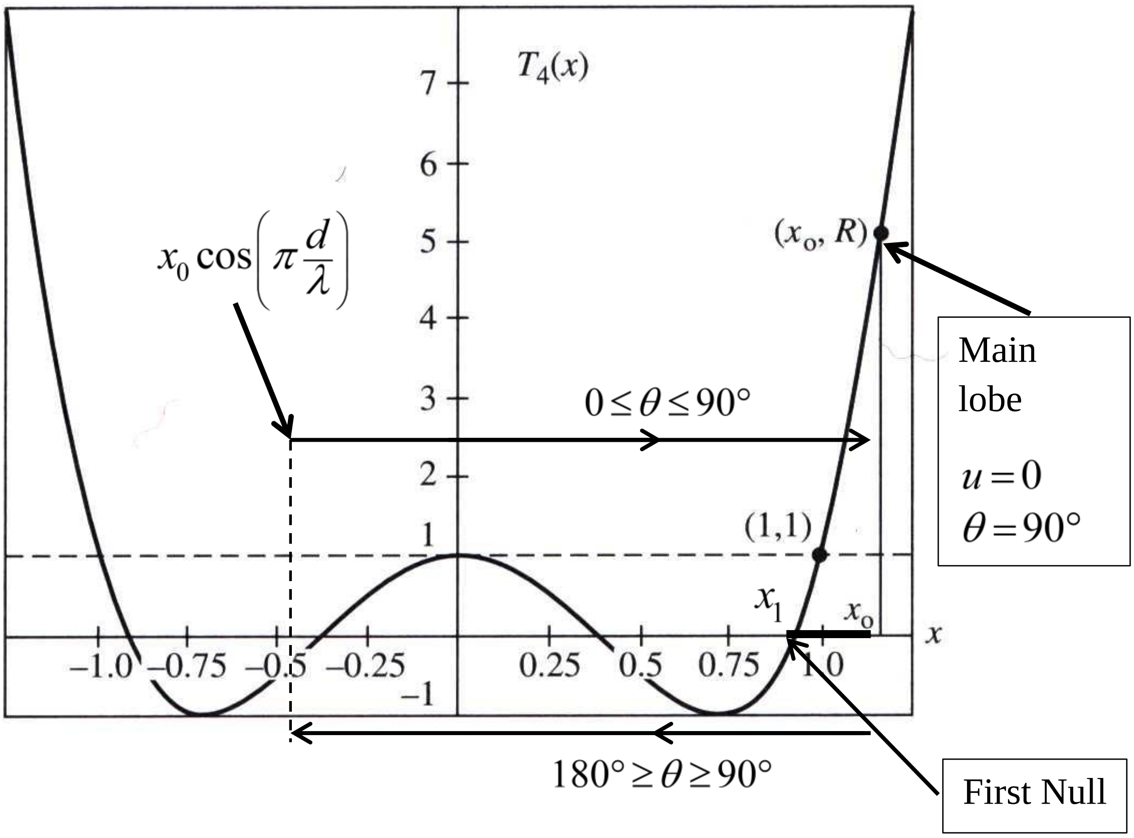
$$T_7(q\cos\theta) = q^7\cos 7\theta + (7q^7 - 7q^5)\cos 5\theta + (21q^7 - 35q^5 + 14q^3)\cos 3\theta +$$
$$+ (35q^7 - 70q^5 + 42q^3 - 7q)\cos\theta$$

$$T_8(q\cos\theta) = q^8\cos 8\theta + (8q^8 - 8q^6)\cos 6\theta + (28q^8 - 48q^6 + 20q^4)\cos 4\theta +$$
$$+ (56q^8 - 120q^6 + 80q^4 - 16q^2)\cos 2\theta + (35q^8 - 80q^6 + 60q^4 - 16q^2 + 1)$$

$$T_9(q\cos\theta) = q^9\cos 9\theta + (9q^9 - 9q^7)\cos 7\theta + (36q^9 - 63q^7 + 27q^5)\cos 5\theta +$$
$$+ (84q^9 - 189q^7 + 135q^5 - 30q^3)\cos 3\theta + (126q^9 - 315q^7 + 270q^5 - 90q^3 + 9q)\cos\theta$$

