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Communications

On Numerical Convergence of Moment Solutions of Moderately Thick Wire Antennas Using Sinusoidal Basis Functions

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Abstract—**Wire antennas are solved using a moments solution where the method of subsectional basis is applied with both the expansion and testing functions being sinusoidal distributions. This allows not only a simplification of near-field terms but also the far-field expression of the radiated field from each segment, regardless of the length L. Using sinusoidal basis functions, the terms of the impedance matrix obtained become equivalent to the mutual impedances between the subsectional dipoles. These impedances are the familiar impedances found using the induced EMF method. In the induced EMF method an equivalent radius is usually used in the evaluation of the self-impedance term to reduce computation time. However, it is shown that only for very thin segments that the correct equivalent radius is independent of length. When the radius to length ratio (a/L) is not small, an expansion for the equivalent radius in terms of a/L is given for the self-impedance term. The use of incorrect self-term, obtained by using a constant equivalent radius term, is shown to be responsible for divergence of numerical solutions as the number of sections is increased. This occurrence is related to the ratio of a/L of the subsections and hence becomes a problem for moderately thick wire antennas even for a reasonably small number of segments per wavelength. Examples are given showing the convergence with the correct self-terms and the divergence when only a length independent equivalent radius is used. The converged solutions are also compared to King's second- and third-order solutions for moderately thick dipoles.**

I. Introduction

The method of moments is applied to wire antennas as discussed in other papers [1], [2], but carried to a higher order of approximation which allows treating the case where the length to radius ratio is small. The theory will discuss the straight wire antenna but the extension to wires of arbitrary shape is straightforward.

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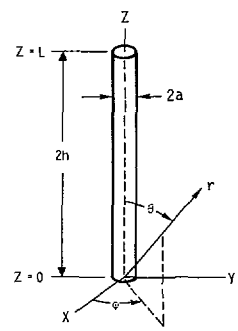

Fig. 1. Straight wire and coordinate system.

Fig. 1 shows a straight section of wire of circular cross section, and defines the coordinate system. The wire extends from $z = 0$ to $z = L$ along the z axis and is of radius a. It is assumed that the radius is small compared to a wavelength but the ratio of a to L need not be small. The only significant component of current on the wire is then the axial component, which can be expressed in terms of the net current $I(z)$ at any point z along the wire. The current distribution will then be modeled as an infinitely thin sheet of current forming a tube of radius a, with the density of current independent of circumferential position on the tube.

An operator equation for the problem is given by

$$\mathcal{L}I(z) = \frac{j}{4\pi\omega\epsilon}\left(\frac{d^2}{dz^2} + k^2\right)\oint_c \int_0^L \frac{e^{-jkR}}{R} I(z')\, dz'\, dc = E_z^i(z) \quad (1)$$

where $E_z^i(z)$ is the z component of the impressed electric field at the wire surface, $I(z')$ is surface current density, $\oint_c dc$ represents the integration around the circumference, and R is the distance from the source point to the field point. The boundary condition for the current is $I(0) = I(L) = 0$.

II. Theory

The procedure is basically one for which the wire is divided into subsections, and a generalized impedance matrix (Z) obtained to describe the electromagnetic interactions between subsections. The problem is thus reduced to a matrix one of the form

$$[Z][I] = [V] \tag{2}$$

where $[I]$ is related to the current on the subsections and $[V]$ to the electromagnetic excitation of the subsections.

Matrix inversion is a simple procedure for high-speed digital computers, and hence the problem is considered solved once a well-conditioned matrix $[Z]$ is obtained. Of considerable importance is the ease and speed of evaluating the matrix elements and the realization of a well-conditioned matrix $[Z]$.

