Chapter 2

Dyadics

Dyadics are linear functions of vectors. In real vector space they can be visualized through their operation on vectors, which for real vectors consists of turning and stretching the vector arrow. In complex vector space they correspondingly rotate and deform ellipses. Dyadic notation was introduced by GIBBS in the same pamphlet as the original vector algebra, in 1884, containing 30 pages of basic operations on dyadics. Double products of dyadics, which give the notation much of its power, were introduced by him in scientific journals (GIBBS 1886, 1891). Gibbs's work on dyadic algebra was compiled from his lectures by WILSON and printed a book *Vector analysis* containing 150 pages of dyadics (GIBBS and WILSON 1909). Of course, not all the formulas given by Gibbs were invented by Gibbs, quite a number of properties of linear vector functions were introduced earlier by Hamilton in his famous book on quaternions. In electromagnetics literature, dyadics and matrices are often used simultaneously. It is well recognized that the dyadic notation is best matched to the vector notation. Nevertheless, often the vector notation is suddenly changed to matrices, for example when inverse dyadics should be constructed, because the corresponding dyadic operations are unknown. The purpose of this section is to introduce the dyadic formalism, and subsequent chapters demonstrate some of its power. The contents of the present chapter are largely based on work given earlier by this author in report form (LINDELL 1968, 1973a, 1981).

2.1 Notation

2.1.1 Dyads and polyads

The dyadic product of two vectors $\mathbf{a}$, $\mathbf{b}$ (complex in general) is denoted without any multiplication sign by $\mathbf{ab}$ and the result is called a dyad. The order of dyadic multiplication is essential, $\mathbf{ab}$ is in general different from $\mathbf{ba}$.

A polyad is a string of vectors multiplying each other by dyadic products and denoted by $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3...\mathbf{a}_n$. For $n = 1$ we have a vector, $n = 2$ a dyad,

$n = 3$ a triad and, in general, an n-ad. Polyads of the same rank n generate a linear space, whose members are polynomials of n-ads. Thus, all polynomials of dyads, or dyadic polynomials, or in short dyadics, are of the form $\mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + ... + \mathbf{a}_k\mathbf{b}_k$. Similarly, n-adic polynomials form a linear space of n-adics. A sum of two n-adics is an n-adic. Here we concentrate on the case $n = 2$, or dyadics, which are denoted by $\overline{\overline{A}}$, $\overline{\overline{B}}$, $\overline{\overline{C}}$, etc.

Dyadics and other polyadics arise in a natural manner in expressions of vector algebra, where a linear operator is separated from the quantity that is being operated upon. For example, projection of a vector $\mathbf{a}$ onto a line which has the direction of the unit vector $\mathbf{u}$ can be written as $\mathbf{u}(\mathbf{u}\cdot\mathbf{a})$. Here, the vectors $\mathbf{u}$ represent the operation on the vector $\mathbf{a}$. Separating these from each other by moving the brackets of vector notation, gives rise to the dyad $\mathbf{uu}$ in the expression $\mathbf{u}(\mathbf{u}\cdot\mathbf{a}) = (\mathbf{uu})\cdot\mathbf{a}$.

A dyad is bilinear in its vector multiplicants:

$$(\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2)\mathbf{b} = \alpha_1(\mathbf{a}_1\mathbf{b}) + \alpha_2(\mathbf{a}_2\mathbf{b}), \tag{2.1}$$

$$\mathbf{a}(\beta_1\mathbf{b}_1 + \beta_2\mathbf{b}_2) = \beta_1(\mathbf{a}\mathbf{b}_1) + \beta_2(\mathbf{a}\mathbf{b}_2). \tag{2.2}$$

This means that the same dyad or dyadic can be written in infinitely many different polynomial forms, just like a vector can be written as a sum of different vectors. Whether two forms in fact represent the same dyadic (the same element in dyadic space), can be asserted if one of them can be obtained from the other through these bilinear operations.

A dyadic $\overline{\overline{A}}$ can be multiplied by a vector $\mathbf{c}$ in many ways. Taking one dyad $\mathbf{ab}$ of the dyadic, the following multiplications are possible:

$$\mathbf{c}\cdot(\mathbf{ab}) = (\mathbf{c}\cdot\mathbf{a})\mathbf{b}, \tag{2.3}$$

$$\mathbf{c}\times(\mathbf{ab}) = (\mathbf{c}\times\mathbf{a})\mathbf{b}, \tag{2.4}$$

$$(\mathbf{ab})\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}), \tag{2.5}$$

$$(\mathbf{ab})\times\mathbf{c} = \mathbf{a}(\mathbf{b}\times\mathbf{c}). \tag{2.6}$$

In dot multiplication of a dyad by a vector, the result is a vector, in cross multiplication, a dyad.

Likewise, double multiplications of a dyad $\mathbf{ab}$ by another dyad $\mathbf{cd}$ are defined as follows:

$$(\mathbf{ab}):(\mathbf{cd}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}), \tag{2.7}$$

$$(\mathbf{ab})\,{}^{\times}_{\times}\,(\mathbf{cd}) = (\mathbf{a}\times\mathbf{c})(\mathbf{b}\times\mathbf{d}), \tag{2.8}$$

$$(\mathbf{ab})\,{}^{\times}_{\cdot}\,(\mathbf{cd}) = (\mathbf{a}\times\mathbf{c})(\mathbf{b}\cdot\mathbf{d}), \tag{2.9}$$

$$(\mathbf{ab})\,{}^{\cdot}_{\times}\,(\mathbf{cd}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\times\mathbf{d}). \tag{2.10}$$

These expressions can be generalized to corresponding double products between dyadics, $\overline{\overline{A}} : \overline{\overline{B}}$, $\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{B}}$, $\overline{\overline{A}} {}^{\times}_{\cdot} \overline{\overline{B}}$ and $\overline{\overline{A}} {}^{\cdot}_{\times} \overline{\overline{B}}$, when dyads are replaced by dyadic polynomials and multiplication is made term by term. The double dot product produces a scalar, the double cross product, a dyadic, and the mixed products, a vector. These products, especially the double dot and double cross products, give more power to the dyadic notation. Their application requires, however, a knowledge of some identities, which are not in common use in the literature. These identities will be introduced later and they are also listed in Appendix A of this book.

The linear space of dyadics contains all polynomials of dyads as its elements. The representation of a dyadic by a dyadic polynomial is, however, not unique. Two polynomials correspond to the same dyadic if their difference can be reduced to the null dyadic by bilinear operations. Because of Gibbs' identity (1.42), any dyadic can be written as a sum of three dyads. In fact, taking three base vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ with their reciprocal base vectors $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, any dyadic polynomial can be written as

$$\sum_{i=1}^{n} \mathbf{a}_i \mathbf{b}_i = \mathbf{a} \sum_{i=1}^{n} (\mathbf{a}' \cdot \mathbf{a}_i) \mathbf{b}_i + \mathbf{b} \sum_{i=1}^{n} (\mathbf{b}' \cdot \mathbf{a}_i) \mathbf{b}_i + \mathbf{c} \sum_{i=1}^{n} (\mathbf{c}' \cdot \mathbf{a}_i) \mathbf{b}_i. \qquad (2.11)$$

This is of the trinomial form $\mathbf{ae} + \mathbf{bf} + \mathbf{cg}$, which is the most general form of dyadic in the three-dimensional vector space. If we can prove a theorem for the general dyadic trinomial, the theorem is valid for any dyadic. A sum sign $\sum$ without index limit values in this text denotes a sum from 1 to 3.