$$T_{10}(q\cos\theta) = q^{10}\cos(10\theta) + (10q^{10} - 10q^8)\cos(8\theta) + (45q^{10} - 80q^8 + 35q^6)\cos(6\theta) +$$
$$(120q^{10} - 280q^8 + 210q^6 - 50q^4)\cos(4\theta) + (210q^{10} - 560q^8 + 525q^6 - 200q^4 + 25q^2)\cos(2\theta)$$
$$(126q^{10} - 350q^8 + 350q^6 - 150q^4 + 25q^2 - 1)$$

$$T_{11}(q\cos\theta) = q^{11}\cos(11\theta) + (11q^{11} - 11q^9)\cos(9\theta) + (55q^{11} - 99q^9 + 44q^7)\cos(7\theta) +$$
$$(165q^{11} - 396q^9 + 308q^7 - 77q^5)\cos(5\theta) + (330q^{11} - 924q^9 + 924q^7 - 385q^5 + 55q^3)\cos(3\theta)$$
$$(462q^{11} - 1386q^9 + 1540q^7 - 770q^5 + 165q^3 - 11q)\cos(\theta)$$



$$x_1 = x_0\cos u_1 = \cos\left(\frac{\pi/2}{N-1}\right)$$

The array factor was derived in class. For $N = 2M + 1$:

$$AF(\psi) = I_0 \left[1 + 2 \sum_{m=1}^{M} \frac{I_m}{I_0} \cos\left(2m\frac{\psi}{2}\right) \right] = I_0 \left[1 + 2 \sum_{m=1}^{M} \frac{I_m}{I_0} T_{2m}\left(\cos\frac{\psi}{2}\right) \right]$$

For $N = 2M$:

$$AF(\psi) = 2I_1 \sum_{m=1}^{M} \frac{I_m}{I_1} \cos\left[(2m-1)\frac{\psi}{2}\right] = 2I_1 \sum_{m=1}^{M} \frac{I_m}{I_1} T_{2m-1}\left(\cos\frac{\psi}{2}\right)$$

$$AF(\psi) = T_{N-1}\left(x_0 \cos\frac{\psi}{2}\right) = T_{N-1}(x_0 \cos u)$$

$$u = \frac{\psi}{2} = \pi\frac{d}{\lambda}\cos\theta$$

For broadside case.

Roots of $T_{N-1}(x)$ are $x_m = \pm\cos\left[\frac{(2m-1)\pi}{2(N-1)}\right]$ for $m = 1,2,3,\cdots,(N-1)/2$ or $m = 1,2,3,\cdots,N/2$

Only positive zeros ($0 \le x_m < 1$) are sufficient. The negative zeros will be automatically taken into account in the following. For each m, the roots in ψ and w plane are:

$$\pm\psi_m = \pm 2\cos^{-1}\left(\frac{x_m}{x_0}\right) \Rightarrow w_m, \bar{w}_m = e^{\pm j\psi_m}$$

Note that each m corresponds to a conjugate pair of zeros in w plane. If N is even, there is always a zero at $\psi_{N/2} = \pi$ or $w_{N/2} = -1$ which only appears in the visible pattern if $d \ge \frac{\lambda}{2}$. If N is odd, there is no zero at $\psi = \pi$. The array polynomial is constructed by

$$f(w) = \prod_{m=1}^{(N-1)/2} (w - w_m)(w - \bar{w}_m) = \prod_{m=1}^{(N-1)/2} (w^2 - 2\cos(\psi_m)\, w + 1)$$

if N is odd or

$$f(w) = (w+1) \prod_{m=1}^{N/2-1} (w - w_m)(w - \bar{w}_m) = (w+1) \prod_{m=1}^{N/2-1} (w^2 - 2\cos(\psi_m)\, w + 1)$$

if N is even.

The array excitations (relative to I_N) are the coefficients of the above polynomial

$$f(w) = \sum_{k=1}^{N} \frac{I_k}{I_N} w^{k-1}$$

It is important to note that we must always construct the complete polynomial of degree $N-1$ regardless of d/λ. What portion of this polynomial (how many nulls) will appear in the visible region is another matter and it will be determined by d/λ.

The array excitations can also be calculated from the Barbiere equations given below.