The solution to be described uses sinusoidal subsectional currents and Galerkin's method [3], which is equivalent to the reaction concept [4], and the variational method [5]. Let the wire be broken up into N segments each of length $2H$ and let $I(z)$ be expanded in a series of sinusoidal functions

$$I(z) \approx \sum_{n=1}^{N-1} I_n S(z - nH) \tag{3}$$

where I_n are constants and

$$S(z) = \begin{cases} \sin k(H - |z|), & |z| < H \\ 0, & |z| > H. \end{cases} \tag{4}$$

Substituting (3) into (1), and using the linearity of $\mathcal{L}$, one has

$$\sum_{n=1}^{N-1} I_n \mathcal{L} S(z - nH) \approx E_z^i(z). \tag{5}$$

Each side of (5) is multiplied by $S(z - mH)$, $m = 1,2,\cdots,N - 1$, and integrated from 0 to L on z. This results in the matrix of (2), where the elements of $[I]$ are I_n, those of $[Z]$ are

$$Z_{mn} = \int_0^L S(z - mH) \mathcal{L} S(z - nH)\, dz \tag{6}$$

and those of $[V]$ are

$$V_m = \int_0^L S(z - mH) E_z^i(z)\, dz. \tag{7}$$

In solving thin wire antennas, the integration around the current tube is normally removed by replacing the integral with the value of the integrand at one point. This then reduces the equation to a single integral and obviates the singularity of the integrand which occurs when the source and field points coincide during the calculation of the self-term and first-adjacent mutual terms. The singularity is of course integrable, and by suitably expanding the integrand, special series for these terms can be obtained and the integration performed in closed form. However, many authors have used an "average" value equal to the radius a. This approximation is described as assuming the current to be totally located on the center axis and the distance a is used to represent an average distance from the current filament to the true current surface. One of the purposes of this communication is to show that the value used in this approximation is critical to the convergence of the solution as the number of segments N and hence the ratio of relative half-length H to radius a becomes comparable to unity. In fact, the use of a single value of the equivalent radius will be shown to be incorrect at any time but less important for a small radius. To show this, first consider the evaluation of the general term Z_{mn} of an infinitely thin current filament. Since $S(z)$ is the same sinusoidal function used in evaluating radiation and impedances via the induced EMF method and $\mathcal{L}S(z)$ is the z directed electric field radiated by the subsectional dipole, it can easily be shown that (see [6], [7], and Fig. 2) Z_{mn} is given by

$$Z_{mn} = 30 \int_{H_{m-1}}^{H_{m+1}} \left(\frac{-j \exp(-jkR_1)}{R_1} - \frac{j \exp(-jkR_2)}{R_2} + 2j \cos kH_n \frac{\exp(jkR_0)}{R_0} \right) \sin k(H_m - |z|)\, dz. \tag{8}$$

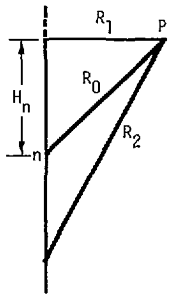

Fig. 2. Geometry for evaluation of Z_{mn}.

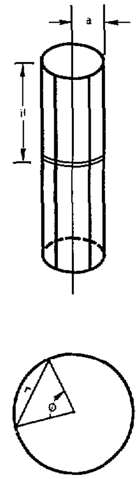

Fig. 3. Geometry for self-term calculation.

The only problem occurs for the self-term and the first-adjacent subdipole when the source and field points coincide and hence the impedance calculation for infinitely thin dipoles would yield a value of infinity. It is evident that in computing the impedance of these terms the finite diameter of the antennas will have to be considered.

The self-impedance of the finite diameter subsegment can be accomplished as follows: Consider the finite diameter segment to be made up of a number of very thin strips of height $2H$ arranged in a circle of radius a as shown in Fig. 3. The strips are all assumed to be center-fed with a voltage V; hence, the voltage can be written as follows:

$$V = Z_{11}I_1 + Z_{12}I_2 + \cdots + Z_{1n}I_n. \tag{9}$$

Since the currents $I_1, I_2, \cdots, I_n$ are all identical and are equal to the total current on the dipole divided by the width of the strip and the impedance $Z_{11}, Z_{12}, \cdots, Z_{1n}$ are the self- and mutual-impedances of the thin strips, (9) can then be rewritten as a sum over the impedances

$$V = \frac{I_T}{2\pi} d\phi \sum_n Z_{1n}. \tag{10}$$

Transferring the sum into an integral using the impedances as a function of $r = (2)^{1/2} a (1 - \cos\phi)^{1/2}$ and also using the definition of self-impedance as being the voltage divided by the current, we have

$$Z_{\text{self}} = \frac{V}{I_T} = \frac{1}{2\pi} \int_0^{2\pi} Z(r)\, d\phi \tag{11}$$

where $Z(r)$ is the mutual impedance between two infinitely thin dipoles separated by a distance r.