2.1.2 Symmetric and antisymmetric dyadics

The transpose operation for dyadics changes the order in all dyadic products:

$$\left(\sum \mathbf{a}_i \mathbf{b}_i\right)^T = \sum (\mathbf{a}_i \mathbf{b}_i)^T = \sum \mathbf{b}_i \mathbf{a}_i. \qquad (2.12)$$

Because $(\overline{\overline{A}}^T)^T = \overline{\overline{A}}$, the eigenvalue problem $\overline{\overline{A}}^T = \lambda \overline{\overline{A}}$ has the eigenvalues $\lambda = \pm 1$ corresponding to symmetric $\overline{\overline{A}}_s$ and antisymmetric $\overline{\overline{A}}_a$ dyadics, which satisfy

$$\overline{\overline{A}}_s^T = \overline{\overline{A}}_s, \qquad \overline{\overline{A}}_a{}^T = -\overline{\overline{A}}_a. \qquad (2.13)$$

Any dyadic can be uniquely decomposed into a symmetric and an antisymmetric part:

$$\overline{\overline{A}} = \frac{1}{2}(\overline{\overline{A}} + \overline{\overline{A}}^T) + \frac{1}{2}(\overline{\overline{A}} - \overline{\overline{A}}^T). \qquad (2.14)$$

Every symmetric dyadic can be written as a polynomial of symmetric dyads:

$$\overline{\overline{A}}_s = \sum \mathbf{a}_i \mathbf{b}_i = \frac{1}{2} \sum (\mathbf{a}_i \mathbf{b}_i + \mathbf{b}_i \mathbf{a}_i) =$$

$$\frac{1}{2} \sum ((\mathbf{a}_i + \mathbf{b}_i)(\mathbf{a}_i + \mathbf{b}_i) - \mathbf{a}_i \mathbf{a}_i - \mathbf{b}_i \mathbf{b}_i) . \quad (2.15)$$

The number of terms in this polynomial is, however, in general higher than 3.

The linear space of dyadics is nine dimensional, in which the antisymmetric dyadics form a three-dimensional and the symmetric dyadics a six-dimensional subspace.

2.2 Dyadics as linear mappings

A dyadic serves as a linear mapping from a vector to another: $\mathbf{a} \rightarrow \mathbf{b} = \overline{\overline{D}} \cdot \mathbf{a}$. Conversely, any such linear mapping can be expressed in terms of a dyadic. This can be seen by expanding in terms of orthonormal basis vectors $\mathbf{u}_i$ and applying the property of linearity of the vector function $\mathbf{f}(\mathbf{a})$:

$$\mathbf{f}(\mathbf{a}) = \sum_i \mathbf{u}_i \mathbf{u}_i \cdot \mathbf{f}(\sum_j \mathbf{u}_j \mathbf{u}_j \cdot \mathbf{a}) = \Big(\sum_i \sum_j \mathbf{u}_i \cdot \mathbf{f}(\mathbf{u}_j) \mathbf{u}_i \mathbf{u}_j\Big) \cdot \mathbf{a}. \quad (2.18)$$

The quantity in brackets is of the dyadic form and it corresponds to the linear function $\mathbf{f}(\mathbf{a})$.

The unit dyadic $\overline{\overline{I}}$ corresponds to the identity mapping $\overline{\overline{I}} \cdot \mathbf{a} = \mathbf{a}$ for any vector $\mathbf{a}$. From Gibbs' identity (1.42) we see that for any base of three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ with the reciprocal base $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, the unit dyadic can be written as

$$\overline{\overline{I}} = \mathbf{a}\mathbf{a}' + \mathbf{b}\mathbf{b}' + \mathbf{c}\mathbf{c}'. \tag{2.19}$$

Taking an orthonormal base $\mathbf{u}_i$ with $\mathbf{u}_i' = \mathbf{u}_i$, the unit dyadic takes the form

$$\overline{\overline{I}} = \sum \mathbf{u}_i \mathbf{u}_i. \tag{2.20}$$

The unit dyadic is symmetric and satisfies $\overline{\overline{I}} \cdot \overline{\overline{D}} = \overline{\overline{D}} \cdot \overline{\overline{I}} = \overline{\overline{D}}$ for any dyadic $\overline{\overline{D}}$. This and (2.19) can be applied to demonstrate the relation between matrix and dyadic notations by writing

$$\overline{\overline{D}} = \overline{\overline{I}} \cdot \overline{\overline{D}} \cdot \overline{\overline{I}} = \sum_i \sum_j (\mathbf{a}_i' \cdot \overline{\overline{D}} \cdot \mathbf{a}_j) \mathbf{a}_i \mathbf{a}_j' = \sum_i \sum_j D_{ij} \mathbf{a}_i \mathbf{a}_j', \tag{2.21}$$

or any dyadic can be written in terms of nine scalars D_{ij}. These scalars can be conceived as matrix components of the dyadic with respect to the base $\{\mathbf{a}_i\}$. The matrix components of the unit dyadic $\overline{\overline{I}}$ are $\{\delta_{ij}\}$ in all bases.

All dyadics can be classified in terms of their mapping properties.

- *Complete dyadics* $\overline{\overline{D}}$ define a linear mapping with an inverse, which is represented by an inverse dyadic $\overline{\overline{D}}^{-1}$. Thus, any vector $\mathbf{b}$ can be reached by mapping a suitable vector $\mathbf{a}$ by $\overline{\overline{D}} \cdot \mathbf{a} = \mathbf{b}$.

- *Planar dyadics* map all vectors in a two-dimensional subspace. If we take a base $\{\mathbf{a}_i\}$, the vectors $\{\overline{\overline{D}} \cdot \mathbf{a}_i\}$ do not form a base, because they are linearly dependent and satisfy $(\overline{\overline{D}}\cdot\mathbf{a}_1)\cdot(\overline{\overline{D}}\cdot\mathbf{a}_2)\times(\overline{\overline{D}}\cdot\mathbf{a}_3) = 0$. Writing the general $\overline{\overline{D}}$ in the trinomial form $\mathbf{ab} + \mathbf{cd} + \mathbf{ef}$, we can show that the vector triple $\mathbf{a}$, $\mathbf{c}$, $\mathbf{e}$ is linearly dependent and one of these vectors can be expressed in terms of other two. Thus, the most general *planar dyadic* can be written as a dyadic binomial $\mathbf{ab} + \mathbf{cd}$.

- *Strictly planar dyadics* are planar dyadics which cannot be written as a single dyad.

- *Linear dyadics* map all vectors in a one-dimensional subspace, i.e. parallel to a vector $\mathbf{c}$: $\overline{\overline{D}} \cdot \mathbf{a} = \alpha\mathbf{c}$. Thus, $\overline{\overline{D}}$ must obviously be of the form $\mathbf{cb}$. Linear dyadics can be written as a single dyad. Finally, we can distinguish between *strictly linear* dyadics and the *null dyadic.*

As examples, we note that the unit dyadic $\overline{\overline{I}}$ is complete, whereas an antisymmetric dyadic is either strictly planar or the null dyadic. The inverse dyadic of a given complete dyadic $\overline{\overline{D}} = \sum \mathbf{a}_i\mathbf{b}_i$ can be written quite straightforwardly in trinomial form. First, to be complete, the vector triples $\{\mathbf{a}_i\}$, $\{\mathbf{b}_i\}$ must be bases because from linear dependence of either base, a planar dyadic would result. Hence, there exist reciprocal bases $\{\mathbf{a}'_i\}$, $\{\mathbf{b}'_i\}$, with which we can write

$$\overline{\overline{D}}^{-1} = \sum \mathbf{b}'_i\mathbf{a}'_i. \tag{2.24}$$

That (2.24) satisfies $\overline{\overline{D}} \cdot \overline{\overline{D}}^{-1} = \overline{\overline{D}}^{-1} \cdot \overline{\overline{D}} = \overline{\overline{I}}$, can be easily verified.

2.3 Products of dyadics

Different products of dyadics play a role similar to dot and cross products of vectors, which introduce the operational power to the vector notation. The products of dyadics obey certain rules which are governed by certain identities summarized in Appendix A.

