III. Extensions of the Procedure

A. Linear Aperture

A similar step procedure for the linear aperture can be used to synthesize patterns. In this case the associated function is $ug(u)$ and is represented simply by a finite Fourier series. The analysis is then the same essentially as given in [8], Case 1.

For a cosecant pattern from an aperture divided into 8 steps, the result of this procedure is illustrated in Fig. 3.

B. Zones of Varying Width

Considering only the circularly symmetrical case and writing r_n, the inner radius of the nth zone, as $\lambda_n r_1$ (4) may be re-written as

$$g(u) = \sum_{n=1}^{N} A_n \lambda_n r_1 \frac{J_1(u\lambda_n r_1)}{u}. \qquad (22)$$

Letting

$$r_1 = \frac{1}{m} \text{ and } \frac{u}{m} = v \text{ we have as before}$$

$$m^2 v g(mv) = \sum_{n=1}^{N} \lambda_n A_n J_1(\lambda_n v). \qquad (23)$$

The λ_n can now be chosen in one of two ways, to give representations by Fourier-Bessel series or Dini series [9].

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Calculation of the Current Distribution on a Thin Linear Antenna

Electromagnetic scattering problems can often be represented by an integral expression with an inhomogeneous source term. Until the advent of the high-speed digital computer, such representations were largely academic, since they could not readily be solved for the scattered field. Now, however, integrals may be approximated by large systems of simultaneous linear algebraic equations which can be solved by computer. The transformation of the pertinent integral equation into its algebraic approximation is recognized in the physical problem as a relaxation of the boundary conditions such that they are enforced only at a discrete set of points over the scatterer, rather than continuously over the surface. Experience has shown that if one chooses a sufficient number of points at which to match boundary conditions, then such approximations can give accurate solutions to problems that heretofore could not be solved.

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One problem that has not previously been considered by the point boundary-value matching method is that of calculating the current distribution on a thin linear antenna with a source at its center.[1] Starting with the proper integral equation, an inhomogeneous system of linear equations can be obtained by forcing the tangential electric field to vanish at $N-1$ points along the axis of the thin hollow wire antenna. The Nth equation is obtained by forcing the current to be unity (or any other nonzero constant) at the center of the wire. The system of equations can then be solved to determine the coefficients of the N terms in the Fourier series expansion of the current.

This method has certain advantages over others in that it is simple in concept, is sufficiently accurate, and is applicable to antennas up to several wavelengths long. In addition, the method can be extended to radiating structures that are not linear, such as the Vee antenna and circular-loop antennas [3].

The field radiated by a current distribution of the form $I(z')$ induced on a thin, hollow, cylindrical wire with open ends and perfect conductivity is given by an integral expression derived by Richmond [4], who considered a plane wave as the exciting source. In the work presented here, the exciting source is a unit current generator across an infinitesimal gap at the wire center. Since the wire is excited symmetrically at the center, the total current (i.e., the sum of the currents on the inner and outer surfaces) may be represented by a Fourier series of odd-ordered even modes. That is,

$$I(z) = \sum_{n=1}^{N} I_n \cos(2n-1)\frac{\pi z}{L}. \qquad (1)$$

The field generated by this current is the same as it would generate if it existed in unbounded free space. Thus, by the vector potential method the required integral expression is

$$E_z(0, 0, z) = \frac{\lambda\sqrt{\mu/\epsilon}}{8j\pi^2 a^2} \sum_{n=1}^{N} I_n \int_{\theta_1'}^{\theta_2'} G(\theta') \cdot \cos(2n-1)\frac{\pi z'}{L}\, d\theta' \qquad (2)$$

[1] Zuhrt has treated the cylindrical antenna as a boundary value problem using the scalar potential to generate an infinite system of linear equations [1]. However, he retained only three modes, and thus his results are inaccurate for long antennas, especially near antiresonance [2].