It is possible for special cases to come up with a closed form approximation for the self-impedance of a subsegment. First, consider the very thin wire case where $a \ll 1$ and $a \ll H$. Writing the impedance $Z = R + jX$ and considering the resistance part first, we can write R as a function of r as follows:

$$R(r) = 60\left[\sin(kH)\cos(kH)\left(\mathrm{Si}(4kH) - \mathrm{Si}\left(\frac{kr^2}{4H}\right) - 2\,\mathrm{Si}(2kH) + 2\,\mathrm{Si}\left(\frac{kr^2}{2H}\right)\right) - \frac{\cos(2kH)}{2}\left(2\,\mathrm{Ci}\left(\frac{kr^2}{2H}\right) - 2\,\mathrm{Ci}(kr) + 2\,\mathrm{Ci}(2kH) - \mathrm{Ci}(4kH) - \mathrm{Ci}\left(\frac{kr^2}{4H}\right)\right) - \left(\mathrm{Ci}\left(\frac{kr^2}{2H}\right) - 2\,\mathrm{Ci}(kr) + \mathrm{Ci}(2kH)\right)\right] \approx C_1 + C_2\left(\frac{kr}{H}\right)^2 \tag{12}$$

where Si and Ci are sine and cosine integrals. Substituting (12) into (11) and integrating, we have

$$R_{\mathrm{self}} = \frac{1}{2\pi}\int_0^{2\pi} R(r)\,d\theta \cong R((2)^{1/2}ka). \tag{13}$$

Equation (13) states that for thin dipoles the self-resistance can be obtained by evaluating the mutual resistance between two infinity thin dipoles at a distance equal to $(2)^{1/2}a$ and is the same result as obtained in [6]. Next, we write the reactive part of the impedance as a function of r as follows:

$$X(r) = -30\left\{\sin 2(kH)\cdot\left[-\gamma + \ln\left(\frac{H}{kr^2}\right) + 2\,\mathrm{Ci}(2kH) - \mathrm{Ci}(4kH)\right] - \cos(2kH)\left[2\,\mathrm{Si}(2kH) - \mathrm{Si}(4kH)\right] - 2\,\mathrm{Si}(2kH)\right\} \tag{14}$$

where $\gamma = 0.5772\cdots$ is Euler's constant. Again, substituting (14) into (11) and integrating we have

$$X_{\mathrm{self}} = \frac{1}{2\pi}\int_0^{2\pi} X(kr)\,d\phi \cong X(ka). \tag{15}$$

Equation (15) states that for thin dipoles the self-reactance can be obtained by evaluating the mutual reactance between two infinitely thin dipoles at a distance equal to the radius a. This is different than the result for the real part and that given in [6] where $(2)^{1/2}\,a$ is used instead.

Continuing on for the case where the radius is still small compared to a wavelength ($a \gg 1$) but comparable to the height H, we can expand the reactance in a series of (r/H) as follows:

$$X(r) \simeq C_1' - C_2'\left[\ln\left(\frac{H}{r}\right)^2 + 3\frac{r}{H} + \cdots\right] \tag{16}$$

where C_1' and C_2' are constants that are not dependent on (r/H). Substituting (16) into (11) and integrating, there follows:

$$X_{\mathrm{self}} = C_1' - C_2'\left[\ln\left(\frac{H}{a}\right)^2 + \frac{12}{\pi}\frac{a}{H} + \cdots\right]. \tag{17}$$