For $N = 2M + 1$:

$$I_{-m} = I_m = \sum_{k=m}^{M} (-1)^{M-k} x_0^{2k} \frac{(2M)(M+k-1)!}{\epsilon_m (M-k)!\ (k-m)!\ (k+m)!}$$

$$m = 0,1,2,3,\cdots,M$$

$$\epsilon_m = \begin{cases} 1 & m \neq 0 \\ 2 & m = 0 \end{cases}$$

For $N = 2M$:

$$I_{-m} = I_m = \sum_{k=m}^{M} (-1)^{M-k} x_0^{2k-1} \frac{(2M-1)(M+k-2)!}{(M-k)!\ (k-m)!\ (k+m-1)!}$$

$$m = 1,2,3,\cdots,M$$

The main beam maximum value is R_0 and it occurs at $x_0 > 1$ which corresponds to $u = 0$ or $\psi = 0$. For broadside case

$$\theta = 0\cdots 90^\circ \cdots 180^\circ \Rightarrow u = \frac{\psi}{2} = \pi\frac{d}{\lambda}\cdots 0 \cdots -\pi\frac{d}{\lambda} \Rightarrow$$
$$\cos u = \cos\left(\pi\frac{d}{\lambda}\right)\cdots 1 \cdots \cos\left(\pi\frac{d}{\lambda}\right) \Rightarrow x = x_0\cos\left(\pi\frac{d}{\lambda}\right)\cdots x_0 \cdots x_0\cos\left(\pi\frac{d}{\lambda}\right)$$

$$\text{Visible region}\quad 0 \le \theta \le \pi \Rightarrow x_0 \cos\left(\pi\frac{d}{\lambda}\right) \le x \le x_0$$

The first null occurs at

$$x_1 = x_0\cos u_1 = \cos\left(\frac{\pi/2}{N-1}\right) \Rightarrow \psi_1 = 2\cos^{-1}\left[\frac{1}{x_0}\cos\left(\frac{\pi/2}{N-1}\right)\right] = kd\cos\theta_1$$

Let $\theta_1 = \pi/2 - \alpha$ where $\Theta_n = 2\alpha$ is the FNBW then

$$x_0 = \frac{\cos\left(\frac{\pi/2}{N-1}\right)}{\cos\left[(\pi d/\lambda)\sin\alpha\right]}$$

The region between $x_1 \le x \le x_0$ is the main beam between first nulls. The SLL is the main beam to side lobe ratio and it is equal to R_0

$$\text{SLL} = -20\log R_0$$

$$R_0 = \cosh\left[(N-1)\cosh^{-1}x_0\right] \Rightarrow x_0 = \cosh\left[\frac{1}{N-1}\cosh^{-1}R_0\right] = \frac{1}{2}\left[\left(R_0 + \sqrt{R_0^2-1}\right)^{1/(N-1)} + \left(R_0 - \sqrt{R_0^2-1}\right)^{1/(N-1)}\right]$$

The optimum element spacing or the maximum allowable element spacing is given by:

$$-1 \le x_0 \cos\left(\pi \frac{d_{max}}{\lambda}\right) \Rightarrow d_{max} \le \lambda \left(1 - \frac{\cos^{-1}\left(\frac{1}{x_0}\right)}{\pi} \right) \quad \text{for broadside array}$$

$$d_{max} \le \frac{\lambda}{2} \left(1 - \frac{\cos^{-1}\left(\frac{1}{x_0}\right)}{\pi} \right) \quad \text{for endfire array}$$

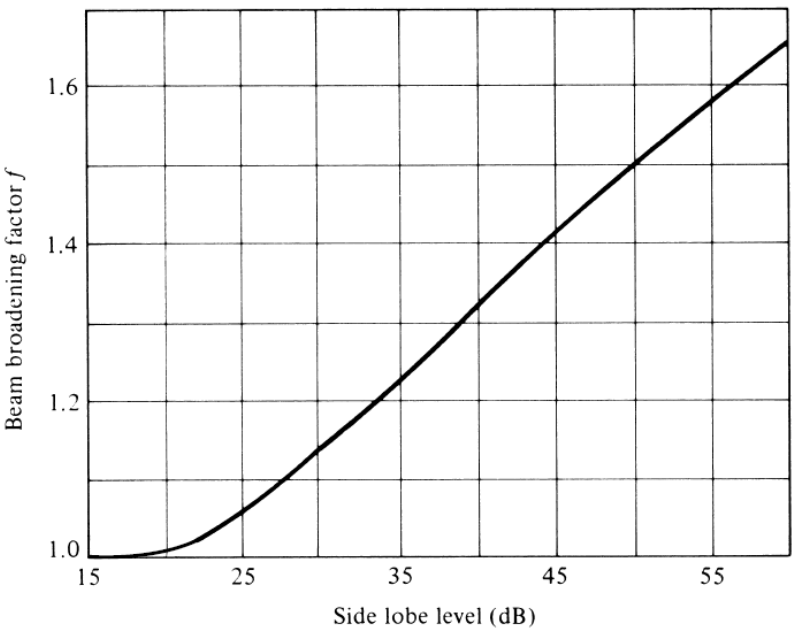
The HPBW of a Dolph-Chebyshev array in general is given by

