(sweeping rate approximately 30 cps), and then applied to the tested network in the form of a signal simultaneously modulated in frequency and swept at a lower frequency rate. The limiter and discriminator circuits remain as described previously. The second harmonic voltage of the demodulated signal is applied to the vertical plates of the cathode-ray oscilloscope. The 10 kc filter should be adjusted to avoid excessive ringing as a consequence of fast changing levels and, at the same time, provide selectivity adequate for suppression of both fundamental and third harmonic voltage.

The attenuation of the filter for 5 kc signals is of the order of 60 db. Oscilloscopic displays obtained for various adjustments of the intermediate-frequency amplifier under test agree well with curves obtained using step-by-step method. Fig. 7 shows the oscillogram taken for four interstages of $kQ=1$ (slightly misaligned).

In conclusion it can be stated that the method of displaying second harmonic distortion or (in another scale) delay derivative versus frequency, provides more distinctive means of recognizing the conditions of network alignment than the conventional magnitude versus frequency display. The authors feel it may provide a useful means of aligning amplifiers intended for the reproduction of FM or transient signals.

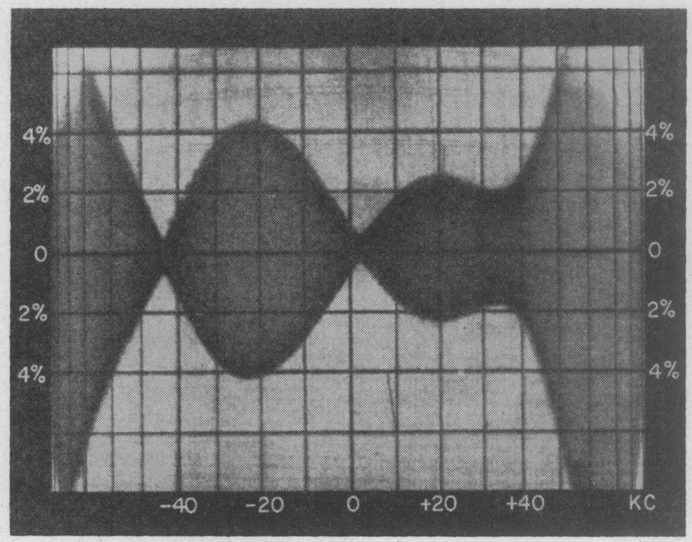


Fig. 7—Typical oscilloscopic presentation.

ACKNOWLEDGMENTS

Acknowledgment is due to Arthur H. Maciszewski who, in earlier discussions with one of the authors, contributed to some of the ideas described here, and to Stanley Torode for suggesting a few useful refinements in the experimental circuits.

Excitation Coefficients and Beamwidths of Tschebyscheff Arrays*

ROBERT J. STEGEN†

***Summary*—In this paper, exact expressions are obtained for the excitation coefficients of a Tschebyscheff array by equating the array space factor to a Fourier series whose coefficients are readily calculated.**

A set of curves is presented showing half-power beam width versus antenna length for various side-lobe levels. An approximate though very accurate expression for the half-power beamwidth is derived.

INTRODUCTION

DOLPH[1] SUGGESTED that a Tschebyscheff polynomial could be made to coincide with the polynomial representing an antenna pattern. He then proved rigorously that the resulting pattern would yield a minimum beamwidth when the side-lobe levels are fixed and a minimum side-lobe level when the beamwidth is specified. He derived the expressions for the currents of the Tschebyscheff arrays as

$$I_q = \frac{1}{A_{2q-1}{}^{2q-1}}\left\{A_{2q-1}{}^{2N-1} Z_0{}^{2q-1} - \sum_{k=q+1}^{N} I_k A_{2q-1}{}^{2k-1}\right\} \quad (1)$$

for $2N$ elements and

$$I_q = \frac{1}{A_{2q}{}^{2q}}\left\{A_{2q}{}^{2N} Z_0{}^{2q} - \sum_{k=q+1}^{N} I_k A_{2q}{}^{2k}\right\} \quad (2)$$

for $2N+1$ elements where

$$A_{2m}{}^{2n} = (-1)^{n-m} \sum_{p=n-m}^{n} \binom{p}{p-n+m}\binom{2n}{2p}, \quad (3)$$

$$Z_0 = \tfrac{1}{2}\left\{\left[r+\sqrt{r^2-1}\right]^{1/M} + \left[r-\sqrt{r^2-1}\right]^{1/M}\right\}, \quad (4)$$

