Mutual Impedance of Unequal Length Antennas in Echelon*

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Summary—**The expression is developed for the mutual impedance between two staggered parallel center-fed, infinitely thin antennas of unequal lengths. Heretofore unpublished curves are presented here which display the mutual impedance characteristics for a variety of unequal antenna lengths in echelon.**

Introduction

The antenna engineer is often confronted with the problem of predetermining the input impedance of each radiator of a directional array. In addition he must frequently determine the interference pattern due to parasitic wires or antennas adjacent to the fed antenna. To find the input impedance and/or to find the induced currents on other wires, it is necessary to determine the mutual impedance between elements. Carter[1] has presented the mutual impedance between two wires in echelon where each antenna is an odd number of $\frac{1}{2}$ wavelengths long. The mutual impedance expression between two identical nonstaggered parallel center-fed antennas of Brown[2] is now a classic. The work of Brown was carried further by Cox[3] who determined the mutual impedance between parallel antennas of unequal lengths.

Thus, sufficient data has been presented in the past to solve the conventional problems. However, the antenna engineer may often require a knowledge of the mutual impedance between staggered parallel antennas of arbitrary length. The purpose of this paper is to present an expression which combines the work of Carter and Cox from which may be calculated the mutual impedance between any two infinitely thin center-fed parallel antennas of any length whether they be nonstaggered, staggered, or collinear.

As with all the formulas cited in the references, the mutual impedance equations apply accurately to infinitely thin antennas only, but nonetheless serve as a practical approximation for real antennas of finite thickness.

Determination of Mutual Impedance

The problem on hand is illustrated in Fig. 1. Two center-fed antennas of half-lengths l_1 and l_2 are shown separated by a distance d, and staggered by the height, h. The mutual impedance between the two antennas of Fig. 1 is defined by

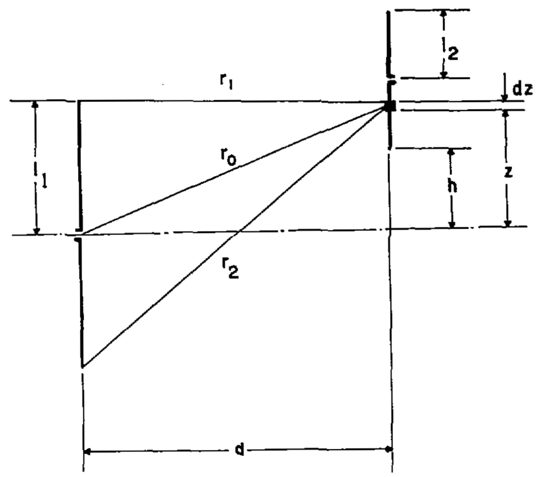

Fig. 1—Two parallel antennas of arbitrary length in echelon.

$$Z_{21} = -\frac{V_{21}}{I_{1b}} \tag{1}$$

where V_{21} is the open-circuit voltage at the terminals of antenna 2, due to a base current I_{1b} in antenna 1. The open-circuit voltage at the terminals of antenna 2 which results from the voltages induced in all the elemental lengths of the antenna may be found, by application of the reciprocity[4] theorem, to be

$$V_{21} = \frac{1}{I_{2b}}\left(\int_h^{l_2+h} E_{z1}I_2(z)dz + \int_{l_2+h}^{2l_2+h} E_{z1}I_2(z)dz\right), \tag{2}$$

where I_{2b} is the base (*i.e.*, feed point) current of antenna 2, and E_{z1} is the component of electric intensity parallel to the axis of the antenna at a point z along antenna 2, due to a current in antenna 1. The antenna current distribution $I_2(z)$ is assumed to be sinusoidal and given by

$$\begin{aligned} I_2(z) &= I_{2m}\sin\beta(z-h) \qquad & h < z < l_2 + h \\ I_2(z) &= I_{2m}\sin\beta(2l_2+h-z) \qquad & h + l_2 < z < 2l_2 + h \end{aligned} \tag{3}$$

where I_{2m} is the value of current at the current loop or current maximum. The expression[2] for parallel component of electric field, is given as

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[1] P. S. Carter, "Circuit relations in radiating systems and application to antenna problems," Proc. IRE, vol. 20, pp. 1004–1041; June, 1932.
[2] G. H. Brown, "Directional antennas," Proc. IRE, vol. 25, pp. 81–145; January, 1937.
[3] C. R. Cox, "Mutual impedance between vertical antennas of unequal heights," Proc. IRE, vol. 35, pp. 1367–1370; November, 1947.
[4] See, for example, E. C. Jordan, "Electromagnetic Waves and Radiating Systems," Prentice-Hall, Inc., New York, N. Y., p. 349; 1950.

$$E_{z1} = 30 I_{1m}\left[\frac{-je^{-j\beta r_1}}{r_1} + \frac{-je^{-j\beta r_2}}{r_2} + \frac{2j\cos\beta l_1 e^{-j\beta r_0}}{r_0}\right]. \quad (4)$$

Eq. (1) gives the mutual impedance referred to the base or feed point. The mutual impedance related to the loop currents will be given by

$$Z_{12\,\mathrm{loop}} = \frac{I_{1b}I_{2b}}{I_{1m}I_{2m}} Z_{12\,\mathrm{base}}. \quad (5)$$

Inserting (2)–(4) into (1), and in view of (5), the expression for the mutual impedance referred to the loop currents becomes

$$Z_{12} = -30\Bigg[\Bigg\{\int_h^{l_2+h} \sin\beta(z-h) + \int_{l_2+h}^{2l_2+h} \sin\beta(2l_2+h-z)\Bigg\} \left(\frac{-je^{-j\beta r_1}}{r_1} + \frac{-je^{-j\beta r_2}}{r_2} + \frac{2j\cos\beta l_1 e^{-j\beta r}}{r_0}\right)dz\Bigg]. \quad (6)$$

The geometry of Fig. 1 reveals that

$$r_0 = \sqrt{d^2 + z^2} \quad (7)$$

$$r_1 = \sqrt{d^2 + (l_1 - z)^2} \quad (8)$$

$$r_2 = \sqrt{d^2 + (l_1 + z)^2}. \quad (9)$$

The real part of the complex expression (6) gives the mutual resistance and the imaginary part gives the mutual reactance. Eq. (6) can be evaluated by mechanical integration; however, to make possible arithmetical computations the integration is done mathematically so that a convenient form is available for calculations.