$$\text{HPBW} = \pi - 2\cos^{-1}\left(\frac{\psi_h}{kd}\right) \qquad \text{Broadside}$$

$$\text{HPBW} = \cos^{-1}\left(1 - \frac{\psi_h}{kd}\right) \qquad \text{Endfire}$$

$$\psi_h = 2\cos^{-1}\left[\frac{\cosh\left(\frac{1}{N-1}\cosh^{-1}\frac{R_0}{\sqrt{2}}\right)}{\cosh\left(\frac{1}{N-1}\cosh^{-1}R_0\right)}\right] = 2\cos^{-1}\left[\frac{1}{x_0}\cosh\left(\frac{1}{N-1}\cosh^{-1}\frac{R_0}{\sqrt{2}}\right)\right]$$

<u>Beam broadening factor:</u> is the ratio by which the beamwidth of a Dolph-Chebyshev array is larger than the corresponding uniformly excited array (with the same number of elements and spacing)



(a) Beam broadening factor

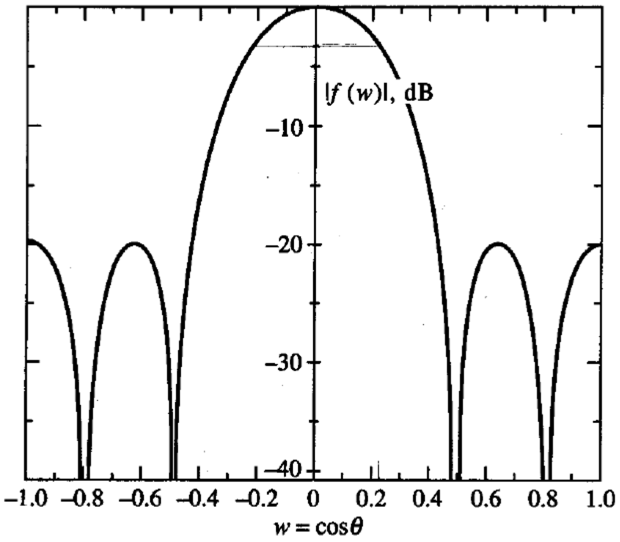
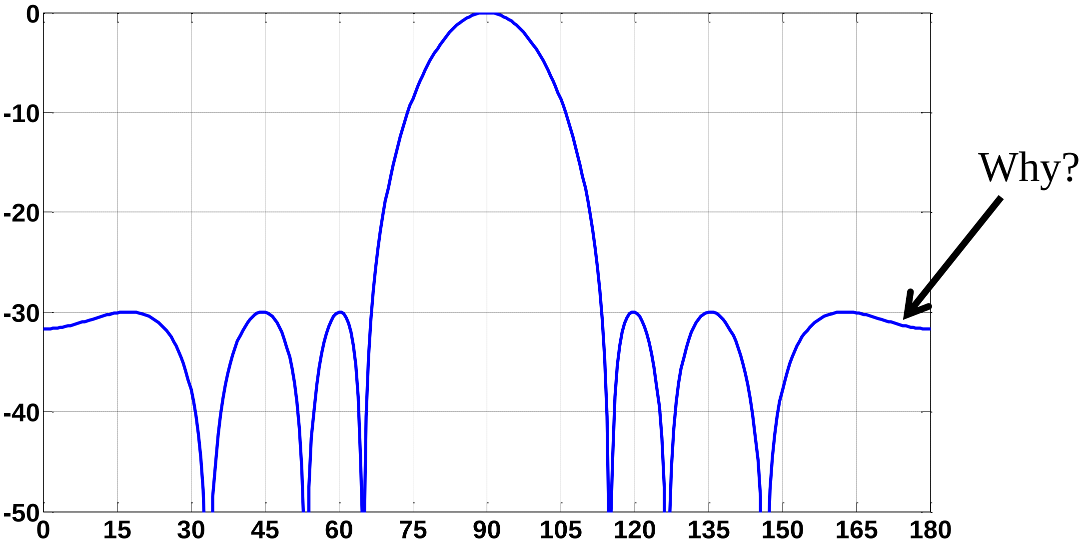
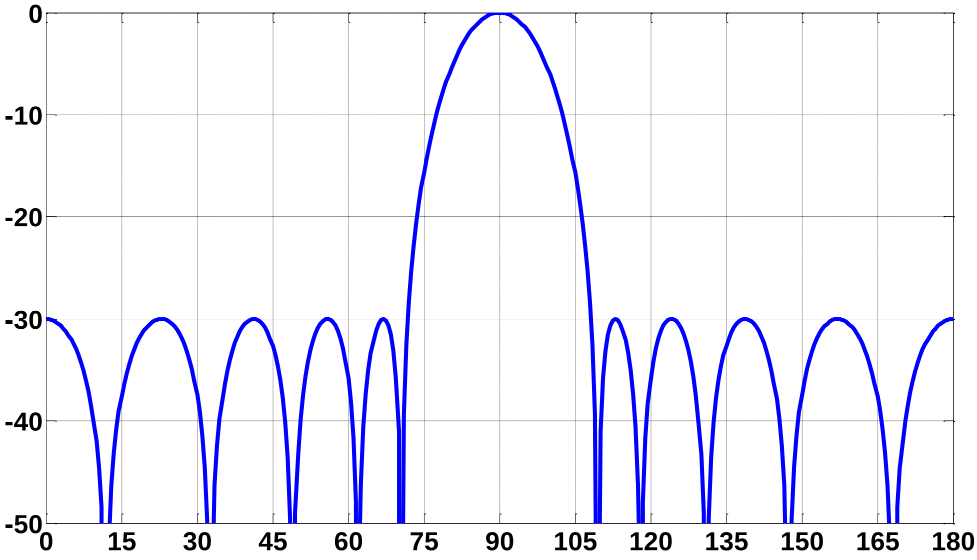