$T_M(Z_0) = r =$ the main beam to side-lobe voltage ratio,

$M =$ one less than the number of elements in the array, (5)

$$T_M(Z) = \cos\,(M \operatorname{arc\,cos} Z), \quad |Z| \leqq 1, \quad (6)$$

$$T_M(Z) = \cosh\,(M \operatorname{arc\,cosh} Z), \quad |Z| \geqq 1, \quad (7)$$

and

$$T_M(Z) = \tfrac{1}{2}\left[(Z+\sqrt{Z^2-1})^M + (Z-\sqrt{Z^2-1})^M\right], \quad \text{all } Z. \quad (8)$$

* Decimal classification: R125.1/R325.11. Original manuscript received by the Institute, February 16, 1953; revised manuscript received June 1, 1953.

† Member, Technical Staff, Hughes Research and Development Laboratories, Culver City, Calif.

[1] C. L. Dolph, "A current distribution which optimizes the relationship between beamwidth and side-lobe level," PROC. I.R.E., vol. 34, pp. 335–348; June 1946.

By clever algebraic manipulation, Barbiere[2] was able to obtain expressions for the currents which were much easier to handle than those of Dolph. The Barbiere current expressions are

$$I_k = \sum_{q=k}^{N} (-1)^{N-q} Z_0^{2q-1} \frac{(2N-1)(q+N-2)!}{(q-k)!(q+k-1)!(N-q)!} \quad (9)$$

for $2N$ elements and

$$I_k = \sum_{q=k}^{N} (-1)^{N-q} Z_0^{2q} \frac{(2N)(q+N-1)!}{(q-k)!(q+k)!(N-q)!} \quad (10)$$

for $2N+1$ elements.

In addition Barbiere also proved that

$$Z_0 = \cosh\left(\frac{1}{M} \operatorname{arc\,cosh} r\right) \quad (11)$$

which is much easier to evaluate than (4) by either using tables or appropriate series expansions. These expressions of Barbiere are a big improvement over those of Dolph; however, they have one serious drawback. This is the factor $(-1)^{N-q}$, in (9) and (10), which results in the currents being the difference of two large and almost equal numbers. For example, take the calculation of I_1 in a 24 element array designed to have -40 db side-lobe level. $Z_0 = 1.02665$ and I_1 then becomes $I_1 = 93{,}040{,}583.6338 - 93{,}040{,}569.0594 = 14.5744$. Six significant figures are lost in this calculation. It was necessary to keep Z_0^n accurate to 12 significant figures to obtain this result, i.e., Z_0 was assumed to be $Z_0 = 1.02665000000$. As the number of elements in the array becomes larger, the number of significant figures required increases.

The use of Dolph's or Barbiere's expressions for calculating the current distributions of Tschebyscheff arrays requires a tremendous amount of calculations for an array having a large number of elements. It is also possible to determine the excitation coefficients of the Tschebyscheff array by equating the array space factor to a Fourier series whose coefficients are readily calculated. This leads to expressions for the currents which are somewhat more amendable to calculations.

The Tschebyscheff-Fourier Coefficients

The space factor of a Tschebyscheff array is given by[3]

$$T_{2N}(Z) = \sum_{m=0}^{N} I_m \cos(2mu) \quad (2N+1 \text{ elements}) \quad (12)$$

and

$$T_{2N-1}(Z) = \sum_{m=0}^{N-1} I_{m+1} \cos(2m+1)u \quad (2N \text{ elements}) \quad (13)$$

where

I_m = the excitation coefficient of the mth element on each side of the array center-line,

$$Z = Z_0 \cos u \quad (14)$$

$$u = \frac{\pi d}{\lambda} \sin\theta \quad (15)$$

and Z_0 is defined by (4) or (7).

The excitation coefficients, I_m, are real and symmetrical about the array center because the power pattern is symmetrical and the space factor is real.

