Unified Structure of Basic UWB Waveforms

Mohammad Ghavami, *Senior Member, IEEE*, Arash Amini, and Farrokh Marvasti, *Senior Member, IEEE*

***Abstract*—In this brief, a generalized expression for the popular ultra wideband waveforms is derived. It is shown that all three waveforms used in ultra wideband (Gaussian, modified Hermite, and prolate spheroidal waveforms) fulfill the Sturm–Liouville differential equation. By using this unified structure, characteristics of the waveforms such as orthogonality, finite duration in time and frequency spectrum are explained.**

***Index Terms*—Gaussian pulses, Hermite polynomials, prolate spheroidal waveforms, ultra-wideband (UWB) signals.**

I. Introduction

ONE OF THE essential functions in communication systems is the representation of a digital symbol by an analog waveform for transmission through a channel. In ultra-wideband (UWB) systems, the conventional analog waveform is a simple pulse that is directly radiated to the air. These waveforms have typical widths of less than 1 ns and thus a bandwidth of over 1 GHz. In this brief, we will examine how to generate waveforms applicable to UWB systems for simple cases of Gaussian, modified Hermite, and prolate spheroidal waveforms. We will show a unified structure for the three mentioned waveforms which is useful in the sense that it helps finding new waveforms that may outperform the previous ones. In fact, it will be demonstrated that the mentioned waveforms belong to the family of solutions of the Sturm–Liouville boundary condition problem. We will also show how the common requirements of UWB waveforms are related to this family. Therefore, by focusing on the subgroup of this family which meet the requirements, new waveforms with more constraints such as in their frequency spectrum (for example, forcing a notch in the frequency domain) can be obtained.

This brief is organized as follows. In Section II, basic UWB waveforms are defined. Section III investigates the unified structure of these wave shapes. Orthogonality, finite duration in time and frequency domain characteristics are considered in Sections IV and V, respectively. Finally, Section VI concludes the paper.

II. Basic UWB Waveforms

There have been several pulse shapes proposed for UWB systems in the literature [1], [2]. The most common waveforms are derived from the Gaussian, Hermitian, and prolate functions.

Manuscript received September 09, 2007; revised January 02, 2008 and July 17, 2008. Current version published December 12, 2008. This paper was recommended by Associate Editor G. M. Maggio.

M. Ghavami is with the Department of Electronic Engineering, King's College London, Strand, London WC2R 2LS, U.K. (e-mail: mohammad.ghavami@kcl.ac.uk).

A. Amini and F. Marvasti are with the Department of Electrical Engineering, Advanced Communication Research Institute (ACRI), Sharif University of Technology, Tehran, Iran (e-mail: arashsil@ee.sharif.edu; marvasti@sharif.edu).

Digital Object Identifier 10.1109/TCSII.2008.2008068

A. Gaussian Waveforms

A class of waveforms are called Gaussian waveforms because their mathematical definition is similar to the Gauss function [1]. The zero-mean Gauss function is described by

$$G(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \tag{1}$$

where σ is the standard deviation. The basis of Gaussian waveforms is a Gaussian pulse represented by the following equation:

$$g_1(t) = Ke^{-(t/\tau)^2} \tag{2}$$

where $-\infty < t < \infty$, τ is the time-scaling factor, and K is a constant. More waveforms can be created by high-pass filtering of this Gaussian pulse; filtering acts in a manner similar to taking the derivative of (2). For example, a Gaussian monocycle, which is the first derivative of a Gaussian pulse, has the form

$$g_2(t) = \dot{g}_1(t) = K\frac{-2t}{\tau^2} e^{-(t/\tau)^2}. \tag{3}$$

By comparing (2) and (3), it can be shown that

$$\dot{g}_1(t) = Atg_1(t) \tag{4}$$

where $A = -2/\tau^2$. A Gaussian monocycle has a single zero crossing. Further derivatives yield additional zero crossings, one additional zero crossing for each additional derivative. If the value of τ is fixed, by taking an additional derivative, the fractional bandwidth decreases, while the center frequency increases.

