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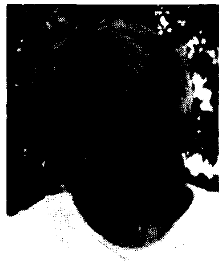

John D. Mahony

Approximations to the Radiation Resistance and Directivity of Circular-Loop Antennas

This interesting and useful analysis by **Dr. John D. Mahony** is his second contribution to our column. John is with the British Aerospace Communication Division (Space Systems), and can be contacted at British Aerospace, Payload Engineering Department, Argyle Way, Stevenage, Herts SG1 2AS, England (Tel: 0438-313456). His first article, which appeared in the October, 1991, issue, was on design equations for rectangular- and circular-patch antennas. Thank you, John, for your interest in this column.

1. Introduction

The purpose of this note is to derive approximate formulas for the radiation resistance, R, and the directivity, D, of circular-loop antennas. Exact, integral expressions for these quantities, valid for all loop sizes, maybe found in Chapter 5 of reference [1], or, in Chapter 6 of reference [2]. For convenience, the formulas are reproduced here, using the usual notation, in the form

$$R = 60\pi^2(ka)\int_0^{2ka} J_2(x)dx, \qquad (1)$$

and

$$D = \frac{120\pi^2(ka)^2}{R}\max\left\{J_1^2[ka\sin(\theta)]\right\}. \qquad (2)$$

Numerical values for the radiation resistance and the directivity are obtained once the input parameter ka is specified; a denotes the radius of the loop, and $k\,(=2\pi/\lambda)$ is the free-space wavenumber (λ denotes the free-space wavelength). The process requires the numerical evaluation/integration of ka-dependent Bessel functions of orders 1 and 2. In the case of small-loop antennas ($0 < ka < 1/3$), the Bessel functions are usually replaced by their small-argument approximations, to simplify the numerical work. In the limiting case, this leads to simple formulas for the quantities of interest. Similarly, in the case of large-loop antennas ($ka > \pi$), asymptotic approximations can be employed and, again, these lead to simple approximate formulas. However, such formulae, which can be found in Table 6-2 of reference [2], do not capture the oscillatory behavior of the Bessel-function integral.

It will be shown here that simple approximations to the Bessel functions can be employed, to accurately model this behavior for both small and intermediate-sized loop antennas. Furthermore, when these approximations are combined with the usual asymptotic contributions to the integral in the case of large ka, **accurate and relatively simple results for R and D can be secured for all loop sizes**. Numerical results can, if necessary, be obtained using a simple pocket calculator. It should be noted in passing that other oscillatory approximations, suitable for values of ka greater than about 5, are also available in reference [2].

2. Approximations

In the case of intermediate-sized loop antennas, the approximations referred to above are, in the usual notation,

$$J_n(x) \cong \frac{1}{n!}(z_{n,1}/\pi)^n \sin^n(\pi x/2z_{n,1}), \quad n = 1,2,3,\ldots \qquad (3)$$

In the above, $z_{n,1}$ denotes the first zero of the derivative of the Bessel function of order n. For $n=1$, $z_{1,1} = 1.84118$, and when $n=2$, $z_{2,1} = 3.05424$ [3]. An appreciation of the accuracy of these approximations can be obtained from Figure 1, which shows a graph of the various Bessel functions and their respective approximations. The approximation in the case when $n=1$ is accurate to within about 5%, for values of the argument not greater than about 3. This approximation was used in a previous design note [4], to simplify the calculations concerning the use of Bessel functions, in the design of circular microstrip-patch antennas. In the case when $n=2$, the same accuracy is sustained at an argument value of about 4.2. It may be noted in passing that the area under the exact curve is more or less the same as that under the approximate one, for argument values not exceeding about 4.75. A graph of the approximation, in the case when $n=3$, is also included in the figure, purely for interest, but it is not needed for the present purpose. Clearly, the accuracy of the approximation diminishes with the increasing order of the Bessel function, and with its increasing argument. Nevertheless, it will be seen to suffice for the present purpose. Thus, on employing this approximation to determine the contribution to the Bessel-function integral for small/intermediate values of the argument, in conjunction with the usual two-term asymptotic approximation to determine the contribution from the