Whittaker and Robinson[4] and Sokolnikoff[5] present a method of finding a sum

$$F(x) = \sum_{m=0}^{r} (a_m \cos mx + b_m \sin mx) \quad (16)$$

which furnishes the best possible representation of a function $u(x)$ when we are given that $u(x)$ takes the values $u_0, u_1, u_2, u_3, \cdots u_{n-1}$ when x takes the values 0, $2\pi/n$, $4\pi/n, \cdots 2(n-1)\pi/n$, respectively, where $n \geqq 2r+1$. The coefficients are evaluated by use of the following equations

$$a_0 = \frac{1}{n} \sum_{k=0}^{n-1} u_k, \quad (17)$$

$$a_m = \frac{2}{n} \sum_{k=0}^{n-1} u_k \cos\frac{2k\pi m}{n}, \quad (18)$$

and

$$b_m = \frac{2}{n} \sum_{k=0}^{n-1} u_k \sin\frac{2k\pi m}{n}. \quad (19)$$

The excitation coefficients of the Tschebyscheff array are easily determined by equating the Tschebyscheff polynomial to (16) and solving for the I_m in terms of the a_m and b_m coefficients. The space factor for the Tschebyscheff array of $2N+1$ elements, (12) may be equated to the Fourier series (16) by setting $2u = x$ and $N = r$. Then

$$I_m = a_m \quad m = 0, 1, 2, \cdots, N, \quad (20)$$

$$b_m = 0, \quad (21)$$

and

$$n = 2r + 1 = 2N + 1. \quad (22)$$

Z_0 may be calculated from (4) or (11).

The values of $u(x)$ are obtained from

$$u(x) = T_{2N}(Z) = T_{2N}(Z_0 \cos u) = T_{2N}\left(Z_0 \cos\frac{x}{2}\right) \quad (23)$$

at the $2N+1$ values of x, namely,

$$x = \frac{2\pi}{2N+1} s, \quad s = 0, 1, 2, \cdots, 2N. \quad (24)$$

$T_{2N}(Z)$ may be computed using the appropriate closed forms (6), (7), or (8).

The expressions for the excitation coefficients of the Tschebyscheff linear array then become

[2] D. Barbiere, "A method for calculating the current distribution of Tschebyscheff arrays," Proc. I.R.E., vol. 40, pp. 78–82; January, 1952.

[3] C. L. Dolph, "Discussion on a current distribution," Proc. I.R.E., vol. 35, pp. 489–492; May, 1947.

[4] E. T. Whittaker and G. Robinson, "The Calculus of Observation," D. Van Nostrand Company, New York, N. Y., pp. 260–267; 1924.

[5] I. S. and E. S. Sokolnikoff, "Higher Mathematics for Engineers and Physicists," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 545–550; 1941.

$$I_0 = \frac{1}{2N+1}\sum_{s=0}^{2N} u_s(x) \tag{25}$$

$$= \frac{1}{2N+1}\left[u_0 + \sum_{s=1}^{N} u_s + \sum_{s=N+1}^{2N} u_s\right] \tag{26}$$

and

$$I_m = \frac{2}{2N+1}\sum_{s=0}^{2N} u_s \cos\frac{2s\pi m}{2N+1} \tag{27}$$

$$= \frac{2}{2N+1}\left[u_0 + \sum_{s=1}^{N} u_s \cos\frac{2s\pi m}{2N+1} + \sum_{s=N+1}^{2N} u_s \cos\frac{2s\pi m}{2N+1}\right]. \tag{28}$$

The $2N+1-k$ term of

$$\sum_{s=N+1}^{2N} T_{2N}\left(Z_0 \cos\frac{s\pi}{2N+1}\right)\cos\frac{s2\pi m}{2N+1} \tag{29}$$

is

$$T_{2N}\left[Z_0 \cos\frac{\pi(2N+1-k)}{2N+1}\right]\cos\frac{2(2N+1-k)\pi m}{2N+1} \tag{30}$$

$$= T_{2N}\left[-Z_0 \cos\frac{\pi k}{2N+1}\right]\cos\frac{2k\pi m}{2N+1} \tag{31}$$

and, since $T_{2N}(Z)=T_{2N}(-Z)$, this $2N+1-k$ term becomes the same as the kth term of

$$\sum_{s=1}^{N} T_{2N}\left(Z_0 \cos\frac{s\pi}{2N+1}\right)\cos\frac{s2\pi m}{2N+1}. \tag{32}$$