A Gaussian doublet which is a common and practical UWB pulse shape [3], [4], is the second derivative of (2) and is defined by

$$g_3(t) = \ddot{g}_1(t) = At\dot{g}_1(t) + Ag_1(t). \tag{5}$$

Differentiation of (5) yields

$$\dot{g}_3(t) = \frac{d\ddot{g}_1(t)}{dt} = A\dot{g}_1(t) + At\ddot{g}_1(t) + A\dot{g}_1(t) = At\ddot{g}_1(t) + 2A\dot{g}_1(t). \tag{6}$$

Since $g_2(t) = \dot{g}_1(t)$, we can write

$$\ddot{g}_2(t) = At\dot{g}_2(t) + 2Ag_2(t). \tag{7}$$

Similarly, we can generalize that

$$\ddot{g}_n(t) - At\dot{g}_n(t) - Ang_n(t) = 0 \tag{8}$$

where $n = 1, 2, 3, \ldots$. The spectrums of the first three Gaussian pulses are shown in Fig. 1. A simple diagram showing the generator circuit of Gaussian pulses is given in Fig. 2.

B. Orthogonal Modified Hermite Waveforms

The functions defined by

$$h_{e_n}(t) = (-\tau)^n e^{t^2/2\tau^2} \frac{d^n}{dt^n}\left(e^{-t^2/2\tau^2}\right) \tag{9}$$

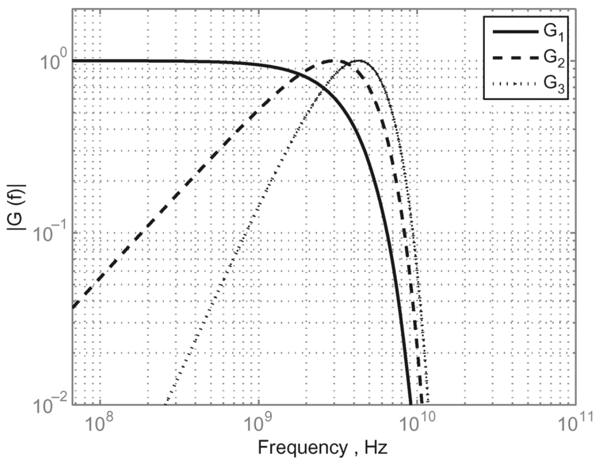

Fig. 1. Spectrum of the first three Gaussian pulse shapes. The effective time duration of the pulses are scaled to 0.5 ns.

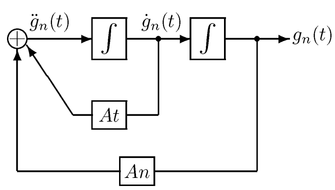

Fig. 2. Analog LTI circuit producing different Gaussian functions.

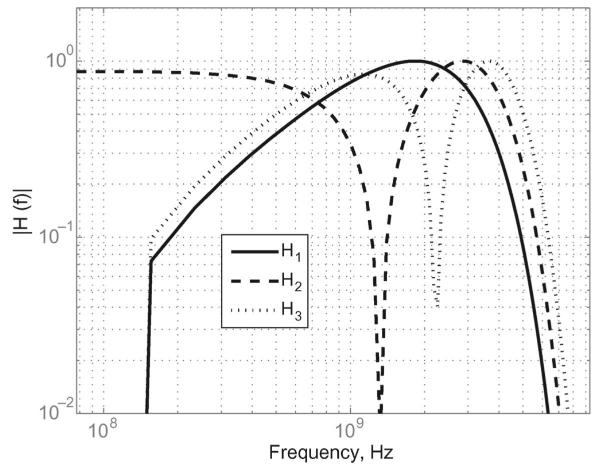

Fig. 3. Spectrum of the first three Hermite pulse shapes. The effective time duration of the pulses are scaled to 0.5 ns.

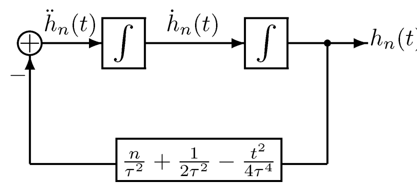

Fig. 4. Analog linear time-variant circuit producing different MHP functions.

where $n = 0, 1, 2, \ldots$ and $-\infty < t < \infty$, are called Hermite polynomials. The parameter τ is the time-scaling factor. It should be mentioned that the definition of (9) is one of many forms of Hermite polynomials used in the literature. An advantage of Hermite pulses is that, with their linear combinations, efficient pulse spectrums could be achieved [5].