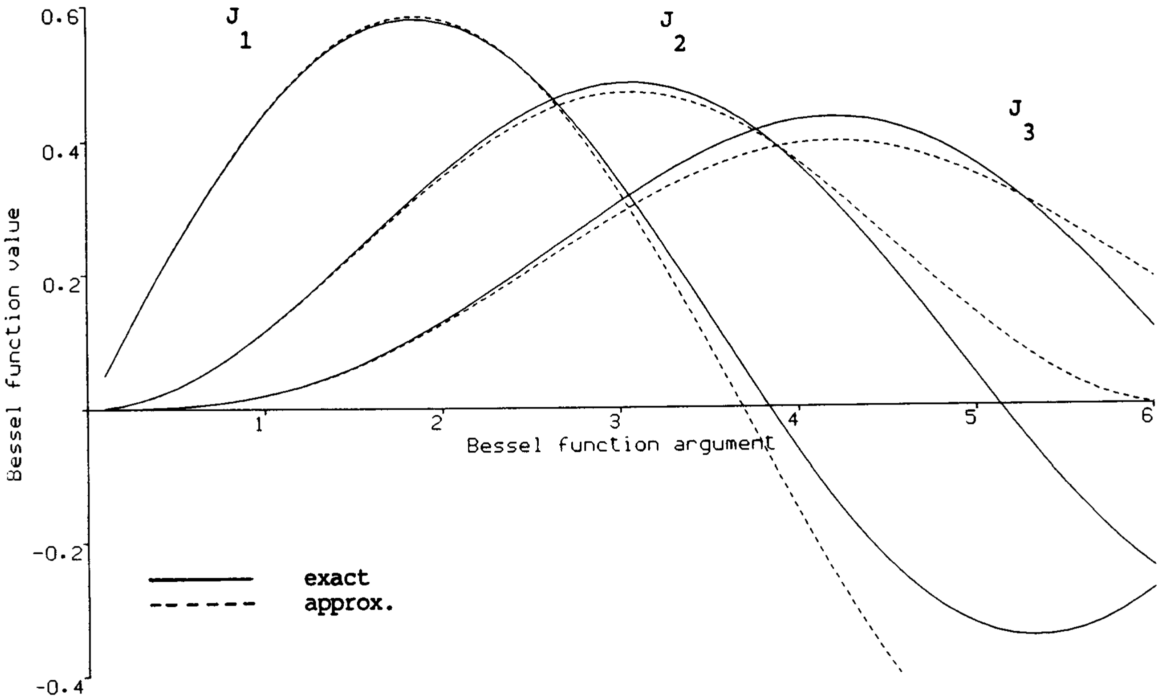

Figure 1. Graphs of Bessel functions and their approximations.