Since there is this asymmetrical term by term correspondence, the two summations are equal. The expressions for the excitation coefficients become

$$I_0 = \frac{1}{2N+1}\left[r + 2\sum_{s=1}^{N} T_{2N}\left(Z_0 \cos\frac{s\pi}{2N+1}\right)\right] \tag{33}$$

and

$$I_m = \frac{2}{2N+1}\left[r + 2\sum_{s=1}^{N} T_{2N}\left(Z_0 \cos\frac{s\pi}{2N+1}\right)\cos\frac{2s\pi m}{2N+1}\right] \tag{34}$$

where

$$m = 1, 2, 3, \cdots, N.$$

A similar procedure for a Tschebyscheff linear array of an even number ($2N$) of elements results in the follow expressions for the excitation coefficients:

$$I_{m+1} = \frac{1}{N}\left[r + 2\sum_{s=1}^{N-1} T_{2N-1}\left(Z_0 \cos\frac{s\pi}{2N}\right)\cos\frac{s\pi(2m+1)}{2N}\right] \tag{35}$$

where

$$m = 0, 1, 2, \cdots, N-1.$$

The excitation coefficients of a 144-element array having -40 db side-lobe level were calculated using (35). The values of I_{72} and I_{71} obtained from (35) were exactly the same as those calculated from (1) or (9). This was used as a check for errors in the calculations of T_{2N-1} $(Z_0 \cos s\pi/2N)$.[6]

The Half-Power Beamwidth

The beamwidth of a Tschebyscheff array may readily be calculated using (5) and (7). The maximum amplitude of the main beam is

$$T_M(Z_0) = r = \cosh\,(M \text{ arc cosh } Z_0) \tag{36}$$

or

$$Z_0 = \cosh\left(\frac{1}{M}\text{ arc cosh } r\right). \tag{37}$$

At the half-power points

$$T_M(Z_1) = \frac{r}{\sqrt{2}} \tag{38}$$

or

$$Z_1 = \cosh\left(\frac{1}{M}\text{ arc cosh }\frac{r}{\sqrt{2}}\right) \tag{39}$$

where

$$Z_1 = Z_0 \cos u_1 \tag{40}$$

$$u_1 = \frac{\pi d}{\lambda}(\sin\theta_1 - \sin\hat{\theta}) \tag{41}$$

$\hat{\theta}$ = the angle of the main beam from broadside

and

d = the element spacing.

Half-power beamwidth for broadside beam ($\hat{\theta}=0°$) is

$$\theta_{\mathrm{HP}} = 2\theta_1, \tag{42}$$

and is shown in Fig. 1 for a spacing

$$d = \tfrac{2}{3}\lambda. \tag{43}$$

The spacing of the elements has only a minor effect on the beamwidth. For other than broadside beams, i.e.,

[6] Method of determining excitation coefficients suggested by C. L. Dolph, "Discussion on a current distribution," Proc. I.R.E., pp. 489–492; May, 1947. Expressions (33), (34), (35), were obtained by Mrs. Wilma Bottaccini and Dr. N. H. Enenstein (unpublished work).

$\hat{\theta} \neq 0°$, (41) may be used to obtain values θ_1^+ and θ_1^- which correspond to the positive and negative values of u_1, respectively. The half-power beamwidth is then