Hermite and Gaussian polynomials are not orthogonal; however, Hermite polynomials can be modified to become orthogonal as follows [6]:

$$h_n(t) = e^{-t^2/4\tau^2} h_{e_n}(t) = (-\tau)^n e^{t^2/4\tau^2} \frac{d^n}{dt^n}\left(e^{-t^2/2\tau^2}\right) \quad (10)$$

where $n = 0, 1, 2, \ldots$ and $-\infty < t < \infty$. The result is a set of orthogonal functions $h_n(t)$ which can be easily derived for all values of n. As examples, we can write

$$\begin{aligned} h_0(t) &= e^{-t^2/4\tau^2} \\ h_1(t) &= \frac{t}{\tau} e^{-t^2/4\tau^2} \\ h_2(t) &= \left(\frac{t^2}{\tau^2} - 1\right) e^{-t^2/4\tau^2}. \end{aligned} \quad (11)$$

We call the functions derived in (10) modified Hermite pulses (MHP), and it can be shown that their general formula is the following [6]:

$$h_n(t) = e^{-t^2/4\tau^2} n! \sum_{i=0}^{[n/2]} \left(-\frac{1}{2}\right)^i \frac{(t/\tau)^{n-2i}}{(n-2i)!i!} \quad (12)$$

where $[n/2]$ denotes the integer part of $n/2$. The spectrums of the first three MHPs are shown in Fig. 3.

It can be shown that all MHPs satisfy the following differential equations [6]:

$$\ddot{h}_n(t) + \left(\frac{n}{\tau^2} + \frac{1}{2\tau^2} - \frac{t^2}{4\tau^4}\right) h_n(t) = 0. \quad (13)$$

Using (13), we can easily draw the feedback diagram for generation of MHPs as in Fig. 4.

C. *Orthogonal Prolate Spheroidal Wave Functions*

One of the waveforms that are practically time and band-limited is referred to as a prolate spheroidal wave function (PSWF) and is famous for its robustness against jitter [7]. This function is the solution of [8]

$$\int_{-T/2}^{T/2} p_n(x) \frac{\sin \Omega(t-x)}{\pi(t-x)} dx = \tilde{\lambda}_n p_n(t) \quad (14)$$

or, alternatively, the solution of the differential equation

$$\frac{d}{dt}(1-t^2)\frac{dp_n(t)}{dt} + (\chi_n - c^2t^2)p_n(t) = 0 \quad (15)$$

where $p_n(t)$ is the prolate spheroidal wave function of order n and χ_n is the eigenvalue of $p_n(t)$. The constant c is

$$c = \frac{\Omega T}{2} \quad (16)$$

where Ω is the bandwidth and T is the pulse duration.

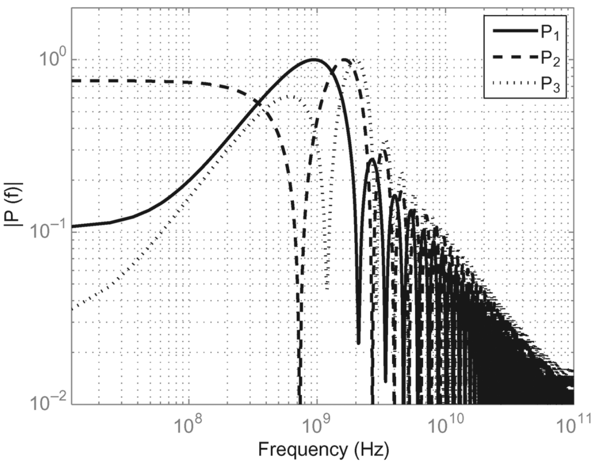

Fig. 5. Spectrum of three orthogonal prolate spheroidal pulse shapes. The effective time duration of the pulses are scaled to 0.5 ns.

In (14), $\tilde{\lambda}$ is the concentration of energy in the interval $[-T/2, T/2]$ given by

$$\tilde{\lambda} = \frac{\int_{-T/2}^{T/2} |p_n(t)|^2 \, dt}{\int_{-\infty}^{\infty} |p_n(t)|^2 \, dt} \tag{17}$$

whose values range from 0 to 1. If we solve the differential equation (15) for the highest derivative, we get [8]:

$$(1-t^2)\frac{d^2 p_n(t)}{dt^2} - 2t\frac{dp_n(t)}{dt} + (\chi_n - c^2 t^2) p_n(t) = 0 \tag{18}$$

and consequently

$$\ddot{p}_n(t) - \frac{2t}{1-t^2}\dot{p}_n(t) + \frac{\chi_n - c^2 t^2}{1-t^2} p_n(t) = 0. \tag{19}$$

As can be seen, different orders of the pulses can be simply obtained by changing the values of χ_n; hence, (19) is the basis of a multi-pulse generator. The spectrum of a few prolate functions are shown in Fig. 5. A simple diagram showing the generator PSWF pulses is given in Fig. 6.