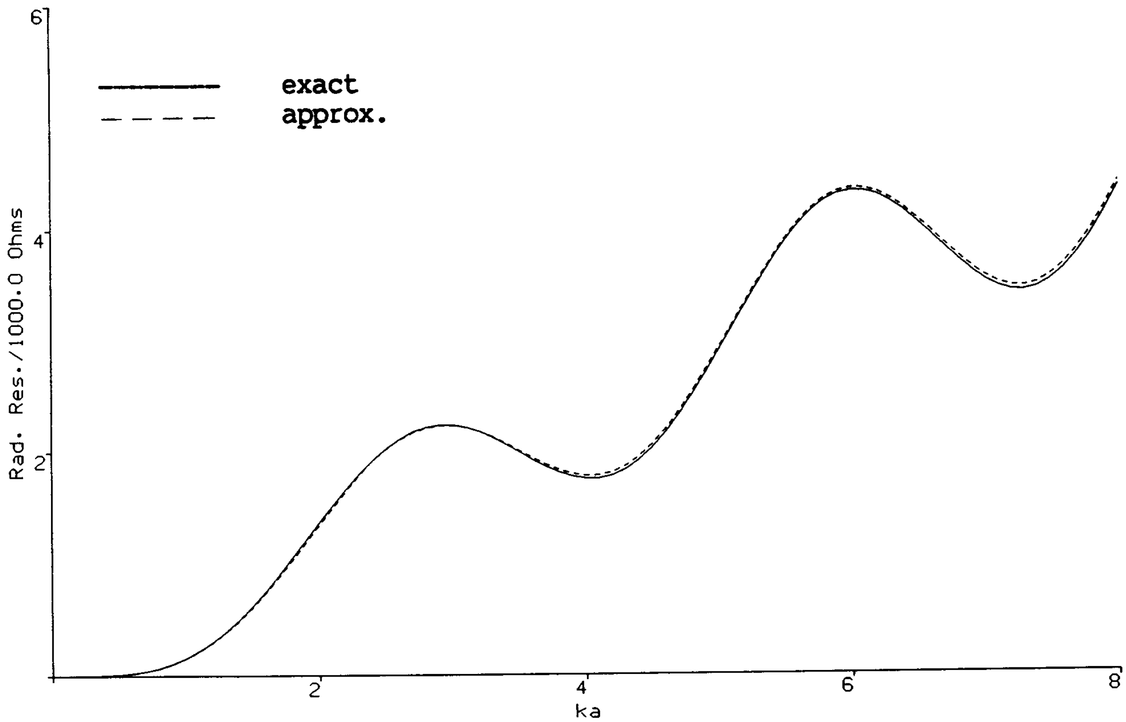

Figure 2. The circular-loop radiation resistance as a function of *ka*.