2.3.1 Dot-product algebra

The dot product between two dyadics has already been mentioned above and is defined in an obvious manner:

$$\overline{\overline{A}} \cdot \overline{\overline{C}} = (\sum \mathbf{a}_i\mathbf{b}_i) \cdot (\sum \mathbf{c}_j\mathbf{d}_j) = \sum\sum(\mathbf{b}_i \cdot \mathbf{c}_j)\mathbf{a}_i\mathbf{d}_j. \tag{2.25}$$

With this dot product, the dyadics form an algebra, where the unit dyadic, null dyadic and inverse dyadics are defined as above. This algebra corresponds to the matrix algebra, because the matrix (with respect to a given base) of $\overline{\overline{A}} \cdot \overline{\overline{B}}$ can be shown to equal the matrix product of the matrices of each dyadic. Thus properties known from matrix algebra are valid to dot-product algebra: associativity

$$\overline{\overline{A}} \cdot (\overline{\overline{B}} \cdot \overline{\overline{C}}) = (\overline{\overline{A}} \cdot \overline{\overline{B}}) \cdot \overline{\overline{C}}, \tag{2.26}$$

and (in general) non-commutativity, $\overline{\overline{A}} \cdot \overline{\overline{B}} \neq \overline{\overline{B}} \cdot \overline{\overline{A}}$. Further, we have

$$(\overline{\overline{A}} \cdot \overline{\overline{B}})^T = \overline{\overline{B}}^T \cdot \overline{\overline{A}}^T, \tag{2.27}$$

$$(\overline{\overline{A}} \cdot \overline{\overline{B}})^{-1} = \overline{\overline{B}}^{-1} \cdot \overline{\overline{A}}^{-1}. \tag{2.28}$$

Powers of dyadics, both positive and negative, are defined through the dot product (negative powers only for complete dyadics) in an obvious manner. For example, the antisymmetric dyadic $\overline{\overline{A}} = \mathbf{u} \times \overline{\overline{I}}$ with an NCP unit vector $\mathbf{u}$, satisfies for all $n > 0$.

$$\overline{\overline{A}}^{4n} = \overline{\overline{I}} - \mathbf{uu}, \tag{2.29}$$

$$\overline{\overline{A}}^{4n+1} = \overline{\overline{A}}, \tag{2.30}$$

$$\overline{\overline{A}}^{4n+2} = -\overline{\overline{I}} + \mathbf{uu}, \tag{2.31}$$

$$\overline{\overline{A}}^{4n+3} = -\overline{\overline{A}}. \tag{2.32}$$

Because for real $\mathbf{u}$, $\overline{\overline{A}}$ can be interpreted as a rotation by $\pi/2$ around $\mathbf{u}$, the powers of $\overline{\overline{A}}$ can be easily understood as multiples of that rotation.

Two dyadics do not commute in general in the dot product. It is easy to see that two antisymmetric dyadics only commute when one can be written as a multiple of the other. This is evident if we expand the dot product of two general antisymmetric dyadics:

$$(\mathbf{a} \times \overline{\overline{I}}) \cdot (\mathbf{b} \times \overline{\overline{I}}) = \mathbf{ba} - (\mathbf{a} \cdot \mathbf{b})\overline{\overline{I}}. \tag{2.33}$$

If this is required to be symmetric in $\mathbf{a}$ and $\mathbf{b}$, we should have $\mathbf{ab} = \mathbf{ba}$ or $(\mathbf{a} \times \mathbf{b}) \times \overline{\overline{I}} = 0$, which implies $\mathbf{a} \times \mathbf{b} = 0$ or $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors.

A dyadic commutes with an antisymmetric dyadic only if its symmetric and antisymmetric parts commute separately. In fact, writing $\overline{\overline{D}} = \overline{\overline{D}}_s + \mathbf{d} \times \overline{\overline{I}}$ in terms of its symmetric and antisymmetric parts, we can write

$$\overline{\overline{D}} \cdot (\mathbf{a} \times \overline{\overline{I}}) - (\mathbf{a} \times \overline{\overline{I}}) \cdot \overline{\overline{D}} = (\overline{\overline{D}}_s \times \mathbf{a}) + (\overline{\overline{D}}_s \times \mathbf{a})^T - (\mathbf{a} \times \mathbf{d}) \times \overline{\overline{I}}. \tag{2.34}$$

Equating the antisymmetric and symmetric parts of (2.34) to zero, shows us that the symmetric and antisymmetric parts of $\overline{\overline{D}}$ must commute with $\mathbf{a}\times\overline{\overline{I}}$ separately. Thus, the antisymmetric part of $\overline{\overline{D}}$ must be a multiple of $\mathbf{a}\times\overline{\overline{I}}$. The symmetric part of $\overline{\overline{D}}$ must be such that $\overline{\overline{D}}_s\times\mathbf{a}$ is antisymmetric, i.e. of the form $\mathbf{b}\times\overline{\overline{I}}$. Multiplying this by $\cdot\mathbf{a}$ gives zero, whence $\mathbf{b}$ must be a multiple of $\mathbf{a}$. It is easy to show that the symmetric dyadic must be of the form $\alpha\overline{\overline{I}}+\beta\mathbf{aa}$. Thus, the most general dyadic, which commutes with the antisymmetric dyadic $\mathbf{a}\times\overline{\overline{I}}$ is necessarily of the form

$$\overline{\overline{D}}=\alpha\overline{\overline{I}}+\beta\mathbf{aa}+\gamma\mathbf{a}\times\overline{\overline{I}}. \tag{2.35}$$

A dyadic of this special form is called *gyrotropic with axis* $\mathbf{a}$, which may also be a complex vector. From this, it is easy to show that if a dyadic commutes with its transpose, it must be either symmetric or gyrotropic.

2.3.2 Double-dot product

The double-dot product of two dyadics $\overline{\overline{A}}=\sum\mathbf{a}_i\mathbf{b}_i$, $\overline{\overline{B}}=\sum\mathbf{c}_j\mathbf{d}_j$ gives the scalar

$$\overline{\overline{A}}:\overline{\overline{B}}=\sum\sum(\mathbf{a}_i\cdot\mathbf{c}_j)(\mathbf{b}_i\cdot\mathbf{d}_j). \tag{2.36}$$

This is symmetric in both dyadics and satisfies

$$\overline{\overline{A}}^T:\overline{\overline{B}}^T=\overline{\overline{A}}:\overline{\overline{B}}, \tag{2.37}$$

$$(\mathbf{a}\times\overline{\overline{I}}):(\mathbf{b}\times\overline{\overline{I}})=2\mathbf{a}\cdot\mathbf{b}, \tag{2.38}$$

$$\overline{\overline{A}}:\overline{\overline{I}}=\sum(\mathbf{a}_i\cdot\mathbf{b}_i)=\mathrm{tr}\overline{\overline{A}}. \tag{2.39}$$

The last operation gives a scalar which can be called the *trace* of $\overline{\overline{A}}$ because it gives the trace of the matrix of $\overline{\overline{A}}$ in any base $\{\mathbf{c}_i\}$. In fact, writing $\overline{\overline{A}}=\sum A_{ij}\mathbf{c}_i\mathbf{c}'_j$ gives us

$$\overline{\overline{A}}:\overline{\overline{I}}=(\sum\sum A_{ij}\mathbf{c}_i\mathbf{c}'_j):(\sum\mathbf{c}'_k\mathbf{c}_k)=\sum A_{kk}. \tag{2.40}$$