$$\theta_{HP} = \theta_1^+ - \theta_1^-. \tag{44}$$

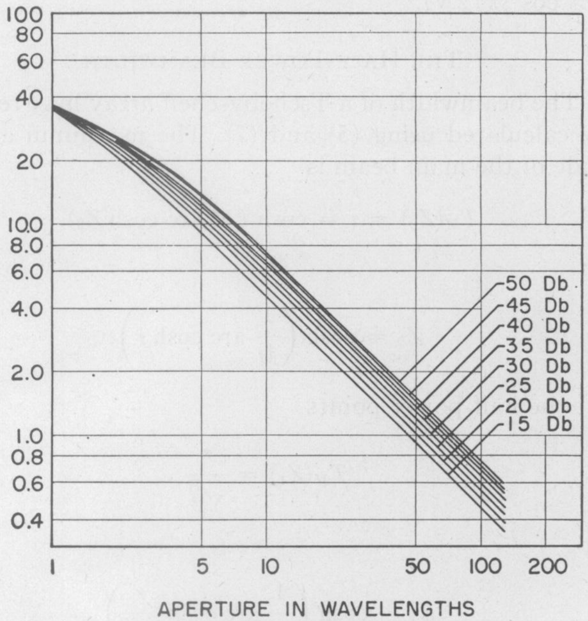


Fig. 1—Beamwidth of a Tschebyscheff array.

An approximate expression for the broadside half-power beamwidth may be obtained, which is quite accurate for the smaller beamwidths, by letting

$$\cos u_1 = 1 - \frac{u_1^2}{2}, \tag{45}$$

$$\cosh v = 1 + \frac{v^2}{2}, \tag{46}$$

$$\text{arc} \cosh r = \log 2r - \frac{1}{4r^2}, \tag{47}$$

and

$$l = Md = \text{the length of the array.} \tag{48}$$

By direct substitution the final expression becomes

$$\sin \frac{\theta_{HP}}{2} = \frac{1}{\pi l/\lambda} \sqrt{\frac{3}{4} \log_e{}^2 2 + 2 \log_e 2 \log_e r + \frac{\log_e r}{2r^2}}. \tag{49}$$

For

$$\theta_{HP} < 12°$$

$$\theta_{HP} = \frac{0.636}{l/\lambda} \sqrt{0.360 + 0.693 \log_e r + \frac{\log_e r}{2r^2}} \tag{50}$$

$$= \frac{A}{l/\lambda} \tag{51}$$

where A depends on the side-lobe level r. Some values are listed below.

r (db)	A (degrees)
−20	51.1
−25	56.0
−30	60.6
−35	65.0
−40	68.7

These may be compared with the value $\theta_{HP} = 50.9/l/\lambda$ for a uniform array.

Contributors to Proceedings of the I.R.E.

For a photograph and biography of Marvin Chodorow, see page 163 of the January, 1953 issue of the Proceedings of the I.R.E.

❖

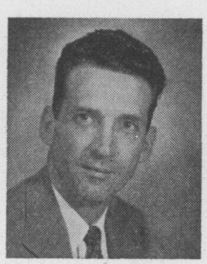
Kenneth W. Goff was born in Salem, West Virginia, in 1928. He received the B.S. degree in electrical engineering from West Virginia University in 1950. After graduation, Mr. Goff joined the Staff of the M.I.T. Acoustics Laboratory and entered the M.I.T. Graduate School. He received his M.S. degree in electrical engineering from M.I.T. in 1952, and is now working toward his D.Sc. degree. His work in the M.I.T. Acoustics Laboratory has been in the field of Acoustical Instrumentation with particular attention given the application of correlation techniques to the localization of sound sources.

Kenneth W. Goff

Mr. Goff is a member of Eta Kappa Nu, Tau Beta Pi and Sigma Xi.

❖

John N. Hines, (S'47–A'50) was born in Tientsin, China, on March 9, 1920. He received the B.S. degree in electrical engineering from the University of Connecticut in 1943 and entered the Armed Forces as a Radio Officer in the Signal Corps until the fall of 1946. Mr. Hines then enrolled in the graduate school of the Ohio State University where he was awarded the M.Sc. degree in electrical engineering in 1949. He is presently employed as a research associate at the Antenna Laboratory there, working on the problems of traveling-wave slot antennas. He is also a member of the technical school staff of Franklin University, Columbus, Ohio.



John N. Hines

Mr. Hines is a member of the AIEE.

❖

Julius J. Hupert (SM'48) was born on May 6, 1910, in Tarnopol, Poland. He received the M.S. (El. Eng.) degree from the Polytechnic Institute of Warsaw, Poland, in 1933 and the Ph.D. degree from Northwestern University in 1951.

J. J. Hupert

Dr. Hupert's considerable experience in research and development in the field of communications was acquired as head of the Transmitter Department,