Substituting the relations (7)–(9) into (6), and following the integrations in a manner similar to that outlined in the Appendix, the most general expressions for the mutual impedance between two thin parallel center-fed antennas of any arbitrary half-length l_1 and l_2, spaced a distance d apart, and staggered by the value h, are:

$$\begin{aligned}
R_{12} = 15\{&\cos\beta(l_1-h)(\mathrm{Ci}\,(u_0) + \mathrm{Ci}\,(v_0) - \mathrm{Ci}\,(u_1) - \mathrm{Ci}\,(v_1)) + \sin\beta(l_1-h)(-\mathrm{Si}\,(u_0) + \mathrm{Si}\,(v_0) + \mathrm{Si}\,(u_1) - \mathrm{Si}\,(v_1))\\
&+ \cos\beta(l_1+h)(\mathrm{Ci}\,(u_0') + \mathrm{Ci}\,(v_0') - \mathrm{Ci}\,(u_2) - \mathrm{Ci}\,(v_2)) + \sin\beta(l_1+h)(-\mathrm{Si}\,(u_0') + \mathrm{Si}\,(v_0') + \mathrm{Si}\,(u_2) - \mathrm{Si}\,(v_2))\\
&+ \cos\beta(l_1-2l_2-h)(-\mathrm{Ci}\,(u_1) - \mathrm{Ci}\,(v_1) + \mathrm{Ci}\,(u_3) + \mathrm{Ci}\,(v_3)) + \sin\beta(l_1-2l_2-h)(\mathrm{Si}\,(u_1) - \mathrm{Si}\,(v_1) - \mathrm{Si}\,(u_3) + \mathrm{Si}\,(v_3))\\
&+ \cos\beta(l_1+2l_2+h)(-\mathrm{Ci}\,(u_2) - \mathrm{Ci}\,(v_2) + \mathrm{Ci}\,(u_4) + \mathrm{Ci}\,(v_4)) + \sin\beta(l_1+2l_2+h)(\mathrm{Si}\,(u_2) - \mathrm{Si}\,(v_2) - \mathrm{Si}\,(u_4) + \mathrm{Si}\,(v_4))\\
&+ 2\cos\beta l_1\cos\beta h(-\mathrm{Ci}\,(w_1) - \mathrm{Ci}\,(y_1) + \mathrm{Ci}\,(w_2) + \mathrm{Ci}\,(y_2)) + 2\cos\beta l_1\sin\beta h\,(\mathrm{Si}\,(w_1) - \mathrm{Si}\,(y_1) - \mathrm{Si}\,(w_2) + \mathrm{Si}\,(y_2))\\
&+ 2\cos\beta l_1\cos\beta(2l_2+h)(\mathrm{Ci}\,(w_2) + \mathrm{Ci}\,(y_2) - \mathrm{Ci}\,(w_3) - \mathrm{Ci}\,(y_3)) + 2\cos\beta l_1\sin\beta h\,(2l_2+h)(-\mathrm{Si}\,(w_2) + \mathrm{Si}\,(y_2) + \mathrm{Si}\,(w_3) - \mathrm{Si}\,(y_3))\} \quad (10)
\end{aligned}$$

$$\begin{aligned}
X_{12} = 15\{&\cos\beta(l_1-h)(-\mathrm{Si}\,(u_0) - \mathrm{Si}\,(v_0) + \mathrm{Si}\,(u_1) + \mathrm{Si}\,(v_1)) + \sin\beta(l_1-h)(-\mathrm{Ci}\,(u_0) + \mathrm{Ci}\,(v_0) + \mathrm{Ci}\,(u_1) - \mathrm{Ci}\,(v_1))\\
&+ \cos\beta(l_1+h)(-\mathrm{Si}\,(u_0') - \mathrm{Si}\,(v_0') + \mathrm{Si}\,(u_2) + \mathrm{Si}\,(v_2)) + \sin\beta(l_1+h)(-\mathrm{Ci}\,(u_0') + \mathrm{Ci}\,(v_0') + \mathrm{Ci}\,(u_2) - \mathrm{Ci}\,(v_2))\\
&+ \cos\beta(l_1-2l_2-h)(\mathrm{Si}\,(u_1) + \mathrm{Si}\,(v_1) - \mathrm{Si}\,(u_3) - \mathrm{Si}\,(v_3)) + \sin\beta(l_1-2l_2-h)(\mathrm{Ci}\,(u_1) - \mathrm{Ci}\,(v_1) - \mathrm{Ci}\,(u_3) + \mathrm{Ci}\,(v_3))\\
&+ \cos\beta(l_1+2l_2+h)(\mathrm{Si}\,(u_2) + \mathrm{Si}\,(v_2) - \mathrm{Si}\,(u_4) - \mathrm{Si}\,(v_4)) + \sin\beta(l_1+2l_2+h)(\mathrm{Ci}\,(u_2) - \mathrm{Ci}\,(v_2) - \mathrm{Ci}\,(u_4) + \mathrm{Ci}\,(v_4))\\
&+ 2\cos\beta l_1\cos\beta h(\mathrm{Si}\,(w_1) + \mathrm{Si}\,(y_1) - \mathrm{Si}\,(w_2) - \mathrm{Si}\,(y_2)) + 2\cos\beta l_1\sin\beta h(\mathrm{Ci}\,(w_1) - \mathrm{Ci}\,(y_1) - \mathrm{Ci}\,(w_2) + \mathrm{Ci}\,(y_2))\\
&+ 2\cos\beta l_1\cos\beta(2l_2+h)\,(-\mathrm{Si}\,(w_2) - \mathrm{Si}\,(y_2) + \mathrm{Si}\,(w_3) + \mathrm{Si}\,(y_3)) + 2\cos\beta l_1\sin\beta(2l_2+h)(-\mathrm{Ci}\,(w_2) + \mathrm{Ci}\,(y_2) + \mathrm{Ci}\,(w_3) - \mathrm{Ci}\,(y_3))\} \quad (11)
\end{aligned}$$