Seeking an expansion of the equivalent radius (i.e., a value of $r = a_e$ which will make (16) and (17) equal) in terms of a/H we find

$$a_e = a\left(1 - 0.40976\frac{a}{H} + \cdots\right). \tag{18}$$

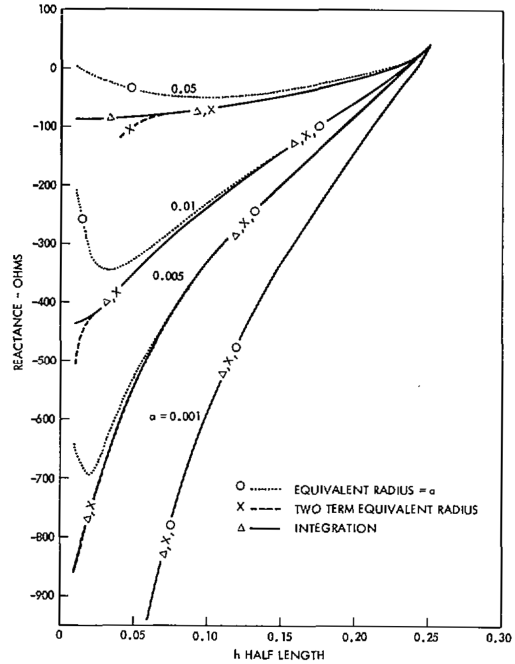

Fig. 4. Reactance of dipole versus half-length comparing results of integration and two term equivalent radius.

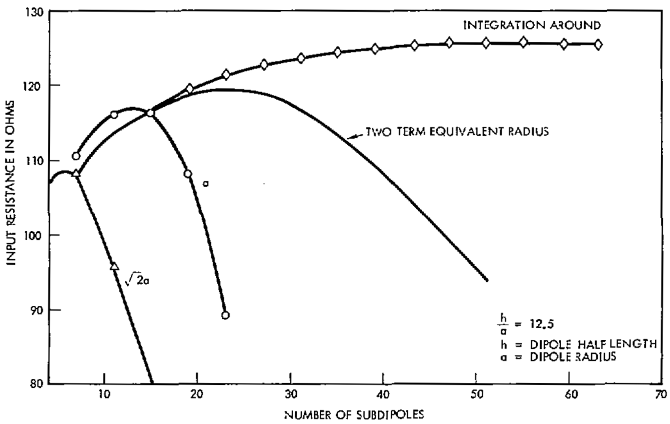

Fig. 5. Input resistance of half-wavelength dipole with $h/a = 12.5$.

Fig. 4 is a plot of the reactance of a dipole versus half-length comparing the results of using the first two terms of the equivalent radius series (18) to the result obtained when the actual integration (11) is performed. For comparison, the corresponding values using a constant equivalent radius a are also shown.

III. Examples

The importance of the utilization of the correct equivalent radius and its effect on the relative convergence of moments solutions is displayed in the following examples.

As a rather extreme case, let us choose a half-wavelength linear dipole with a half-length to radius $h/a = 12.5$. When the dipole is divided into 12 segments the H/a per segment is approximately equal to unity. The calculation of the input impedance for various equivalent radii and different numbers of subdipoles is shown in Fig. 5. Note that for this h/a it was necessary to use the complete integration for the self-term to get a solution that converged as the number of subdipoles increased. Using an equivalent radius of $(2)^{1/2}a, a$, and even the two term equivalent radius was not sufficient to yield a convergent solution.

The same type of plot (Fig. 6) is made for a $h/a = 50$. Here, the two term equivalent radius is sufficient whereas using a and

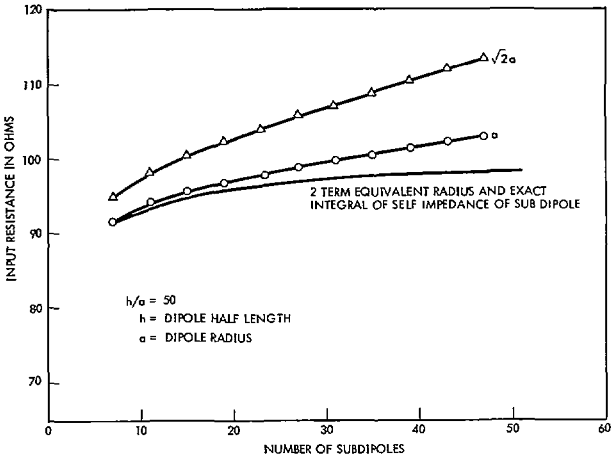

Fig. 6. Input resistance of half-wavelength dipole with $h/a = 50$.