As special cases we have $\overline{\overline{I}}:\overline{\overline{I}}=3$ and $\overline{\overline{A}}:\overline{\overline{B}}=(\overline{\overline{A}}\cdot\overline{\overline{B}}^T):\overline{\overline{I}}=(\overline{\overline{B}}\cdot\overline{\overline{A}}^T):\overline{\overline{I}}$. A dyadic whose trace is zero is called trace free. Any dyadic can be written as a sum of a multiple of $\overline{\overline{I}}$ and a trace-free dyadic:

$$\overline{\overline{D}}=\frac{1}{3}(\overline{\overline{D}}:\overline{\overline{I}})\overline{\overline{I}}+(\overline{\overline{D}}-\frac{1}{3}(\overline{\overline{D}}:\overline{\overline{I}})\overline{\overline{I}}). \tag{2.41}$$

Antisymmetric dyadics are trace free. In fact, more generally, if $\overline{\overline{S}}$ is symmetric and $\overline{\overline{A}}$ antisymmetric, we have from (2.37)

$$\overline{\overline{A}} : \overline{\overline{S}} = \overline{\overline{A}}^T : \overline{\overline{S}}^T = -\overline{\overline{A}} : \overline{\overline{S}} = 0. \tag{2.42}$$

The scalar $\overline{\overline{D}} : \overline{\overline{D}}^*$ is a non-negative real number for any complex dyadic $\overline{\overline{D}}$. It is zero only for $\overline{\overline{D}} = 0$, which can be shown from the following:

$$\overline{\overline{D}} : \overline{\overline{D}}^* = (\overline{\overline{D}}^T \cdot \overline{\overline{D}}^*) : \overline{\overline{I}} = \sum \mathbf{u}_i \cdot \overline{\overline{D}}^T \cdot \overline{\overline{D}}^* \cdot \mathbf{u}_i = \sum |\overline{\overline{D}} \cdot \mathbf{u}_i|^2. \tag{2.43}$$

Here, $\{\mathbf{u}_i\}$ is a real orthonormal base. (2.43) is seen to give a non-negative number and vanish only if all $\overline{\overline{D}} \cdot \mathbf{u}_i$ vanish, whence $\overline{\overline{D}} = \sum \overline{\overline{D}} \cdot \mathbf{u}_i \mathbf{u}_i = 0$. We can define the norm of $\overline{\overline{D}}$ as

$$||\overline{\overline{D}}|| = \sqrt{\overline{\overline{D}} : \overline{\overline{D}}^*}. \tag{2.44}$$

2.3.3 Double-cross product

The double-cross product of two dyadics produces a third dyadic. Thus, it defines a double-cross algebra. Unlike the dot-product algebra, the double-cross algebra is commutative:

$$\overline{\overline{A}} \overset{\times}{\times} \overline{\overline{B}} = \overline{\overline{B}} \overset{\times}{\times} \overline{\overline{A}}, \tag{2.45}$$

and non-associative, because $\overline{\overline{A}} \overset{\times}{\times} (\overline{\overline{B}} \overset{\times}{\times} \overline{\overline{C}}) \neq (\overline{\overline{A}} \overset{\times}{\times} \overline{\overline{B}}) \overset{\times}{\times} \overline{\overline{C}}$ in general. The commutative property follows directly from the anticommutativity of the cross product: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, as is easy to see. It is also easy to show that there does not exist a unit element in this algebra.

A most useful formula for the expansion of dyadic expressions can be obtained from the following evaluation with dyads:

$$(\mathbf{ab}) \overset{\times}{\times} [(\mathbf{cd}) \overset{\times}{\times} (\mathbf{ef})] = [\mathbf{a} \times (\mathbf{c} \times \mathbf{e})][\mathbf{b} \times (\mathbf{d} \times \mathbf{f})] =$$

$$[\mathbf{ca} \cdot \mathbf{e} - \mathbf{a} \cdot \mathbf{ce}][\mathbf{db} \cdot \mathbf{f} - \mathbf{b} \cdot \mathbf{df}] =$$

$$(\mathbf{ab} : \mathbf{cd})\mathbf{ef} + (\mathbf{ab} : \mathbf{ef})\mathbf{cd} - \mathbf{ef} \cdot (\mathbf{ab})^T \cdot \mathbf{cd} - \mathbf{cd} \cdot (\mathbf{ab})^T \cdot \mathbf{ef}. \tag{2.46}$$

This expression is a trilinear identity for dyads. Thus, every dyad can be replaced by any dyadic polynomial because of linearity, whence (2.46) may be written for general dyadics:

$$\overline{\overline{A}} \overset{\times}{\times} (\overline{\overline{B}} \overset{\times}{\times} \overline{\overline{C}}) = (\overline{\overline{A}} : \overline{\overline{B}})\overline{\overline{C}} + (\overline{\overline{A}} : \overline{\overline{C}})\overline{\overline{B}} - \overline{\overline{B}} \cdot \overline{\overline{A}}^T \cdot \overline{\overline{C}} - \overline{\overline{C}} \cdot \overline{\overline{A}}^T \cdot \overline{\overline{B}}. \tag{2.47}$$

Use of this dyadic identity adds more power to the dyadic calculus. Its memorizing is aided by the fact that, because of the commutative property (2.45), $\overline{\overline{B}}$ and $\overline{\overline{C}}$ are symmetrical in (2.47).

The method used above to obtain a dyadic identity from vector identities can be generalized by the following procedure.

1. A multilinear dyadic expression (linear in every dyadic) is written in terms of dyads, i.e. every dyadic is replaced by a dyad.

2. Vector identities are applied to change the expression into another form.

3. The result is grouped in such a way that the original dyads are formed.

4. The dyads are replaced by the original dyadics.

To demonstrate this procedure, let us expand the dyadic expression $(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}}):\overline{\overline{I}}$, which is linear in $\overline{\overline{A}}$ and $\overline{\overline{B}}$. Hence, we start by replacing them by $\mathbf{ab}$ and $\mathbf{cd}$, respectively, and applying the well-known vector identity $(\mathbf{ab}{}^{\times}_{\times}\mathbf{cd}):\overline{\overline{I}} = (\mathbf{a}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$. This can be grouped as $(\mathbf{ab}:\overline{\overline{I}})(\mathbf{cd}:\overline{\overline{I}}) - (\mathbf{ab}):(\mathbf{cd})^T$. Finally, going back to $\overline{\overline{A}}$ and $\overline{\overline{B}}$ leaves us with the dyadic identity

$$(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}}):\overline{\overline{I}} = (\overline{\overline{A}}:\overline{\overline{I}})(\overline{\overline{B}}:\overline{\overline{I}}) - \overline{\overline{A}}:\overline{\overline{B}}^T, \tag{2.48}$$

or trace of $\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}}$ equals $\mathrm{tr}\overline{\overline{A}}\ \mathrm{tr}\overline{\overline{B}} - \mathrm{tr}(\overline{\overline{A}}\cdot\overline{\overline{B}})$.

New dyadic identities can also be obtained from old identities. As an example, let us write (2.48) in the form

$$[\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{I}} - (\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{I}} + \overline{\overline{A}}^T]:\overline{\overline{B}} = 0. \tag{2.49}$$

To obtain this, we have applied the invariance in any permutation of the triple scalar product of dyadics, $\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}}:\overline{\overline{C}} = \overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{C}}:\overline{\overline{B}} = \overline{\overline{B}}{}^{\times}_{\times}\overline{\overline{A}}:\overline{\overline{C}} = \cdots$. (Of course, the double-cross product must always be performed first.) Because (2.49) is valid for any dyadic $\overline{\overline{B}}$, the bracketed dyadic must be the null dyadic, and the following identity is obtained:

$$\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{I}} = (\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{I}} - \overline{\overline{A}}^T. \tag{2.50}$$

That $\overline{\overline{D}}:\overline{\overline{B}} = 0$ for all $\overline{\overline{B}}$ implies $\overline{\overline{D}} = 0$, is easily seen by taking $\overline{\overline{B}} = \mathbf{u}_i\mathbf{u}_j$ from an orthonormal base $\{\mathbf{u}_i\}$, whence all matrix coefficients D_{ij} of $\overline{\overline{D}}$ can be shown to vanish. The identity (2.48) is obtained from (2.50) as a special case by operating it by $:\overline{\overline{B}}$.