where

$$\begin{aligned}
u_0 &= \beta(\sqrt{d^2 + (h-l_1)^2} + (h-l_1))\\
v_0 &= \beta(\sqrt{d^2 + (h-l_1)^2} - (h-l_1))\\
\mu_0' &= \beta(\sqrt{d^2 + (h+l_1)^2} - (h+l_1))\\
v_0' &= \beta(\sqrt{d^2 + (h+l_1)^2} + (h+l_1))\\
\mu_1 &= \beta(\sqrt{d^2 + (h-l_1+l_2)^2} + (h-l_1+l_2))\\
v_1 &= \beta(\sqrt{d^2 + (h-l_1+l_2)^2} - (h-l_1+l_2))\\
\mu_2 &= \beta(\sqrt{d^2 + (h+l_1+l_2)^2} - (h+l_1+l_2))\\
v_2 &= \beta(\sqrt{d^2 + (h+l_1+l_2)^2} + (h+l_1+l_2))\\
u_3 &= \beta(\sqrt{d^2 + (h-l_1+2l_2)^2} + (h-l_1+2l_2))\\
v_3 &= \beta(\sqrt{d^2 + (h-l_1+2l_2)^2} - (h-l_1+2l_2))\\
u_4 &= \beta(\sqrt{d^2 + (h+l_1+2l_2)^2} - (h+l_1+2l_2))\\
v_4 &= \beta(\sqrt{d^2 + (h+l_1+2l_2)^2} + (h+l_1+2l_2))\\
w_1 &= \beta(\sqrt{d^2 + h^2} - h)\\
y_1 &= \beta(\sqrt{d^2 + h^2} + h)\\
w_2 &= \beta(\sqrt{d^2 + (h+l_2)^2} - (h+l_2))\\
y_2 &= \beta(\sqrt{d^2 + (h+l_2)^2} + (h+l_2))\\
w_3 &= \beta(\sqrt{d^2 + (h+2l_2)^2} - (h+2l_2))\\
y_3 &= \beta(\sqrt{d^2 + (h+2l_2)^2} + (h+2l_2)).
\end{aligned}$$

Considerable reduction in the length of (10) and (11) results if the length $l_1 = \lambda/4$. Furthermore, note that once R_{12} is obtained one can find X_{12} by the following relations. Let

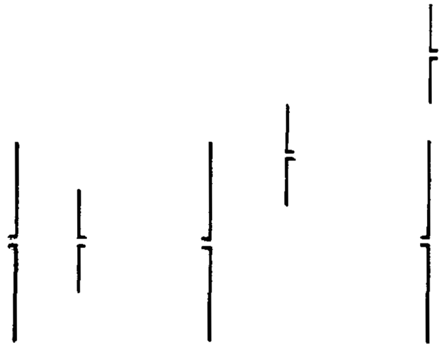
$$R_{12} = X_{12}$$

$$\mathrm{Ci}\,(x) = -\mathrm{Si}\,(x)$$

$$\mathrm{Si}\,(y) = \mathrm{Ci}\,(y).$$

Mutual Impedance Curves

The antenna configurations used in this presentation are illustrated in Fig. 2. In all the curves the mutual



(a) (b) (c)

Fig. 2—Two parallel antennas in (a) nonstaggered, (b) staggered, and (c) collinear arrangement.

impedance is referred to the loop currents. Fig. 3 (opposite) shows the mutual impedance between two $\lambda/2$ dipoles (half-length $l_1=l_2=\lambda/4$). Note that Fig. 3(a) for the nonstaggered case (or $h=-\lambda/4$) is the plot normally seen in all standard references for two parallel antennas. Figs. 3(b) and 3(c) show the deviation from Fig. 3(a) when the antennas are in echelon, *i.e.*, staggered by $h=0$ and $h=\lambda/4$, respectively. The shape is generally the same with a decrease in magnitude.

Fig. 4 (pp. 309–310) illustrates the mutual impedance between two antennas of half-lengths $l_1=\lambda/4$ and $l_2=\lambda/6$ for the nonstaggered case ($h=-l_1=-\lambda/4$), and for two staggered cases ($h=0$, and $h=l_1=\lambda/4$). Fig. 5 (pp. 310–311) depicts corresponding results for the case where the half-lengths are $l_1=\lambda/4$ and $l_2=\lambda/8$.

Special Cases

Collinear Array

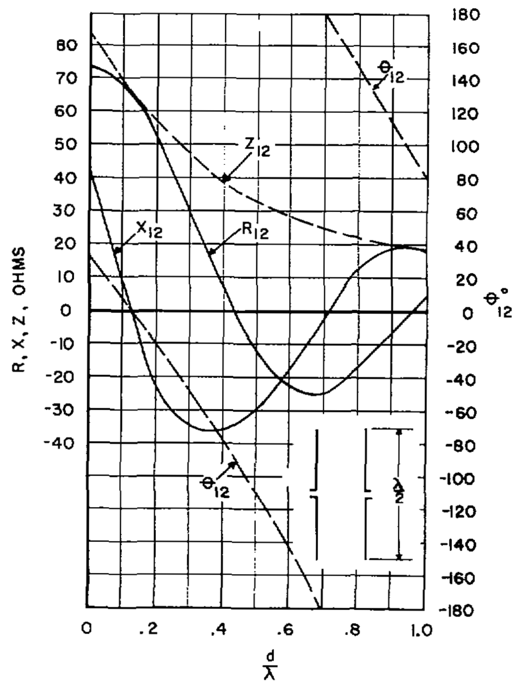
A special case of echelon antennas is the arrangement of Fig. 2(c) where $d=0$ in (10) and (11). This gives rise to an indeterminate form of $\infty-\infty$. By taking the limit of the expression as d approaches zero, the mutual resistance and reactance are obtained as