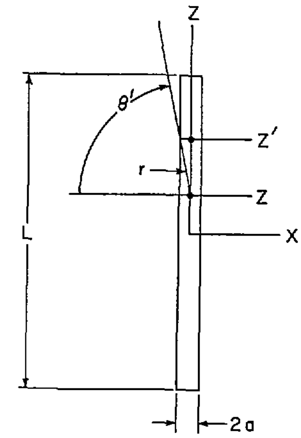

Fig. 1. Perfectly conducting hollow wire of length L and radius a.

where

$$G(\theta') = e^{-jka/\cos\theta'} [\cos\theta' + jka)(2 - 3\cos^2\theta') + k^2a^2\cos\theta'] \qquad (3)$$

and the limits of integration as illustrated in Fig. 1 are given by

$$\theta_1' = -\tan^{-1}\frac{0.5L + z}{a} \qquad (4)$$

and

$$\theta_2' = \tan^{-1}\frac{0.5L - z}{a}. \qquad (5)$$

There are, however, certain important assumptions implied by these equations. First, the walls of the hollow wire are taken to be very thin, which means that the total current goes essentially to zero at the ends of the wire. Second, the points at which the radiated electric field is set to zero are, for convenience, chosen on the axis of the hollow wire, whereas, strictly speaking, this boundary condition should be enforced at the surface of the wire. It can be shown that the difference in the electric field evaluated on the axis of the wire and that on its surface is negligibly small for radii less than about 0.01λ, except within 3 or 4 radii of the open ends. Third, the current is assumed to be independent of ϕ by virtue of the symmetrical geometry.

When the wire is treated as a scatterer with N modes in the current expansion, N inhomogeneous linear equations are generated by forcing the scattered field to equal the negative of the incident field at N points along the wire shown in Fig. 2. However, if we treat the wire as a radiator and wish to determine the current distribution on the wire when it is driven at some point along its length (taken to be at the center of the wire in this work), then the procedure mentioned above must be altered to account for this source. One method is to assume a unit current generator acting in an infinitesimally narrow gap in the wire, and require that the sum of all the mode currents at the gap sum to unity.

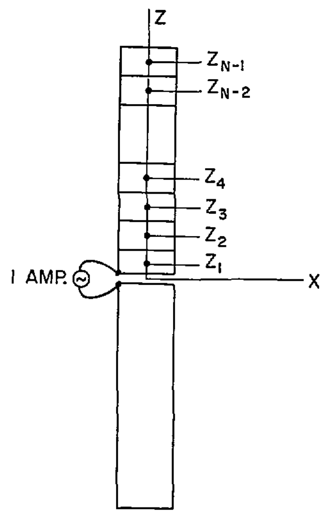

Fig. 2. Hollow wire with $N-1$ matching points along its axis.

If we force the tangential electric field to vanish at $N-1$ equally spaced points along the axis with $0<z_N<L/2-4a$, by using (2) we generate $N-1$ homogeneous equations of the form

$$\sum_{n=1}^{N} C_{mn}I_n = 0, \quad (m = 1, 2, \cdots, N-1), \qquad (6)$$

and one inhomogeneous equation of the form

$$\sum_{n=1}^{N} I_n = 1. \qquad (7)$$

This inhomogeneous system of N equations in N unknowns can be solved by a digital computer with no severe limitations on the number of modes that may be used to represent the current. It is important, however, in any case where the antenna is several wavelengths long that a sufficient number of modes be used to ensure that the points at which boundary conditions are matched are not too far apart. Also, it is important when evaluating (2) that sufficient intervals be taken in the numerical integration, especially when the wire radius is small (i.e., less than 0.001λ).

The far-zone field of the thin finite length wire shown in Fig. 1 has only a θ component, which is given by

$$E(\theta) = \frac{j\omega\mu}{4\pi r_0} e^{-jkr_0} \cdot \sin\theta \int_{-L/2}^{L/2} I(z') e^{jkz \cos\theta} dz'. \qquad (8)$$

For the case of symmetrical excitation, with the current expressed as a Fourier series, (8) becomes

$$E(\theta) = \frac{-jL\sqrt{\mu/\epsilon}}{\pi r_0} e^{-jkr_0} \sin\theta \cdot \sum_{n=1}^{N} (-1)^n \frac{(2n-1)I_n \cos(\pi L' \cos\theta)}{(2n-1)^2 - (2L'\cos\theta)^2}, \qquad (9)$$