An important identity for the double-cross product can be obtained by expanding the expression $(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}}){}^{\times}_{\times}(\overline{\overline{C}}{}^{\times}_{\times}\overline{\overline{D}})$ twice through (2.47) by considering one of the bracketed dyadics as a single dyadic, and equating the expressions. Setting $\overline{\overline{C}} = \overline{\overline{D}} = \overline{\overline{I}}$ we obtain

$$\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}} =$$

$$[(\overline{\overline{A}}:\overline{\overline{I}})(\overline{\overline{B}}:\overline{\overline{I}})-\overline{\overline{A}}:\overline{\overline{B}}^T]\overline{\overline{I}}-(\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{B}}^T-(\overline{\overline{B}}:\overline{\overline{I}})\overline{\overline{A}}^T+(\overline{\overline{A}}\cdot\overline{\overline{B}}+\overline{\overline{B}}\cdot\overline{\overline{A}})^T. \quad (2.51)$$

This identity could be also conceived as the definition of the double-cross product in terms of single-dot and double-dot products and the transpose operation. It is easily seen that (2.50) is a special case of (2.51). Also, $\overline{\overline{I}}{}^{\times}_{\times}\overline{\overline{I}} = 2\overline{\overline{I}}$ is obtained as a further special case. The operation $\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{I}}$ is in fact a mapping from dyadic to dyadic. Its properties can be examined through the following dyadic eigenproblem:

$$\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{I}} = \lambda\overline{\overline{A}}. \quad (2.52)$$

Taking the trace of (2.52) leaves us with $(2-\lambda)\overline{\overline{A}}:\overline{\overline{I}} = 0$, whence either $\lambda = 2$ or $\overline{\overline{A}}$ is trace free. Substituting (2.50) in (2.52) gives us the following different solutions:

- $\lambda = 2$ and $\overline{\overline{A}} = \alpha\overline{\overline{I}}$ where α is any scalar;
- $\lambda = 1$ and $\overline{\overline{A}}$ is antisymmetric;
- $\lambda = -1$ and $\overline{\overline{A}}$ is symmetric and trace free.

Any dyadic can be written uniquely as a sum of three components: a multiple of $\overline{\overline{I}}$, an antisymmetric dyadic and a trace-free symmetric dyadic,

$$\overline{\overline{A}} = \frac{1}{3}(\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{I}} + \frac{1}{2}(\overline{\overline{A}} - \overline{\overline{A}}^T) + [\frac{1}{2}(\overline{\overline{A}} + \overline{\overline{A}}^T) - \frac{1}{3}(\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{I}}], \quad (2.53)$$

respectively. It is a simple matter to check that the right-hand side of (2.53), each term multiplied by the corresponding eigenvalue λ, gives the same result as (2.50). There exists an inverse mapping to $\overline{\overline{I}}{}^{\times}_{\times}$ which, denoted by $\overline{\overline{L}}(\overline{\overline{A}})$ and satisfying $\overline{\overline{L}}(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{I}}) = \overline{\overline{A}}$ for every $\overline{\overline{A}}$, can be written in terms of inverse eigenvalues, or in the simple form

$$\overline{\overline{L}}(\overline{\overline{A}}) = \frac{1}{2}(\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{I}} - \overline{\overline{A}}^T. \quad (2.54)$$

2.6 The eigenvalue problem

Right and left eigenvalue problems with eigenvalues and eigenvectors α_i, $\mathbf{a}_i$ and, respectively, β_i, $\mathbf{b}_i$, are of the form

$$\overline{\overline{A}} \cdot \mathbf{a}_i = \alpha_i \mathbf{a}_i, \tag{2.105}$$

$$\mathbf{b}_i \cdot \overline{\overline{A}} = \beta_i \mathbf{b}_i. \tag{2.106}$$

Because the dyadic $\overline{\overline{A}} - \gamma \overline{\overline{I}}$ is planar when γ equals α_i or β_i, the eigenvalues satisfy the equation

$$-\det(\overline{\overline{A}} - \gamma \overline{\overline{I}}) = \gamma^3 - \gamma^2 \mathrm{tr}\overline{\overline{A}} + \gamma\ \mathrm{spm}\overline{\overline{A}} - \det\overline{\overline{A}} = 0. \tag{2.107}$$

Because both right and left eigenvalues satisfy the same problem, they have the same values which are denoted by α_i. There are either one, two or three different values for α_i. Because $\mathbf{b}_i \cdot \overline{\overline{A}} \cdot \mathbf{a}_j = (\alpha_i \mathbf{b}_i) \cdot \mathbf{a}_j = \mathbf{b}_i \cdot (\alpha_j \mathbf{a}_j)$, we see that if $\alpha_i \neq \alpha_j$, the left and right eigenvectors are orthogonal, i.e. they satisfy $\mathbf{b}_i \cdot \mathbf{a}_j = 0$.

Eigenvectors $\mathbf{b}_i$ and $\mathbf{a}_i$ corresponding to a solution α_i of (2.107) can be constructed using dyadic methods. The construction depends on the multitude of the particular eigenvalue, which depends on whether the dyadic $\overline{\overline{A}} - \alpha_i \overline{\overline{I}}$ is strictly planar, strictly linear or null. Let us consider these cases separately. For this we need the following identities:

$$\mathbf{a} \times (\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{B}}) = \overline{\overline{B}} \times (\mathbf{a} \cdot \overline{\overline{A}}) + \overline{\overline{A}} \times (\mathbf{a} \cdot \overline{\overline{B}}), \tag{2.108}$$

$$(\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{B}}) \times \mathbf{a} = (\overline{\overline{A}} \cdot \mathbf{a}) \times \overline{\overline{B}} + (\overline{\overline{B}} \cdot \mathbf{a}) \times \overline{\overline{A}}, \tag{2.109}$$

with the special cases

$$\mathbf{a} \times (\overline{\overline{A}} {}^\times_\times \overline{\overline{A}}) = 2\overline{\overline{A}} \times (\mathbf{a} \cdot \overline{\overline{A}}), \tag{2.110}$$

$$(\overline{\overline{A}} {}^\times_\times \overline{\overline{A}}) \times \mathbf{a} = 2(\overline{\overline{A}} \cdot \mathbf{a}) \times \overline{\overline{A}}. \tag{2.111}$$

These can be derived with the general method described in Section 2.3, for creating dyadic identities.

- *Strictly planar* $\overline{\overline{A}} - \alpha_i\overline{\overline{I}}$. Defining $\overline{\overline{B}}_i = (\overline{\overline{A}} - \alpha_i\overline{\overline{I}}) {}^\times_\times (\overline{\overline{A}} - \alpha_i\overline{\overline{I}}) \neq 0$, from (2.110), (2.111) we see that $\mathbf{b}_i \times \overline{\overline{B}}_i = \overline{\overline{B}}_i \times \mathbf{a}_i = 0$, whence there exists a scalar $\xi_i \neq 0$ such that $\overline{\overline{B}}_i = \xi_i \mathbf{b}_i \mathbf{a}_i$. Thus, from the knowledge of α_i the dyadic $\overline{\overline{B}}_i$ is known and the eigenvectors can be written in the form $\mathbf{a}_i = \mathbf{c} \cdot \overline{\overline{B}}_i$ and $\mathbf{b}_i = \overline{\overline{B}}_i \cdot \mathbf{c}$ with suitable vector $\mathbf{c}$. In this case α_i is a single root of (2.107).