$$\begin{aligned}
R_{12} = 15\Bigg\{&\cos\beta(h-l_1)\left[\mathrm{Ci}\,(u_0)-\mathrm{Ci}\,(u_1)+\ln\frac{h-l_1+l_2}{h-l_1}\right]+\sin\beta(h-l_1)\left[\mathrm{Si}\,(u_0)-\mathrm{Si}(u_1)\right]\\
&+\cos\beta(h+l_1)\left[\mathrm{Ci}\,(v_0')-\mathrm{Ci}\,(v_2)+\ln\frac{h+l_1+l_2}{h+l_1}\right]+\sin\beta(h+l_1)\left[\mathrm{Si}\,(v_0')-\mathrm{Si}\,(v_2)\right]\\
&+\cos\beta(h-l_1+2l_2)\left[-\mathrm{Ci}\,(u_1)+\mathrm{Ci}\,(u_3)+\ln\frac{h-l_1+l_2}{h-l_1+2l_2}\right]+\sin\beta(h-l_1+2l_2)\left[-\mathrm{Si}\,(u_1)+\mathrm{Si}\,(u_3)\right]\\
&+\cos\beta(h+l_1+2l_2)\left[-\mathrm{Ci}\,(v_2)+\mathrm{Ci}(v_4)+\ln\frac{h+l_1+l_2}{h+l_1+2l_2}\right]+\sin\beta(h+l_1+2l_2)\left[-\mathrm{Si}\,(v_2)+\mathrm{Si}\,(v_4)\right]\\
&+2\cos\beta l_1\cos\beta h\left[-\mathrm{Ci}\,(y_1)+\mathrm{Ci}\,(y_2)+\ln\frac{h}{h+l_2}\right]+2\cos\beta l_1\sin\beta h\left[-\mathrm{Si}\,(y_1)+\mathrm{Si}\,(y_2)\right]\\
&+2\cos\beta l_1\cos\beta(h+2l_2)\left[\mathrm{Ci}\,(y_2)-\mathrm{Ci}(y_3)+\ln\frac{h+2l_2}{h+l_2}\right]+2\cos\beta l_1\sin\beta(h+2l_2)\left[\mathrm{Si}\,(y_2)-\mathrm{Si}(y_3)\right]\Bigg\}
\end{aligned}\tag{12}$$

and

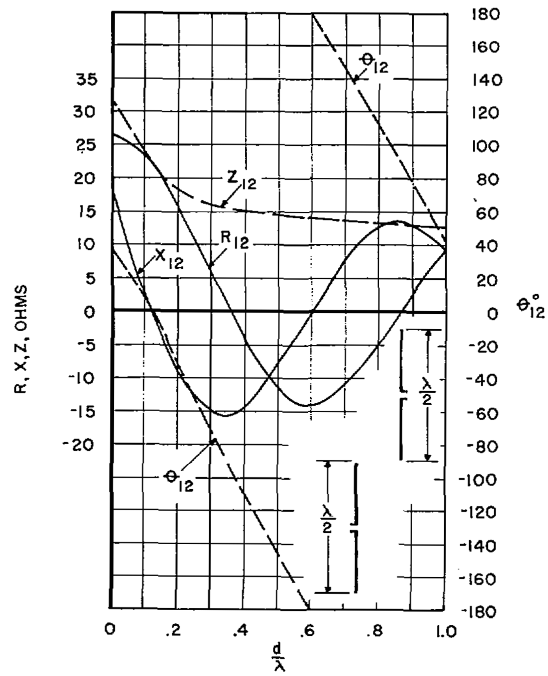
$$\begin{aligned}
X_{12} = 15\Bigg\{&\cos\beta(h-l_1)\left[-\mathrm{Si}\,(u_0)+\mathrm{Si}\,(u_1)\right]+\sin\beta(h-l_1)\left[\mathrm{Ci}\,(u_0)-\mathrm{Ci}\,(u_1)+\ln\frac{h-l_1}{h-l_1+l_2}\right]\\
&+\cos\beta(h+l_1)\left[-\mathrm{Si}\,(v_0')+\mathrm{Si}\,(v_2)\right]+\sin\beta(h+l_1)\left[\mathrm{Ci}\,(v_0')-\mathrm{Ci}\,(v_2)+\ln\frac{h+l_1}{h+l_1+l_2}\right]\\
&+\cos\beta(h-l_1+2l_2)\left[\mathrm{Si}(u_1)-\mathrm{Si}\,(u_3)\right]+\sin\beta(h-l_1+2l_2)\left[-\mathrm{Ci}\,(u_1)+\mathrm{Ci}\,(u_3)+\ln\frac{h-l_1+2l_2}{h-l_1+l_2}\right]\\
&+\cos\beta(h+l_1+2l_2)\left[\mathrm{Si}\,(v_2)-\mathrm{Si}\,(v_4)\right]+\sin\beta(h+l_1+2l_2)\left[-\mathrm{Ci}\,(v_2)+\mathrm{Ci}\,(v_4)+\ln\frac{h+l_1+2l_2}{h+l_1+l_2}\right]\\
&+2\cos\beta l_1\cos\beta h\left[\mathrm{Si}\,(y_1)-\mathrm{Si}\,(y_2)\right]+2\cos\beta l_1\sin\beta h\left[-\mathrm{Ci}\,(y_1)+\mathrm{Ci}\,(y_2)+\ln\frac{h+l_2}{h}\right]\\
&+2\cos\beta l_1\cos\beta(h+2l_2)\left[-\mathrm{Si}\,(y_2)+\mathrm{Si}\,(y_3)\right]+2\cos\beta l_1\sin\beta(h+2l_2)\left[\mathrm{Ci}(y_2)-\mathrm{Ci}\,(y_3)+\ln\frac{h+l_2}{h+2l_2}\right]\Bigg\}.
\end{aligned}\tag{13}$$

Eqs. (12) and (13) represent the mutual impedance for two collinear antennas of any arbitrary lengths l_1 and l_2, providing $h>l_1$. Figs. 6–8 (p. 311) represent the mu-



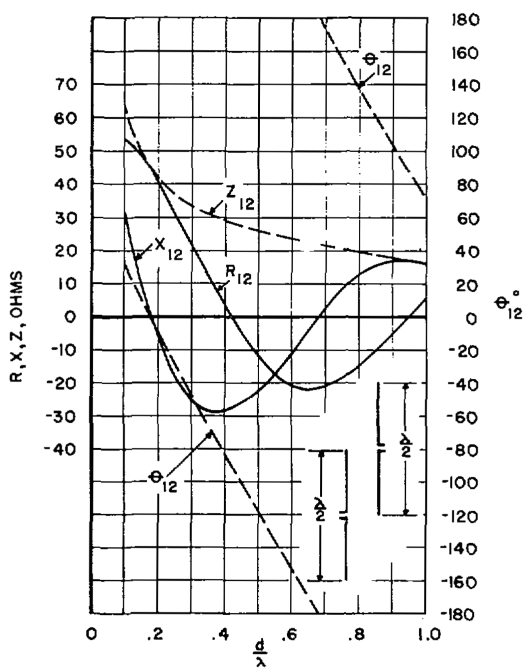
(a)

Fig. 3(a)—Mutual impedance curves for two parallel half-wavelength antennas, nonstaggered.