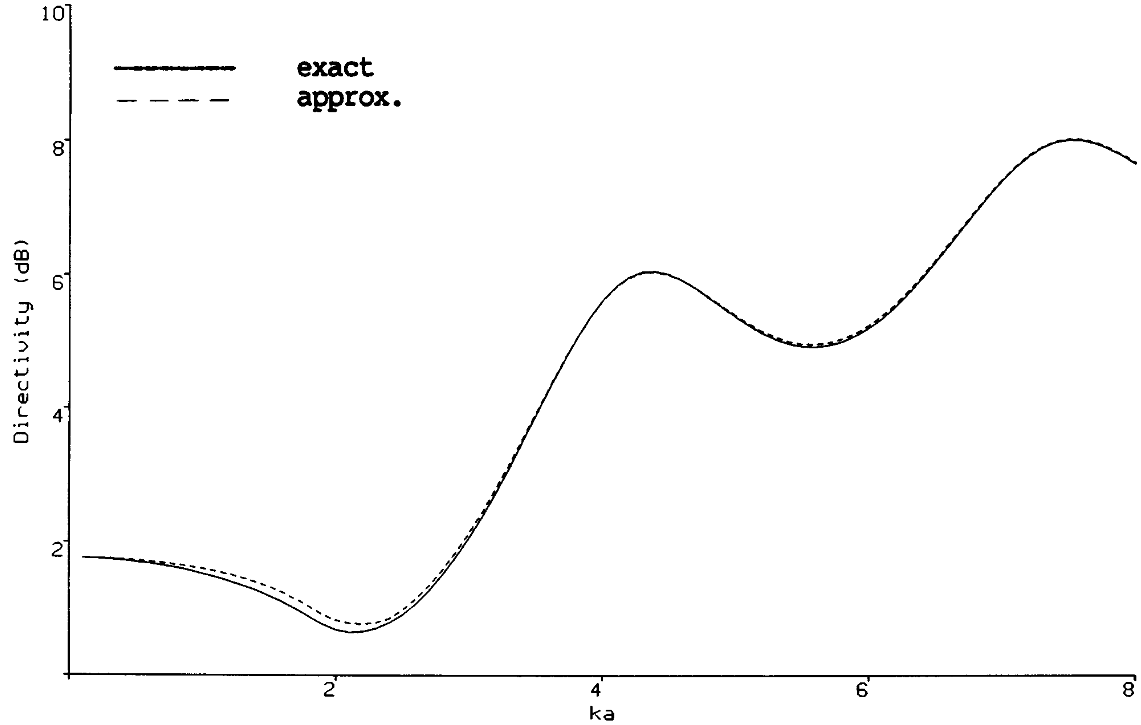

Figure 3. The circular-loop directivity (in dB) as a function of *ka*.

remainder, it is possible to approximate the integral by the expression,

$$\int_0^{2ka} J_2(x)dx = \begin{bmatrix} f(ka), & ka \le u_1/2, \\ f(u_1/2) + g(u_1/2) - g(ka), & ka \ge u_1/2, \end{bmatrix} \quad (4)$$

where f, the small/intermediate contribution, and g, the asymptotic contribution, are respectively given by

$$f(t) = (t/2)(z_{2,1}/\pi)^2\left[1 - \mathrm{sinc}(2\pi t/z_{2,1})\right], \quad (5)$$

and

$$g(t) = \sqrt{1/\pi t}\left[\sin(2t - \pi/4) + \frac{11}{16}\frac{\cos(2t-\pi/4)}{t}\right]. \quad (6)$$

Here, u_1 denotes a value for the variable of integration, beyond which it is appropriate to use the asymptotic form for the integrand. In the light of the above comment, concerning the equality of areas under the curve, a suitable value for u_1 is about 4.75. The term "$\mathrm{sinc}(x)$" has been used to denote $\sin(x)/(x)$. Accordingly, approximate expressions for R and D can now be written down, in the form

$$R = 60\pi^2(ka)\begin{bmatrix} f(ka), & ka \le u_1/2, \\ f(u_1/2) + g(u_1/2) - g(ka), & ka \ge u_1/2, \end{bmatrix} \quad (7)$$

and

$$D = \frac{120(ka)^2(z_{1,1})^2}{R}\max\left\{\sin^2\left[\pi ka \sin(\theta)/(2z_{1,1})\right]\right\}. \quad (8)$$

In this form, the computations required to produce values for R and D do not rely explicitly upon numerical procedures to evaluate/integrate Bessel functions.

3. Discussion

Calculations based on Equations (7) and (8) were carried out, to determine approximate values of R and D for a range of values of the input parameter, *ka*. The results are shown, respectively, in Figures 2 and 3. Also shown in the figures are the results of "exact" calculations. From these figures, it may be seen that the approximate results closely track the exact results, for all values of *ka*. In the case of the directivity calculations, the discrepancy between the two sets of results for small/intermediate values of *ka* is, at worst, no more than about 0.2 dB. The particular choice of value for u_1 (in this case, 4.75) governs the closeness of agreement between the two sets of results. At large values of *ka*, for example, a value for u_1 in excess of 4.75 will under estimate the directivity, whereas a value below this will result in an over-estimate. Although it is possible to evaluate the above approximations for a given *ka* using only a pocket calculator, it is a time-consuming business to do so for a range of *ka* values. Accordingly, for the sake of completeness, a short *FORTRAN* computer code, which will predict values of R and D based upon these approximations, is given in the Appendix.