- *Strictly linear* $\overline{\overline{A}} - \alpha_i\overline{\overline{I}}$. There exist non-null vectors $\mathbf{c}$, $\mathbf{d}$ such that $\overline{\overline{A}}$ is of the form $\alpha\overline{\overline{I}} + \mathbf{cd}$. This is a special type of dyadic, which is called *uniaxial*. (2.107) leaves us with the equation $(\gamma - \alpha)^2(\gamma - \alpha - \mathbf{c}\cdot\mathbf{d}) = 0$, which shows us that the eigenvalue $\alpha_i = \alpha$ is a double root of (2.107). Assuming $\mathbf{c} \cdot \mathbf{d} \neq 0$, the third root is a simple one, $\alpha_j = \alpha + \mathbf{c} \cdot \mathbf{d}$, for which the eigenvectors can be obtained through the expression above. The left and right eigenvectors corresponding to the double root are any vectors satisfying the conditions $\mathbf{b} \cdot \mathbf{c} = 0$ and $\mathbf{d} \cdot \mathbf{a} = 0$, respectively. For $\mathbf{c} \cdot \mathbf{d} = 0$, all three eigenvalues are the same, but there only exist two linearly independent eigenvectors, those just mentioned.

- *Null dyadic* $\overline{\overline{A}} - \alpha_i\overline{\overline{I}}$. In this case $\overline{\overline{A}} = \alpha\overline{\overline{I}}$, there is a triple eigenvalue α and any vector is an eigenvector. This happens if $\overline{\overline{A}}$ is a multiple of the unit dyadic $\overline{\overline{I}}$.

The previous classification was made in terms of the dyadic $\overline{\overline{A}} - \alpha_i\overline{\overline{I}}$, or multitude of a particular eigenvalue α_i. Let us now consider the number of different eigenvalues of a dyadic $\overline{\overline{A}}$, which can be 1, 2 or 3. Because $\mathrm{det}\overline{\overline{A}} = \alpha_1\alpha_2\alpha_3$, $\mathrm{spm}\overline{\overline{A}} = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$ and $\mathrm{tr}\overline{\overline{A}} = \alpha_1 + \alpha_2 + \alpha_3$, the Cayley–Hamilton equation (2.65) can be written as

$$(\overline{\overline{A}} - \alpha_1\overline{\overline{I}}) \cdot (\overline{\overline{A}} - \alpha_2\overline{\overline{I}}) \cdot (\overline{\overline{A}} - \alpha_3\overline{\overline{I}}) = 0. \tag{2.112}$$

The order of terms is immaterial here.

- *One eigenvalue* $\alpha_1 = \alpha_2 = \alpha_3$. Because $(\overline{\overline{A}} - \alpha_1\overline{\overline{I}})^3 = 0$, from the solution of equation (2.100) we conclude that $\overline{\overline{A}} - \alpha_1\overline{\overline{I}}$ is a trace-free shearer, whence the most general form for $\overline{\overline{A}}$ is $\alpha_1\overline{\overline{I}} + (\mathbf{ab} + \mathbf{cc}) \times \mathbf{a}$. Taking the trace operation it is seen that $\alpha_1 = \mathrm{tr}\overline{\overline{A}}/3$. The number of eigenvectors is obviously that of the shearer term. Here we can separate the cases.

  - Strictly planar trace-free shearer with $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} \neq 0$. It is easy to show that there exists only one eigenvector (left and right), which can be obtained from the expression $(\overline{\overline{A}} - \alpha_1\overline{\overline{I}})^{\times}_{\times}(\overline{\overline{A}} - \alpha_1\overline{\overline{I}}) = 2(\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})(\mathbf{a} \times \mathbf{c})\mathbf{a}$. Thus, the left eigenvector is $\mathbf{a} \times \mathbf{c}$ and the right eigenvector $\mathbf{a}$. Only this type of dyadic has just one eigenvector.

  - Linear shearer, which can be written with $\mathbf{c} = 0$ in the above expression. Now $\overline{\overline{A}} - \alpha_1\overline{\overline{I}}$ satisfies (2.99). There exist two eigenvectors, which are orthogonal to vectors $\mathbf{a}$ from the left and $\mathbf{b} \times \mathbf{a}$ from the right.

- *Two eigenvalues* $\alpha_1 \neq \alpha_2 = \alpha_3$. The dyadic $\overline{\overline{A}}$ satisfies an equation of the form (2.103): $\overline{\overline{B}}^2(\overline{\overline{B}} - \mathrm{tr}\overline{\overline{B}}\ \overline{\overline{I}}) = 0$ with $\overline{\overline{B}} = \overline{\overline{A}} - \alpha_2\overline{\overline{I}}$, as is easily verified. Thus, the dyadic $\overline{\overline{B}}$ must be a general shearer and $\overline{\overline{A}}$ is of the form $\alpha_2\overline{\overline{I}} + \mathbf{a} \times (\mathbf{bc} + \mathbf{da})$. The eigenvalue α_1 equals $\alpha_2 + \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$, whence the shearer here cannot be trace free in order that the two eigenvalues do not coincide. In this case there exist two linearly independent eigenvectors.

- *Three eigenvalues* $\alpha_1 \neq \alpha_2 \neq \alpha_3$. In this case there exist three linearly independent eigenvectors. In fact, assuming the eigenvector $\mathbf{a}_3$ to be a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, which are linearly independent, (2.105) will lead to the contradictory conditions $\alpha_3 = \alpha_1$, $\alpha_3 = \alpha_2$.

As a summary, the following table presents the different cases of dyadics with different numbers of eigenvalues (N in horizontal lines) and eigenvectors (M in vertical columns).

$M \mid N$	1	2	3
1	$\alpha\overline{\overline{I}} + (\mathbf{ab} + \mathbf{cc}) \times \mathbf{a}$	none	none
2	$\alpha\overline{\overline{I}} + \mathbf{aa} \times \mathbf{b}$	$\alpha\overline{\overline{I}} + (\mathbf{ab} + \mathbf{cd}) \times \mathbf{a}$	none
3	$\alpha\overline{\overline{I}}$	$\alpha\overline{\overline{I}} + \mathbf{ab}$	$\mathbf{ab} + \mathbf{cd} + \mathbf{ef}$

Because the left or right eigenvectors of a dyadic with three linearly independent eigenvectors form two bases $\{\mathbf{b}_i\}$, $\{\mathbf{a}_i\}$, the dyadic can be written in either base as

$$\overline{\overline{A}} = \overline{\overline{A}} \cdot \sum \mathbf{a}_i \mathbf{a}'_i = \sum \alpha_i \mathbf{a}_i \mathbf{a}'_i = \sum \alpha_i \mathbf{b}'_i \mathbf{b}_i. \tag{2.113}$$

From this we conclude that the left and right eigenvectors are in fact reciprocals of each other: $\mathbf{a}'_i = \mathbf{b}_i$, $\mathbf{b}'_i = \mathbf{a}_i$. If two dyadics commute, they have the same eigenvectors.

2.7 Hermitian and positive definite dyadics

Hermitian and positive definite dyadics are often encountered in electromagnetics. In fact, lossless medium parameter dyadics are hermitian or antihermitian depending on the definition. Also, from power considerations in a medium, positive definiteness of dyadics often follows.

2.7.1 Hermitian dyadics

By definition, the hermitian dyadic satisfies

$$\overline{\overline{A}}^T = \overline{\overline{A}}^*, \tag{2.114}$$

whereas the antihermitian dyadic is defined by

$$\overline{\overline{A}}^T = -\overline{\overline{A}}^*. \tag{2.115}$$

Any dyadic can be written as a sum of a hermitian and an antihermitian dyadic in the form

$$\overline{\overline{A}} = \frac{1}{2}(\overline{\overline{A}} + \overline{\overline{A}}^{T*}) + \frac{1}{2}(\overline{\overline{A}} - \overline{\overline{A}}^{T*}). \tag{2.116}$$

Any hermitian dyadic can be written in the form $\sum \pm \mathbf{c}\mathbf{c}^*$ and antihermitian, in the form $\sum \pm j\mathbf{c}\mathbf{c}^*$. Conversely, these kinds of dyadics are always hermitian and antihermitian, respectively.