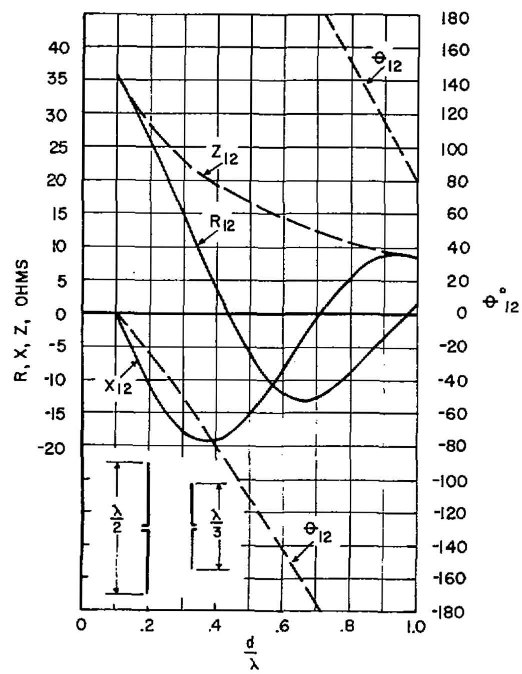
(c)

Fig. 3(c)—Mutual impedance curves for two parallel half-wavelength antennas staggered by $h=\lambda/4$.



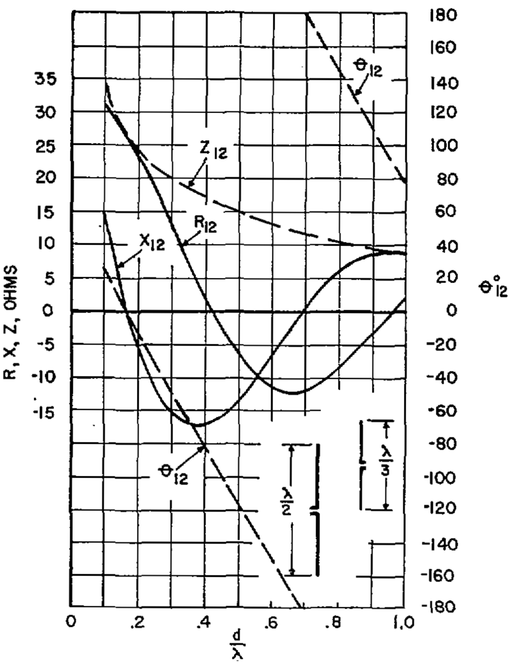
(b)

Fig. 3(b)—Mutual impedance curves for two parallel half-wavelength antennas staggered by $h=0$.



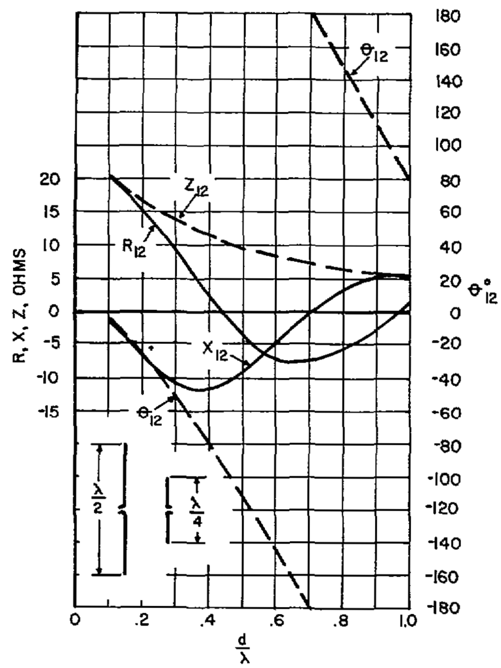
(a)

Fig. 4(a)—Mutual impedance curves between two parallel antennas of lengths $\lambda/2$ and $\lambda/3$, nonstaggered (referred to the loop currents).



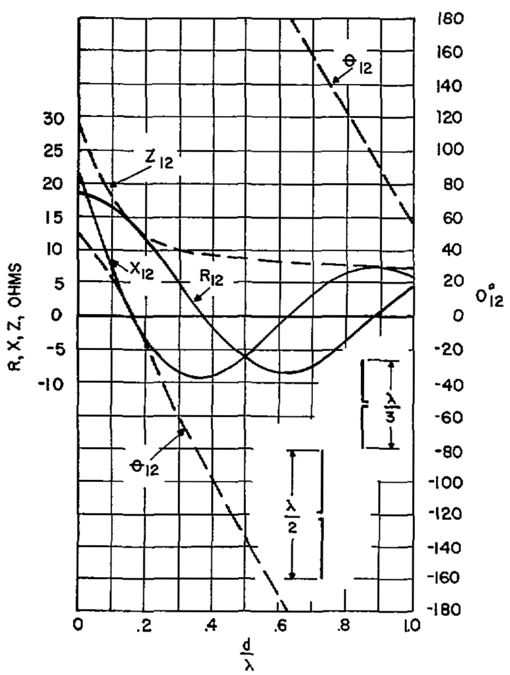
(b)

Fig. 4(b)—Mutual impedance curves between two parallel antennas of lengths $\lambda/2$ and $\lambda/3$, staggered by $h=0$ (referred to the loop currents).