Finally, it is appropriate to comment on an equivalence relationship between intermediate-sized circular- and square-loop antennas. Specifically, with reference to the expressions for the electric far field in the principal planes of such loops (see Chapter 5 of reference [1] for details), and to the approximations contained herein, it may be shown that the ratio, of the electric far-field component of the circular loop to that of the square loop, ρ, can be approximated by the expression

$$\rho = \frac{z_{1,1}L}{\pi a} \frac{\sin\left[\pi ka \sin(\theta) / \left(2z_{1,1}\right)\right]}{\sin\left[kL\sin(\theta)/2\right]} \frac{A_c}{A_s}, \tag{9}$$

where L denotes the side length of the square loop, and A_c and A_s denote the areas of the circular and square loops, respectively. For small loops, the ratio reduces simply to the ratio of the respective loop areas, irrespective of the relative loop dimensions. However, in the case of intermediate-sized loops, a ratio of areas is obtained when $a/L = z_{1,1}/\pi$ (= 0.5861), the value for the ratio ρ then being approximately unity. Incidentally, it is possible to discern an identical equivalence relationship between circular and square microstrip-patch antennas, excited in their fundamental modes [5].

4. References

1. C. Balanis, *Antenna theory analysis and design*, New York, Harper and Row, 1982.

2. J. D. Kraus, *Antennas*, New York, McGraw-Hill Book Company, 1950.

3. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, New York, Dover Publications Inc., November 1970.

4. J. D. Mahony, "Similarities in the design equations of rectangular and circular patches," *IEEE Antennas and Propagation Magazine*, **33**, 5, October 1991.

5. J. D. Mahony, "Simplified analysis of circular microstrip patches," *IEE Colloquium Digest 1992*/023, February 1993.

5. Acknowledgment

The author is pleased to acknowledge the column Editor for his help and encouragement.

6. Appendix

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c                              APPENDIX
c
c       A FORTRAN programme to calculate approximate values for the
c       Radiation Resistance and Directivity of circular loop antennas
c                        ___________________
c       'wavena' is the inuput parameter ka which runs from 0.1 to 24.0
c       'valintap' is the approximate value of the integral[see eqn.(4)]
c       'radresap' is the approximate value of the radiation resistance
c       'dirapp' is the approximate value of the directivity in dB
c                other terms are self explanatory
c       ________________________________________________________________

        pi=3.141592653589793
        z11=1.84118
        z21=3.05424
        ul=4.75

        wavena=0.1
44      term=60.0*pi*pi*wavena

        if(wavena.ge.ul/2.0)go to 12

        arg=2.0*wavena*pi/z21
        valintap=wavena*z21*z21/pi/pi*(1.0-sin(arg)/arg)/2.0

        go to 13

12      arg=ul*pi/z21
        valintap=ul*z21*z21/pi/pi/4.0*(1-sin(arg)/arg)

        addon1=sin(ul-pi/4.0)/sqrt(ul)
        addon2=-sin(2.0*wavena-pi/4.0)/sqrt(2.0*wavena)
        addon3=11.0/8.0*cos(ul-pi/4.0)/(ul**1.5)
        addon4=-11.0/8.0*cos(2.0*wavena-pi/4.0)/((2.0*wavena)**1.5)

        addon=(addon1+addon2+addon3+addon4)*sqrt(2.0/pi)

        valintap=valintap+addon

13      radresap=term*valintap

        if(wavena.lt.z11)factorap=z11*sin(pi*wavena/2.0/z11)
        if(wavena.ge.z11)factorap=z11
        dirapp=10.0*alog10(120.0*wavena*wavena*factorap*factorap/radresap)

        write(6,45)wavena,radresap/1000.0,dirapp

45      format(f6.2,2x,f5.2,2x,f5.2)

        if(wavena.gt.24.0)stop
        wavena=wavena+0.1
        go to 44

        end
c       ________________________________________________________________
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Ideas for Antenna Designer's Notebook

Ideas are needed for future issues of the *Magazine*. Please send your suggestions to Hal Schrank and they will be considered for publication as quickly as possible. Topics can include antenna design tips, equations, nomographs, or shortcuts, as well as ideas to improve or facilitate measurements.

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Updated Table of Contents

An updated Table of Contents for the Antenna Designer's Notebook was published in the August, 1993, issue of the *Magazine*.