From (2.114), (2.115) it follows that the symmetric part of a hermitian dyadic is real and the antisymmetric part imaginary, whence the most general hermitian dyadic $\overline{\overline{H}}$ can be written in the form

$$\overline{\overline{H}} = \overline{\overline{S}} + j\mathbf{h} \times \overline{\overline{I}}, \tag{2.117}$$

with real and symmetric $\overline{\overline{S}}$ and real $\mathbf{h}$. Any antihermitian dyadic can be written as $j\overline{\overline{H}}$, where $\overline{\overline{H}}$ is a hermitian dyadic.

Hermitian dyadics form a subspace in the linear space of dyadics. The dot product of two hermitian dyadics is not necessarily hermitian, but the double-cross product is, as is seen from

$$(\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{B}})^T = \overline{\overline{A}}^T {}^{\times}_{\times} \overline{\overline{B}}^T = \overline{\overline{A}}^* {}^{\times}_{\times} \overline{\overline{B}}^* = (\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{B}})^*, \tag{2.118}$$

where $\overline{\overline{A}}$ and $\overline{\overline{B}}$ are hermitian. Also, the double-dot product of two hermitian dyadics is a real number, as is easy to see from the sum expression. Thus, the scalars $\mathrm{tr}\overline{\overline{A}}$, $\mathrm{spm}\overline{\overline{A}}$, $\det\overline{\overline{A}}$ are real if $\overline{\overline{A}}$ is hermitian. This implies that the inverse of a hermitian dyadic is hermitian.

The following theorem is very useful when deriving identities for hermitian dyadics:

$$\overline{\overline{A}} : \mathbf{a}\mathbf{a}^* = 0 \quad \text{for all } \mathbf{a}, \ \Rightarrow \ \overline{\overline{A}} = 0. \tag{2.119}$$

This can be proved by setting first $\mathbf{a} = \mathbf{b} + \mathbf{c}$ and then $\mathbf{a} = \mathbf{b} + j\mathbf{c}$, whence the condition $\overline{\overline{A}} : \mathbf{b}\mathbf{c} = 0$ for all vectors $\mathbf{b}$, $\mathbf{c}$ will result from (2.119), making the matrix components of $\overline{\overline{A}}$ vanish. For comparison, $\overline{\overline{A}} : \mathbf{a}\mathbf{a} = 0$ for all $\mathbf{a}$ implies $\overline{\overline{A}}$ antisymmetric, as is easy to prove. From (2.119) we can show that if $\overline{\overline{A}} : \mathbf{a}\mathbf{a}^*$ is real for all vectors $\mathbf{a}$, $\overline{\overline{A}}$ is hermitian. In fact, this implies $\overline{\overline{A}} : \mathbf{a}\mathbf{a}^* - \overline{\overline{A}}^* : \mathbf{a}^*\mathbf{a} = 0$ or $(\overline{\overline{A}} - \overline{\overline{A}}^{*T}) : \mathbf{a}\mathbf{a}^* = 0$, whence $\overline{\overline{A}}$ is hermitian.

A hermitian dyadic always has three eigenvectors no matter how many eigenvalues it has, as can be shown. The right and left eigenvectors corresponding to the same eigenvalues are complex conjugates of each other, because from $\overline{\overline{A}} \cdot \mathbf{a} = \alpha\mathbf{a}$ we have $\overline{\overline{A}}^* \cdot \mathbf{a}^* = \mathbf{a}^* \cdot \overline{\overline{A}} = \alpha^*\mathbf{a}^*$. But eigenvalues are real and eigenvectors conjugate orthogonal, because $(\alpha_i - \alpha_j^*)\mathbf{a}_i \cdot \mathbf{a}_j^* = 0$, whence $\alpha_i - \alpha_i^* = 0$ and $\mathbf{a}_i \cdot \mathbf{a}_j^* = 0$ for $\alpha_i \neq \alpha_j$. Thus, the general hermitian dyadic can be written in terms of its eigenvalues and eigenvectors as

$$\overline{\overline{A}} = \sum \alpha_i \frac{\mathbf{a}_i\mathbf{a}_i^*}{\mathbf{a}_i \cdot \mathbf{a}_i^*}. \tag{2.120}$$

2.7.2 Positive definite dyadics

By definition, a dyadic $\overline{\overline{D}}$ is positive definite (PD), if it satisfies

$$\overline{\overline{D}} : \mathbf{a}\mathbf{a}^* > 0, \quad \text{for all } \ \mathbf{a} \neq 0. \tag{2.121}$$

A PD dyadic is always hermitian, as is evident. Other properties, whose proofs are partly omitted, follow.

- PD dyadics are complete. If $\overline{\overline{D}}$ were planar, there would exist a vector a such that $\overline{\overline{D}}\cdot\mathbf{a} = 0$, in contradiction with (2.121). Thus, the inverse of a PD dyadic always exists.

- PD dyadics possess positive eigenvalues. This is seen by dot multiplying the expansion (2.120) by $\mathbf{a}_j^*\mathbf{a}_j/\mathbf{a}_j\cdot\mathbf{a}_j^*$, and the result is α_j, which must be greater than 0 because of (2.121).

- if $\overline{\overline{A}}$ is PD, its symmetric part is PD.

- $\overline{\overline{A}}$ is PD exactly when its invariants $\mathrm{tr}\overline{\overline{A}}$, $\mathrm{spm}\overline{\overline{A}}$, $\det\overline{\overline{A}}$ are real and positive.

- $\overline{\overline{A}}$ and $\overline{\overline{B}}$ PD implies $\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}}$ PD.

- A dyadic of the form $\overline{\overline{A}}\cdot\overline{\overline{A}}^{*T}$ is positive semidefinite and PD if $\overline{\overline{A}}$ is complete.

2.8 Special dyadics

In this section we consider some special classes of dyadics appearing in practical electromagnetic problems. Rotation and reflection dyadics emerge in symmetries of various structures whereas uniaxial and gyrotropic dyadics are encountered when electromagnetic fields in special materials are analysed. Parameters of some media like the sea ice can be approximated in terms of a uniaxial dyadic, while others like magnetized ferrite or magnetoplasma may exhibit properties which can be analysed using gyrotropic dyadics.

2.8.1 Rotation dyadics

In real vector space, the rotation of a vector by an angle θ in the right-hand direction around the axis defined by the unit vector $\mathbf{u}$ can be written in terms of the following dyadic:

$$\overline{\overline{R}}(\mathbf{u},\theta) = \mathbf{u}\mathbf{u} + \sin\theta(\mathbf{u}\times\overline{\overline{I}}) + \cos\theta(\overline{\overline{I}} - \mathbf{u}\mathbf{u}), \qquad (2.122)$$

It can also be written in the form $e^{\mathbf{u}\times\overline{\overline{I}}\theta}$, as is seen if the dyadic exponential function is written as a Taylor series. The rotation dyadic obeys the properties

$$\overline{\overline{R}}(-\mathbf{u},\theta) = \overline{\overline{R}}(\mathbf{u},-\theta) = \overline{\overline{R}}^T(\mathbf{u},\theta) = \overline{\overline{R}}^{-1}(\mathbf{u},\theta), \qquad (2.123)$$

$$\overline{\overline{R}}(\mathbf{u},\theta_1)\cdot\overline{\overline{R}}(\mathbf{u},\theta_2) = \overline{\overline{R}}(\mathbf{u},\theta_1+\theta_2), \tag{2.124}$$

$$\overline{\overline{R}}(\mathbf{u},\theta)^{\times}_{\times}\overline{\overline{R}}(\mathbf{u},\theta) = 2\overline{\overline{R}}(\mathbf{u},\theta), \tag{2.125}$$

$$\det\overline{\overline{R}}(\mathbf{u},\theta) = \frac{1}{3}\overline{\overline{R}}(\mathbf{u},\theta) : \overline{\overline{R}}(\mathbf{u},\theta) = 1. \tag{2.126}$$

It is not difficult to show that the properties $\det\overline{\overline{R}} = 1$ and $\overline{\overline{R}}^T = \overline{\overline{R}}^{-1}$ uniquely define the form (2.122) of the dyadic $\overline{\overline{R}}(\mathbf{u},\theta)$, so that they could be given as the definition. Also, (2.125) with $\overline{\overline{R}} \neq 0$ would do. In general, θ and/or $\mathbf{u}$ may be complex, which means that the geometrical interpretation is lost. The resulting dyadic is also called the rotation dyadic in the complex case.