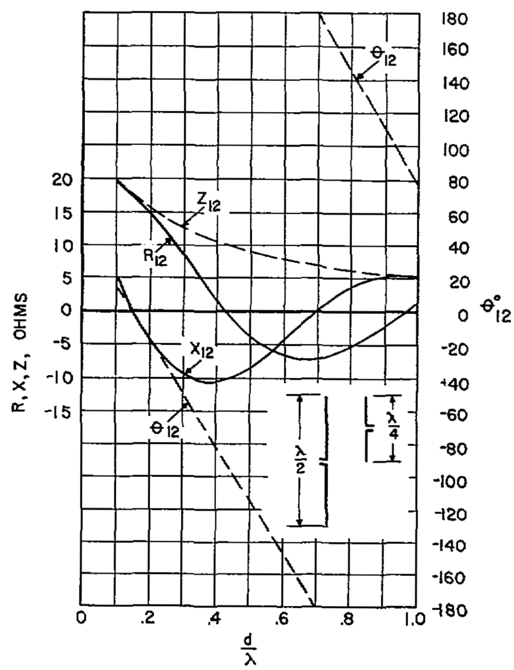
(a)

Fig. 5(a)—Mutual impedance curves between two parallel antennas of lengths $\lambda/2$ and $\lambda/4$, nonstaggered (referred to the loop currents).



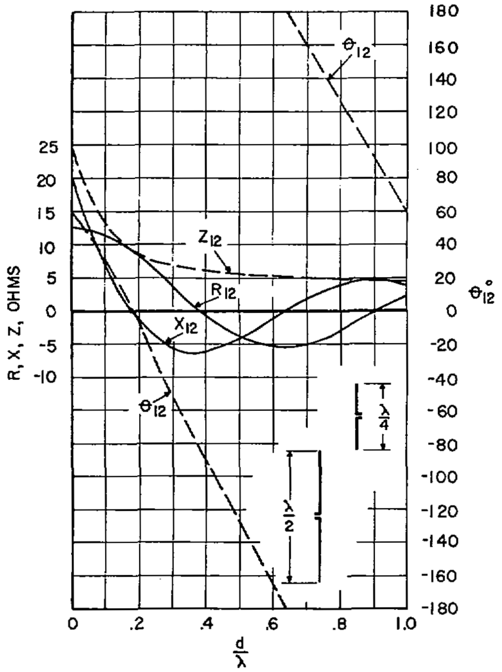
(c)

Fig. 4(c)—Mutual impedance curves between two parallel antennas of lengths $\lambda/2$ and $\lambda/3$, staggered by $h=\lambda/4$ (referred to the loop currents).



(b)

Fig. 5(b)—Mutual impedance curves between two parallel antennas of lengths $\lambda/2$ and $\lambda/4$, staggered by $h=0$ (referred to the loop currents).



(c)

Fig. 5(c)—Mutual impedance curves between two parallel antennas of lengths $\lambda/2$ and $\lambda/4$, staggered by $h=\lambda/4$ (referred to the loop currents).

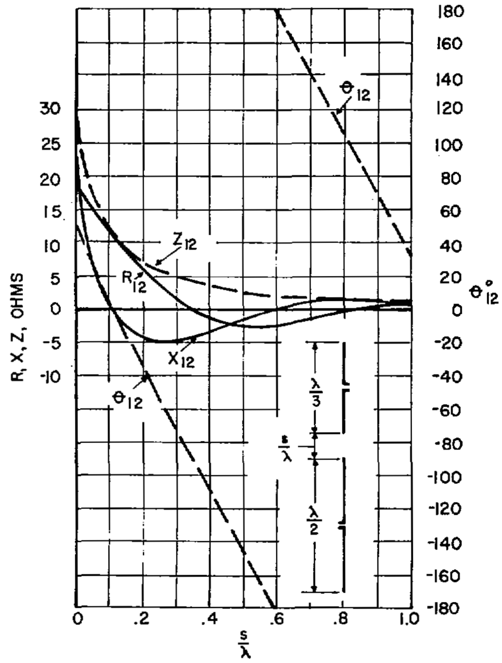


Fig. 7—Mutual impedance curves between two antennas of lengths $\lambda/2$ and $\lambda/3$ in a collinear arrangement.

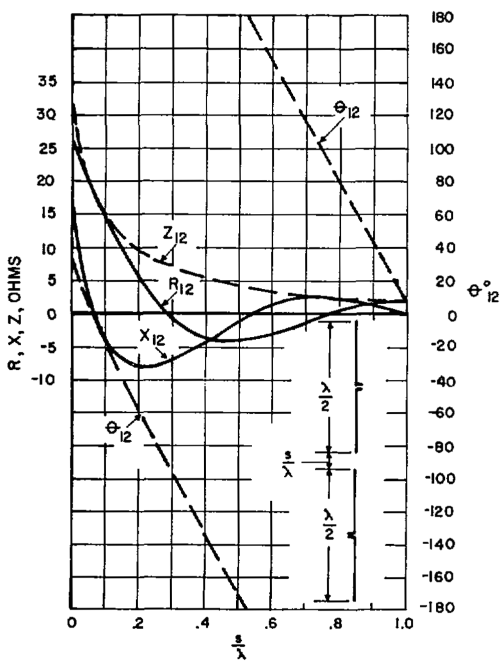


Fig. 6—Mutual impedance curves between two half-wavelength antennas in a collinear arrangement.

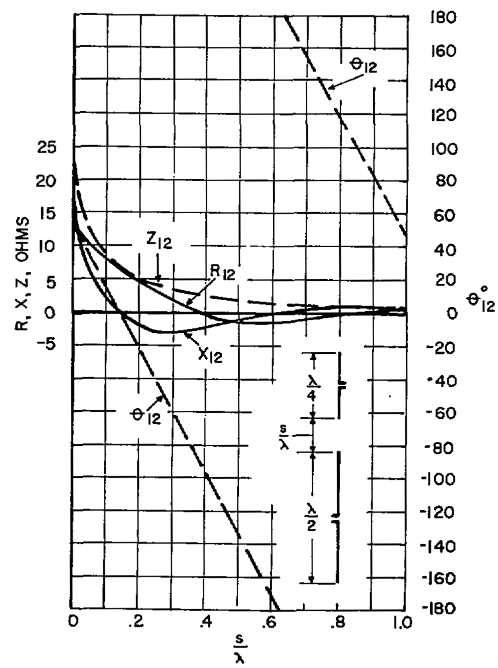


Fig. 8—Mutual impedance curves between two antennas of lengths $\lambda/2$ and $\lambda/4$ in a collinear arrangement.

tual impedances between an antenna with its half-length $l_1=\lambda/4$, and antennas of half-lengths corresponding to $l_2=\lambda/4$, $\lambda/6$, and $\lambda/8$.