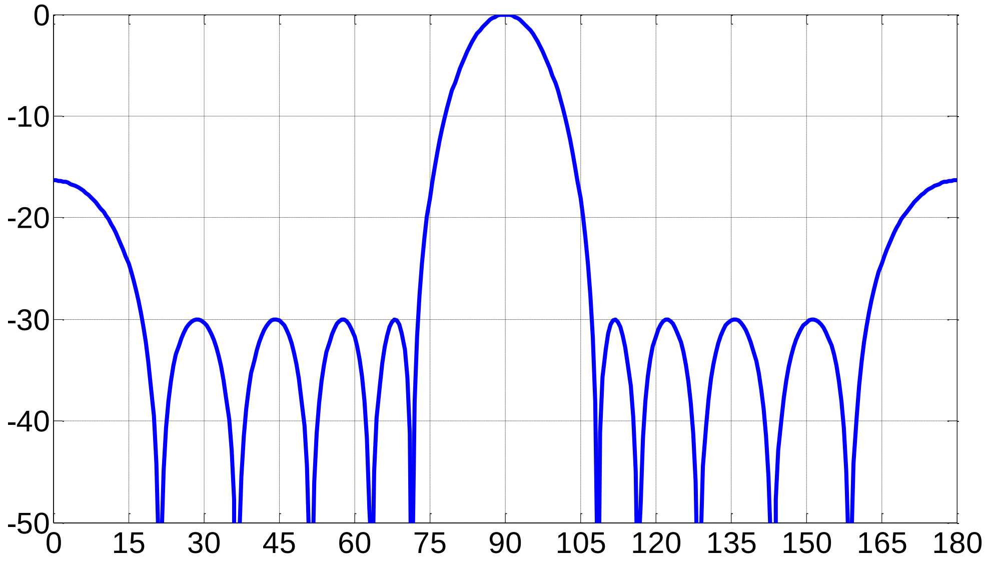
Figure 8-7 Dolph–Chebyshev synthesized array factor for a five-element, λ/2 spaced, broadside array with −20-dB side lobes (Example 8-5).



N=6 element Dolph-Chebyshev array with SLL = -30dB and d=0.6λ



N=6 element Dolph-Chebyshev array with SLL = -30dB and $d = d_{max} = 0.7169\lambda$



N=6 element Dolph-Chebyshev array with SLL = -30dB and d=0.8λ