III. Unified Structure of Basic UWB Waveforms

By comparing the expressions for the above three basic pulses, it can be observed that they satisfy a unified structure

$$\frac{d}{dt}\left[r(t)\dot{y}(t)\right] + \left[q(t) + \lambda s(t)\right] y(t) = 0 \tag{20}$$

or

$$\ddot{y}(t) + \frac{\dot{r}(t)}{r(t)}\dot{y}(t) + \frac{q(t) + \lambda s(t)}{r(t)} y(t) = 0. \tag{21}$$

This is a well-known differential equation called Sturm–Liouville boundary value problem where $r(t) > 0$. Since the above differential equation is of the second order, it should be accompanied with two boundary conditions. These conditions are usually chosen as

$$\begin{cases} a_1 y(a) + a_2 \dot{y}(a) = 0 \\ b_1 y(b) + b_2 \dot{y}(b) = 0. \end{cases} \tag{22}$$

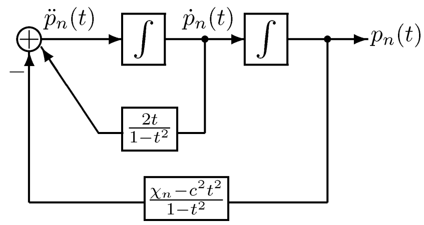

Fig. 6. Analog linear time-variant circuit producing different PSWF wave shapes.

One or both of the endpoints a, b can be $\pm\infty$ as well. The values of λ in (21) which lead to a nontrivial solution (the solution should fulfill the boundary conditions) are called eigenvalues, and the respective nontrivial solutions are called eigenfunctions of the equation. The following theorem, which is referred to as the main principle of the Sturm–Liouville theory, briefly describes the proper characteristics of the eigenvalues and eigenfunctions [9].

Theorem 1: Assuming the following properties in the differential equation of (21).

1) All three functions $r(t)$, $q(t)$, and $s(t)$ are continuous on the closed interval $[a, b]$; in addition, $r(t)$ is differentiable and $\dot{r}(t)$ is continuous on the same interval.
2) Both functions $r(t)$ and $s(t)$ are positive on $[a, b]$.

We have the following relations.

1) Equation (21) with conditions (22) has countable infinite number of eigen-values which are real and monotonically increasing

$$\lambda_0 < \lambda_1 < \lambda_2 < \ldots \in \mathbb{R}. \tag{23}$$

2) For the eigenvalue λ_n, except for a constant coefficient, there exists a unique eigen-function $(y_n(t))$ and it has exactly n zeros in the interval (a, b). Moreover, eigenfunctions are mutually orthogonal with respect to the weight function $s(t)$

$$m \neq n : \int_a^b y_n(t) y_m(t) s(t) dt = 0. \tag{24}$$

Now it is demonstrated that different UWB pulse shapes can be derived from (21) in a straightforward way.

A. Generation of Gaussian Waveforms

By comparing (21) and (8), we can write

$$\frac{\dot{r}(t)}{r(t)} = -\,At \tag{25}$$

$$\frac{q(t) + \lambda_n s(t)}{r(t)} = -\,An. \tag{26}$$

Equation (25) has a solution in the form of

$$r(t) = k_1 e^{-\frac{At^2}{2}} \tag{27}$$

where k_1 is a constant. As a solution for (26), we consider $q(t) = 0$, $\lambda_n = n$ and

$$s(t) = -Ar(t) = -k_1 A e^{-\frac{At^2}{2}}. \tag{28}$$

TABLE I
SUMMARY OF THE PARAMETERS FOR GENERATION OF BASIC UWB WAVEFORMS

Pulse shape	$r(t)$	$q(t)$	λ_n	$s(t)$
Gaussian	$k_1 e^{-\frac{At^2}{2}}$	0	n	$-k_1 A e^{-\frac{At^2}{2}}$
MHP	k_2	$\frac{k_2}{2\tau^2} - \frac{k_2 t^2}{4\tau^4}$	n	$\frac{k_2}{\tau^2}$
PSWF	$k_3(1-t^2)$	$-k_3 c^2 t^2$	χ_n	1