The mutual resistance and reactance of two collinear $\lambda/2$ dipoles ($l_1=l_2=\lambda/4$) can be written from (12) and (13), respectively, as

$$R_{12} = 15\left\{\sin\beta h\left[\mathrm{Ci}\,2\beta(h-l_1) - 2\,\mathrm{Ci}\,2\beta(h+l_1) + \mathrm{Ci}\,2\beta(h+3l_1) + \ln\frac{(h+l_1)^2}{h^2+2hl_1-3l_1^2}\right]\right.$$
$$\left. + \cos\beta h[-\mathrm{Si}\,2\beta(h-l_1) + 2\,\mathrm{Si}\,2\beta(h+l_1) - \mathrm{Si}\,2\beta(h+3l_1)]\right\} \quad (14)$$

and

$$X_{12} = 15\left\{\sin\beta h[-\mathrm{Si}\,2\beta(h-l_1) + 2\,\mathrm{Si}\,2\beta(h+l_1) - \mathrm{Si}\,2\beta(h+3l_1)]\right.$$
$$\left. + \cos\beta h\left[-\mathrm{Ci}\,2\beta(h-l_1) + 2\,\mathrm{Ci}\,2\beta(h+l_1) - \mathrm{Ci}\,2\beta(h+3l_1) + \ln\frac{(h+l_1)^2}{h^2+2hl_1-3l_1^2}\right]\right\}. \quad (15)$$

Non-Staggered Arrays

[See Fig. 2(a).] Referring to (10) and (11), if $h=-l_2$, then R_{12} and X_{12} will reduce to the equations given by Cox for parallel antennas of unequal height which are

$$\begin{aligned} R_{12} = 30\{&\cos\beta(l_1+l_2)[\mathrm{Ci}\,(u_0) + \mathrm{Ci}\,(v_0) - \mathrm{Ci}\,(u_1) \\ &- \mathrm{Ci}\,(v_1) - \mathrm{Ci}\,(w_1) - \mathrm{Ci}\,(y_1) + 2\,\mathrm{Ci}\,(\beta d)] \\ &+ \cos\beta(l_1-l_2)[\mathrm{Ci}\,(u_0') + \mathrm{Ci}\,(v_0') - \mathrm{Ci}\,(\mu_1) \\ &- \mathrm{Ci}\,(v_1) - \mathrm{Ci}\,(w_1) - \mathrm{Ci}\,(y_1) + 2\,\mathrm{Ci}\,(\beta d)] \\ &+ \sin\beta(l_1+l_2)[-\mathrm{Si}\,(u_0) + \mathrm{Si}\,(v_0) + \mathrm{Si}\,(u_1) \\ &- \mathrm{Si}\,(v_1) - \mathrm{Si}\,(w_1) + \mathrm{Si}\,(y_1)] \\ &+ \sin\beta(l_1-l_2)[-\mathrm{Si}\,(u_0') + \mathrm{Si}\,(v_0') + \mathrm{Si}\,(u_1) \\ &- \mathrm{Si}(v_1) + \mathrm{Si}(w_1) - \mathrm{Si}\,(y_1)]\} \end{aligned} \quad (16)$$

and

$$\begin{aligned} X_{12} = 30\{&\cos\beta(l_1+l_2)[-\mathrm{Si}\,(u_0) - \mathrm{Si}\,(v_0) + \mathrm{Si}\,(u_1) \\ &+ \mathrm{Si}\,(v_1) + \mathrm{Si}\,(w_1) + \mathrm{Si}\,(y_1) - 2\,\mathrm{Si}\,(\beta d)] \\ &+ \cos\beta(l_1-l_2)[-\mathrm{Si}\,(u_0') - \mathrm{Si}\,(v_0') + \mathrm{Si}\,(u_1) \\ &+ \mathrm{Si}\,(v_1) + \mathrm{Si}\,(w_1) + \mathrm{Si}\,(y_1) - 2\mathrm{Si}\,(\beta d)] \\ &+ \sin\beta(l_1+l_2)[-\mathrm{Ci}\,(u_0) + \mathrm{Ci}\,(v_0) + \mathrm{Ci}(u_1) \\ &- \mathrm{Ci}\,(v_1) - \mathrm{Ci}\,(w_1) + \mathrm{Ci}(y_1)] \\ &+ \sin\beta(l_1-l_2)[-\mathrm{Ci}\,(u_0') + \mathrm{Ci}\,(v_0') + \mathrm{Ci}\,(u_1) \\ &- \mathrm{Ci}\,(v_1) + \mathrm{Ci}\,(w_1) - \mathrm{Ci}\,(y_1)]\}. \end{aligned} \quad (17)$$

If the radiators are equal in length ($l_1=l_2$) and with $h=-l_2$, then the mutual resistance and reactance for two parallel center-fed thin antennas simplify to

$$\begin{aligned} R_{12}=30\{&\cos 2\beta l_1[\mathrm{Ci}\,(u_0)+\mathrm{Ci}\,(v_0)-2\,\mathrm{Ci}\,(u_1) \\ &-2\,\mathrm{Ci}\,(v_1)+2\,\mathrm{Ci}\,(\beta d)] \\ &+\sin 2\beta l_1[-\mathrm{Si}\,(u_0)+\mathrm{Si}\,(v_0)+2\,\mathrm{Si}\,(u_1)-2\,\mathrm{Si}\,(v_1)] \\ &+[-2\,\mathrm{Ci}\,(u_1)-2\,\mathrm{Ci}\,(v_1)+4\,\mathrm{Ci}\,(\beta d)]\} \end{aligned} \quad (18)$$

and

$$\begin{aligned} X_{12}=30\{&\cos 2\beta l_1[-\mathrm{Si}\,(u_0)-\mathrm{Si}\,(v_0)+2\,\mathrm{Si}\,(u_1) \\ &+2\,\mathrm{Si}\,(v_1)-2\,\mathrm{Si}\,(\beta d)] \\ &+\sin 2\beta l_1[-\mathrm{Ci}\,(u_0)+\mathrm{Ci}\,(v_0)+2\,\mathrm{Ci}\,(u_1)-2\,\mathrm{Ci}\,(v_1)] \\ &+[2\,\mathrm{Si}\,(u_1)+2\,\mathrm{Si}\,(v_1)-4\,\mathrm{Si}\,(\beta d)]\}, \end{aligned} \quad (19)$$

which are the formulas given by Brown.