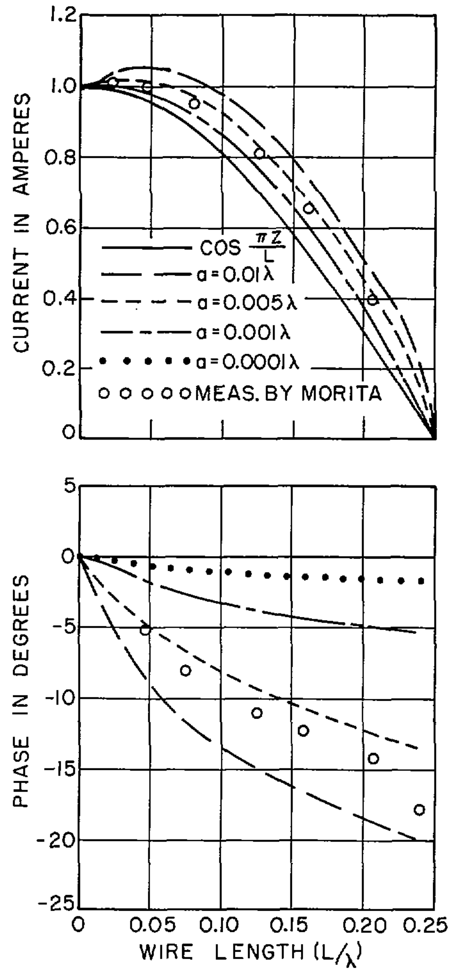

Fig. 3. Current distribution on a half-wave thin linear antenna as a function of its radius.

from which a pattern factor $F(\theta)$ can be defined as

$$F(\theta) = L \sin\theta \sum_{n=1}^{N} (-1)^n \frac{(2n-1)I_n \cos(\pi L' \cos\theta)}{(2n-1)^2 - (2L'\cos\theta)^2} \qquad (10)$$

where $L' = L/\lambda$.

The results obtained by the point boundary-value matching method generally compare quite well with results obtained by other methods used to treat the thin linear antenna. Figure 3 shows the amplitude and phase variation on a half-wave dipole for various wire radii. It is interesting to note that as the radius becomes smaller, the current amplitude approaches the usually assumed sinusoidal distribution and the phase becomes more nearly constant. The amplitude plot for $a=0.0001\lambda$ is not shown because it lies only slightly above the cosine plot. Twelve modes were used in calculating each curve and 1000 intervals were used in evaluating the integral in (2) by the fifth-order Newton-Cotes formula. However, the curve for $a=0.0001\lambda$ required the use of 2000 intervals since the integrand varies quite rapidly in the vicinity of the observation point. For comparison, the experimental results obtained by Morita are also shown in Fig. 3. They were taken from the coarse graphs in the literature [5], [6] and scaled to a current amplitude of unity at the wire center.

Further excellent results were obtained for cases for which antenna length L was 1.5λ, 1.25λ, and 1.46λ. The first two cases were compared with the measured results of Morita ($a \approx 0.003\lambda$) and the last case with the calculated results of Einarsson ($a=0.01\lambda$) [7]. The current in all three cases was scaled to unity at the center.

In the anti-resonant case, in which the wire length is 1.0λ, good correlation exists between the phase calculated by point boundary-value matching and that obtained by Morita and Einarsson. Although the positions of current maxima and minima also agree with theirs, the variation in the current amplitude differs slightly, even when the number of modes is doubled to 24, tending to agree slightly better with the above-mentioned measured results than with those calculated. Unfortunately, results reported in the literature were limited and did not permit a more thorough investigation of the anti-resonant case.

The calculation of the far-field pattern in the E-plane was accomplished by calculating the pattern factor $F(\theta)$ defined by (10). Excellent agreement was obtained with Einarsson's results for the pattern shape with $a=0.01\lambda$. Since the excitation source in the present work is a unit current source, while that of Einarsson is a unit voltage source, comparisons between relative pattern amplitudes for antennas of different length are not possible. Such a direct comparison could be made if the gap voltage were calculated by the point-matching method. This will be attempted in the future.

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