B. Generation of Orthogonal Modified Hermite Waveforms

By comparing (21) and (13), we can write

$$\frac{\dot{r}(t)}{r(t)} = 0 \quad (29)$$

$$\frac{q(t)+\lambda_n s(t)}{r(t)} = \frac{n}{\tau^2} + \frac{1}{2\tau^2} - \frac{t^2}{4\tau^4}. \quad (30)$$

From (29) it is clear that $r(t) = k_2$, where k_2 is a constant. Substituting into (30), we get

$$q(t) + \lambda_n s(t) = \frac{k_2 n}{\tau^2} + \frac{k_2}{2\tau^2} - \frac{k_2 t^2}{4\tau^4}. \quad (31)$$

Consequently, $\lambda_n = n$,

$$s(t) = \frac{k_2}{\tau^2} \quad (32)$$

$$q(t) = \frac{k_2}{2\tau^2} - \frac{k_2 t^2}{4\tau^4}. \quad (33)$$

C. Generation of Orthogonal Prolate Spheroidal Wave Functions

By comparing (21) and (19), we can write

$$\frac{\dot{r}(t)}{r(t)} = -\frac{2t}{1-t^2} \quad (34)$$

$$\frac{q(t)+\lambda_n s(t)}{r(t)} = \frac{\chi_n - c^2 t^2}{1-t^2.} \quad (35)$$

Equation (34) can be solved to derive

$$r(t) = k_3(1-t^2) \quad (36)$$

where k_3 is a constant. Using (36) in (35), we derive

$$q(t) + \lambda_n s(t) = k_3 \chi_n - k_3 c^2 t^2. \quad (37)$$

Hence, $\lambda_n = \chi_n$, $s(t) = 1$, and $q(t) = -k_3 c^2 t^2$. To prevent confusion, it should be mentioned that although λ_n and $\tilde{\lambda}_n$ are similar in notation, they are different in concept and value; however, the value of any of them could be generated by knowing the other one.

The result of this section is summarized in Table I. In the next sections, we explain the similar properties of the pulses using the above unified structure.

IV. ORTHOGONALITY

The eigenfunctions of a Sturm–Liouville equation are orthogonal when a weight function is considered [part 2) in Theorem 1]; However, in the engineering literature, the orthogonality that prevents inter symbol interference (ISI) between different users is defined as

$$m \neq n : \int_a^b y_n(t) y_m(t) dt = 0. \quad (38)$$

Recalling that $s(t)$ is positive, the nonorthogonal eigenfunctions can be made orthogonal using the following change:

$$z_n(t) = \sqrt{s(t)} y_n(t). \quad (39)$$

Now. it is easy to verify that

$$m \neq n : \int_a^b z_n(t) z_m(t) dt = 0. \quad (40)$$

The only thing left to prove is that the produced set $\{z_n(t)\}$, is itself the set of eigenfunctions of a Sturm–Liouville equation

$$\begin{cases} y_n(t) = s(t)^{-0.5} z_n(t) \\ \dot{y}_n(t) = s(t)^{-0.5} \dot{z}_n(t) - \frac{1}{2}\dot{s}(t) s(t)^{-1.5} z_n(t) \\ \ddot{y}_n(t) = s(t)^{-0.5} \ddot{z}_n(t) - \dot{s}(t) s(t)^{-1.5} \dot{z}_n(t) \\ \quad + \frac{1}{2} s(t)^{-2.5} \left(\frac{3}{2}\dot{s}(t)^2 - s(t)\ddot{s}(t)\right) z_n(t) \end{cases}. \quad (41)$$

Combining the above results with (21), we get

$$p(t)\ddot{z}_n(t) + \dot{p}(t)\dot{z}_n(t) + \left(k(t) + \frac{q(t)}{s(t)} + \lambda\right) z_n(t) = 0 \quad (42)$$

where

$$\begin{cases} p(t) = \frac{r(t)}{s(t)} \\ k(t) = \frac{d}{dt}\left(r(t)\frac{d}{dt}\left(s(t)^{-0.5}\right)\right) \cdot s(t)^{-0.5} \end{cases}. \quad (43)$$

Therefore, the new function set $\{z_n(t)\}$ is the set of eigenfunctions of a Sturm–Liouville equation with $s(t) = 1$; consequently, these functions are mutually orthogonal.