The dot product of two rotation dyadics with arbitrary axes and angles is another rotation dyadic. This is seen from

$$(\overline{\overline{R}}_1\cdot\overline{\overline{R}}_2)^{\times}_{\times}(\overline{\overline{R}}_1\cdot\overline{\overline{R}}_2) = \frac{1}{2}(\overline{\overline{R}}_1{}^{\times}_{\times}\overline{\overline{R}}_1)\cdot(\overline{\overline{R}}_2{}^{\times}_{\times}\overline{\overline{R}}_2) = 2(\overline{\overline{R}}_1\cdot\overline{\overline{R}}_2), \tag{2.127}$$

where use has been made of the identity (2.67). It is not very easy to find the axis and angle of the resulting rotation dyadic. This can be done perhaps most easily by using a representation in terms of a special gyrotropic dyadic. The gyrotropic dyadic was defined in (2.35) and it is the most general non-symmetric dyadic which commutes with its own transpose. The special gyrotropic dyadic of interest here is of the type

$$\overline{\overline{G}}(\mathbf{q}) = \overline{\overline{I}} + \mathbf{q}\times\overline{\overline{I}}, \tag{2.128}$$

and the rotation dyadic can be written as

$$\overline{\overline{R}} = \overline{\overline{G}}\cdot(\overline{\overline{G}}^T)^{-1} = (\overline{\overline{I}} + \mathbf{q}\times\overline{\overline{I}})\cdot(\overline{\overline{I}} - \mathbf{q}\times\overline{\overline{I}})^{-1}. \tag{2.129}$$

In fact, because

$$(\overline{\overline{I}} - \mathbf{q}\times\overline{\overline{I}})^{-1} = \frac{1}{1+q^2}(\overline{\overline{I}} + \mathbf{q}\mathbf{q} + \mathbf{q}\times\overline{\overline{I}}), \tag{2.130}$$

(2.129) can be seen to be of the form (2.122) if we write $\mathbf{q} = \mathbf{u}\tan(\theta/2)$, or $\mathbf{u} = \mathbf{q}/\sqrt{\mathbf{q}\cdot\mathbf{q}}$ and $\theta = 2\tan^{-1}\sqrt{\mathbf{q}\cdot\mathbf{q}}$. Thus, $\mathbf{q}$ is not a CP vector. For real $\mathbf{q}$, its length q determines the angle of rotation. $q = 0$ corresponds to $\theta = 0$, $q = \infty$ to $\theta = \pi$ and $q = 1$ to $\theta = \pi/2$.

It is straightforward, although a bit tedious, to prove the following identity between the dot product of two rotation dyadics $\overline{\overline{R}}_1$, $\overline{\overline{R}}_2$ and the corresponding $\mathbf{q}$ vectors:

$$\mathbf{q}(\overline{\overline{R}}_1\cdot\overline{\overline{R}}_2) = \frac{\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_2\times\mathbf{q}_1}{1 - \mathbf{q}_1\cdot\mathbf{q}_2}. \tag{2.131}$$

This equation shows us that two rotations do not commute in general, since $\overline{\overline{R}}_1 \cdot \overline{\overline{R}}_2$ and $\overline{\overline{R}}_2 \cdot \overline{\overline{R}}_1$ lead to the same $\mathbf{q}$ vector only if $\mathbf{q}_1$ and $\mathbf{q}_2$ are parallel vectors, i.e. the two rotation dyadics have the same axes.

2.8.2 Reflection dyadics

The symmetric dyadic of the form

$$\overline{\overline{C}}(\mathbf{u}) = \overline{\overline{I}} - 2\mathbf{u}\mathbf{u} \tag{2.132}$$

with a unit vector $\mathbf{u}$ is called the reflection dyadic because, when $\mathbf{u}$ is real, mapping the position vector $\mathbf{r}$ through $\overline{\overline{C}} \cdot \mathbf{r} = \mathbf{r} - 2\mathbf{u}(\mathbf{u} \cdot \mathbf{r})$ obviously performs a reflection in the plane $\mathbf{u} \cdot \mathbf{r} = 0$. The reflection dyadic can be also presented as a negative rotation by an angle π around $\mathbf{u}$ as the axis

$$\overline{\overline{C}}(\mathbf{u}) = -\overline{\overline{R}}(\mathbf{u}, \pi), \tag{2.133}$$

as is seen from the definition of the rotation dyadic (2.122). In fact, the unit dyadic $\overline{\overline{I}}$ and the negative of the reflection dyadic are the only rotation dyadics that are symmetric.

The reflection dyadic satisfies

$$\overline{\overline{C}}^2(\mathbf{u}) = \overline{\overline{I}}, \quad \text{or} \quad \overline{\overline{C}}^{-1} = \overline{\overline{C}}, \tag{2.134}$$

$$\overline{\overline{C}}(\mathbf{u}) {}^\times_\times \overline{\overline{C}}(\mathbf{u}) = -2\overline{\overline{C}}(\mathbf{u}), \tag{2.135}$$

$$\mathrm{tr}\overline{\overline{C}} = 1, \quad \mathrm{spm}\overline{\overline{C}} = -1, \quad \det\overline{\overline{C}} = -1, \tag{2.136}$$

The most general square root of the unit dyadic is not the reflection dyadic, but a dyadic of the form $\pm(\overline{\overline{I}} - 2\mathbf{ab})$ with either $\mathbf{a} \cdot \mathbf{b} = 1$ or $\mathbf{ab} = 0$.

It is easy to see that both rotation and reflection dyadics preserve the inner product of two vectors. In fact, because they both satisfy $\overline{\overline{A}}^T \cdot \overline{\overline{A}} = \overline{\overline{I}}$, we have

$$(\overline{\overline{A}} \cdot \mathbf{a}) \cdot (\overline{\overline{A}} \cdot \mathbf{b}) = \mathbf{a} \cdot (\overline{\overline{A}}^T \cdot \overline{\overline{A}}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}, \tag{2.137}$$

for any vectors $\mathbf{a}$, $\mathbf{b}$. The cross product is transformed differently through rotation

$$(\overline{\overline{R}} \cdot \mathbf{a}) \times (\overline{\overline{R}} \cdot \mathbf{b}) = \frac{1}{2}(\overline{\overline{R}} {}^\times_\times \overline{\overline{R}}) \cdot (\mathbf{a} \times \mathbf{b}) = \overline{\overline{R}} \cdot (\mathbf{a} \times \mathbf{b}), \tag{2.138}$$

than through reflection

$$(\overline{\overline{C}} \cdot \mathbf{a}) \times (\overline{\overline{C}} \cdot \mathbf{b}) = -\overline{\overline{C}} \cdot (\mathbf{a} \times \mathbf{b}). \tag{2.139}$$

This equation shows us that a reflection transformed electromagnetic field is not an electromagnetic field, because if the electric and magnetic fields are transformed through reflection, the Poynting vector is not. The converse is, however, true for the rotation transformation.