Conclusion

The mutual impedance relations for collinear or echelon arrays with elements of arbitrary lengths were presented. The expressions are only approximate because sinusoidal current distributions on both elements were assumed. However, the information in this paper should be useful for practical antenna array design.

Appendix

A typical term of (6) reduces to

$$\begin{aligned} A &= 30\int_h^{l_2+h}\frac{\sin\beta r_1}{r_1}\sin\beta(z-h)dz \\ &= 30\int_h^{l_2+h}\frac{\sin\beta\sqrt{d^2+(l_1-z)^2}}{\sqrt{d^2+(l_1-z)^2}}\sin\beta(z-h)dz. \end{aligned} \quad (20)$$

By change in the variable of $x=l_1-z$, and use of trigonometric identity for the product of angles one obtains

$$\begin{aligned} A = 15\int_{l_1-h}^{l_1-l_2-h}&[\cos\beta(l_1-h)\cos\beta(\sqrt{d^2+x^2}-x) \\ &-\sin\beta(l_1-h)\sin\beta(\sqrt{d^2+x^2}-x) \\ &-\cos\beta(l_1-h)\cos\beta(\sqrt{d^2+x^2}+x) \\ &-\sin\beta(l_1-h)\sin\beta(\sqrt{d^2+x^2}+x) \\ &\cdot\frac{dx}{\sqrt{d^2+x^2}}. \end{aligned} \quad (21)$$

A second change in the variables of

$$u = \beta(\sqrt{d^2 + x^2} - x)$$
$$v = \beta(\sqrt{d^2 + x^2} + x) \tag{22}$$

gives

$$A = 15 \int_{u_0}^{u_1} \left[-\cos\beta(l_1 - h) \frac{\cos u}{u} du + \sin\beta(l_1 - h) \frac{\sin u}{u} du \right]$$
$$+ 15 \int_{v_0}^{v_1} \left[-\cos\beta(l_1 - h) \frac{\cos v}{v} dv - \sin\beta(l_1 - h) \frac{\sin v}{v} dv \right] \tag{23}$$

where u_0, v_0, u_1, and v_1 are defined in (10) and (11). The integrals of (23) are recognized as cosine and sine integrals, and it can be rewritten as

$$A = 15\{\cos\beta(l_1 - h)[\mathrm{Ci}(u_0) + \mathrm{Ci}(v_0) - \mathrm{Ci}(u_1) - \mathrm{Ci}(v_1)] + \sin\beta(l_1 - h)[-\mathrm{Si}(u_0) + \mathrm{Si}(v_0) + \mathrm{Si}(u_1) - \mathrm{Si}(v_1)]\}. \tag{24}$$

Following this procedure, (10) and (11) were derived for mutual resistance and reactance.

Corrections

The following correction to "Exterior Electromagnetic Boundary Value Problems for Spheres and Cones," by L. L. Bailin and S. Silver, which appeared on pages 5–16 in the January, 1956 issue of these TRANSACTIONS, has been called to the attention of the authors by Dr. Leopold Felsen.

The expressions for the field produced by a slot in a cone given in the above paper are incomplete. We overlooked the contribution made to the TE modes in the external field by an excitation in the radial direction along the cone. It is necessary to add a term to our function Π^*, given by (52) on page 12, that represents the contribution of the E_R component of excitation in the aperture. The complete expression is

$$\Pi^*(r, \theta, \phi) = \left(\frac{\epsilon}{\mu}\right)^{1/2} \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} \frac{(2\nu_i' + 1) P_{\nu_i'}{}^m(\cos\theta)}{\nu_i'(\nu_i' + 1)\sin^2\theta_0 \left.\dfrac{\partial^2 P_\nu{}^m}{\partial\theta\partial\nu}\right|_{\substack{\theta=\theta_0 \\ \nu=\nu_i'}} \pi(1 + \delta_{0m})}$$
$$\left\{ \int_{r_1}^{r_2} \int_{\phi_1}^{\phi_2} m f_1(r', \phi') \Gamma_1(r, r') \sin m(\phi' - \phi) dr' d\phi' + \nu_i'(\nu_i' + 1) \sin\theta_0 \int_{r_1}^{r_2} \int_{\phi_1}^{\phi_2} f_2(r', \phi') \Gamma_2(r, r') \cdot \cos m(\phi' - \phi) dr' d\phi' \right\}$$

where $f_1(r', \phi')$ is the E_R component of the excitation in the aperture and $f_2(r', \phi')$ is the E_ϕ component; and

$$\Gamma_1(r, r') = j_{\nu_i'}(kr) \frac{d}{dr'} [r' h_{\nu_i'}{}^{(2)}(kr')] \qquad r < r'$$
$$= \frac{j_{\nu_i'}(kr') \dfrac{d}{dr'} [r' h_{\nu_i'}{}^{(2)}(kr')] h_{\nu_i'}{}^{(2)}(kr)}{h_{\nu_i'}{}^{(2)}(kr')} \qquad r > r'$$

and

$$\Gamma_2(r, r') = j_{\nu_i'}(kr) h_{\nu_i'}{}^{(2)}(kr') \qquad r < r'$$
$$= j_{\nu_i'}(kr') h_{\nu_i'}{}^{(2)}(kr) \qquad r > r'.$$

A circumferential slot such as is shown on page 13 in Fig. 3(a) of the paper, in general generates both TM and TE type modes. The total field contains terms derived from the function Π and the function Π^*. When the slot runs completely around the circumference and is excited uniformly (*i.e.*, has no variation in ϕ), the function Π^* reduces to zero and the field consists of only TM waves. A slot along a generator of the cone such as is shown in Fig. 3(b) gives rise to only TE waves and its field is obtained from the function Π^*; the results for such a slot remain as in the paper.

James R. Wait, author of "The Transient Behavior of the Electromagnetic Ground Wave on a Spherical Earth," which appeared on pages 198–202 of the April, 1957 issue of these TRANSACTIONS, has brought the following corrections to the attention of the editors.

In (3), $(2\pi)^{1/2}X$ should be replaced by $(2\pi X)^{1/2}$.

In (12), a should be replaced by 2.

In Fig. 3, the curve for 1500 mi. was misplotted; it should be essentially the same as the corresponding curve in Fig. 4.