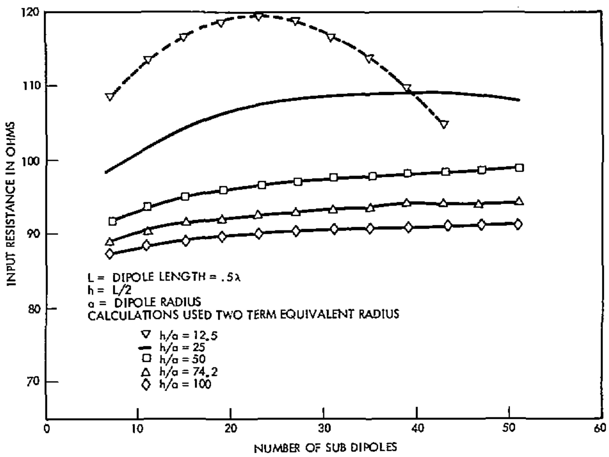

Fig. 7. Input resistance of half-wavelength dipoles versus number of subsections.

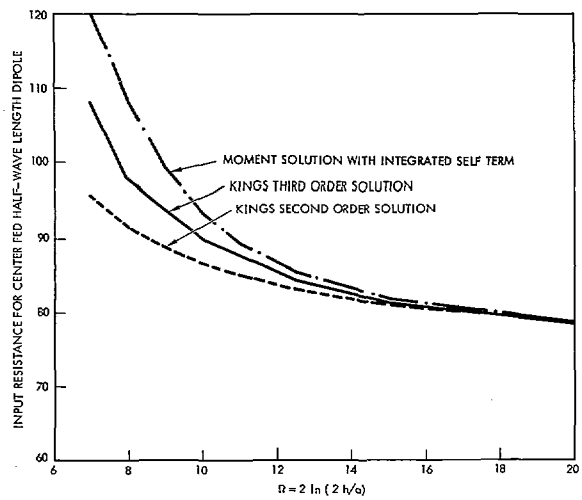

Fig. 8. Input resistance of half-wavelength dipoles for varying h/a.

$(2)^{1/2}a$ is not. Plots of the input impedance versus number of subdipoles for various h/a ratios are shown in Fig. 7 using the two term equivalent radius for the self-term. Fig. 8 compares the results of the converged solutions to results given by [9].

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Comparison Between the Peak Sidelobe of the Random Array and Algorithmically Designed Aperiodic Arrays

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***Abstract*—Thinned arrays (mean interelement spacing greater than one-half wavelength) are made aperiodic to suppress grating lobes. Many thinning algorithms were created in the 1960's and tested by computer simulation. Seventy such algorithmically designed aperiodic arrays are examined and the distribution of their peak sidelobes, relative to the expected values for random arrays having the same parameters, is obtained. The distribution is compared to that of a set of 170 random arrays. Both distributions are found to be nearly log normal with the same average and median values. They differ markedly in their standard deviations, however, the standard deviation of the random array distribution (1.1 dB) is approximately half that of the algorithmic group. The compactness of the random distribution almost guarantees against selection of a random array with catastrophically large peak sidelobes. Among the several algorithms examined, the method of dynamic programming produced the lowest peak sidelobe on the average.**

Introduction

A first-order property of an antenna array is the width of its main lobe, which is approximately the reciprocal of the number of wavelengths across the array. Thinning the array, i.e., mean spacing between elements is larger than one-half wavelength, can materially reduce the number of elements and, therefore, the cost with little effect upon the beamwidth. Thinning introduces grating lobes into the radiation pattern unless the periodicity in the locations of the antenna elements is destroyed [1]. Many aperiodic designs have been created for this purpose (e.g., [2]–[13]). Randomizing the element locations also eliminates periodicities [14]. In this paper the peak sidelobe resulting from random design is compared with the results of many algorithmic procedures developed during the last decade.

In a random array, the location of the nth element is a random variable drawn from a population described by a probability density

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