It can be shown [10] that the Hermite polynomials introduced in (9), for $r = 1$, satisfy the following differential equation:

$$\ddot{h}_{e_n}(t) - t\dot{h}_{e_n}(t) + n h_{e_n}(t) = 0. \quad (44)$$

The equivalent Sturm-Liuoville form is obtained by

$$\begin{cases} r(t) = e^{-\frac{t^2}{2}} \\ q(t) = 0 \\ s(t) = e^{-\frac{t^2}{2}} \end{cases} \quad (45)$$

which suggests the orthogonality with respect to a Gaussian weight function. Now it becomes evident that the multiplier $\sqrt{s(t)} = e^{-(t^2/4)}$ in (10) is mainly introduced to provide the orthogonality of the modified Hermite waveforms. Similarly, we can introduce modified Gaussian waveforms (modified to be orthogonal). Comparing (3) and (4), we see $A = -(2/\tau^2)$. By recalling Table I, the weight function for Gaussian waveforms is

$$s(t) = -k_1 A e^{-A\frac{t^2}{2}} = \frac{2k_1}{\tau^2} e^{\frac{t^2}{\tau^2}}. \quad (46)$$

Thus the modified Gaussian waveforms are

$$g_n^{(\mathrm{mod})}(t) = e^{\frac{t^2}{2\tau^2}} g_n(t) = e^{\frac{t^2}{2\tau^2}} \frac{d^n}{dt^n}\left(e^{-\frac{t^2}{\tau^2}}\right) \quad (47)$$

which are exactly the same as modified Hermite functions if we neglect the constant coefficient $(-\tau)^n$ (amplitude scaling) in (10) in addition to the replacement of τ with $\sqrt{2}\tau$ (time scaling). For the case of prolate spherical wave functions, since $s(t) = 1$, no modification is required.

V. Practical Issues

Two significant requirements of the pulses which can be considered even more important than orthogonality are duration in time and frequency shape. For proper transmission of the waveforms (and also for prevention of inter symbol interference), the pulse should be limited in time or at least should have the capability to be appropriately approximated by a finite length function. In our case, since the pulses are eigen-functions of a differential equation, theoretically they will not vanish. To overcome this undesired effect, one way is to use finite end points (a and b in Theorem 1). In this case, transmitting pulses are formed by considering the eigen-functions between the end points; i.e., if we represent the end points by a, b and the set of eigenfunctions by $\{f_n(t)\}$, then the pulses are

$$p_n(t) = \begin{cases} f_n(t), & a < t < b \\ 0, & \text{otherwise} \end{cases}. \quad (48)$$

Orthogonality is obviously preserved in this way. Prolate spheroidal wave functions are examples of this case; though the eigenfunctions are not finite in time, they are only included on $[-(T/2), (T/2)]$ for transmission. The other option is to use fast decaying pulses. Even though these functions theoretically spread over time, their effective part is limited. Although the original Hermite functions (9) are not showing such a property, however, the modifying part decays exponentially. Therefore, the resultant functions have exponential decay rate which makes them good approximates of a time-limited pulse. In fact, the modifying part is useful for both orthogonality and finite duration.

The other issue of concern is the frequency shape of the pulse. Usually the generated basic waveforms are changed to fulfill some desired frequency constraints using shapers such as filters or modulators [11], [12]. The primary step for designing the shapers is to find the Fourier transform of the basic pulses. Since the modified Gaussian waveforms are the same as modified Hermites, we only investigate the spectrum of modified Hermite and prolate spheroidal waveforms.

We denote the Fourier transform of $y(t)$ by $Y(\omega)$. It is well known that the transforms of $\dot{y}(t)$ and $ty(t)$ are $j\omega Y(\omega)$ and $j\dot{Y}(\omega)$, respectively. Using these simple equalities, if we take the Fourier transform of both sides of (13), after simplification, we have

$$\ddot{H}_n(\omega) + \left(4n + 2 - (2\tau\omega)^2\right) H_n(\omega) = 0. \quad (49)$$

Hence, $\{H_n(\omega)\}$ are real valued functions which form a Sturm–Liouville eigenfunction set. Similar results are obtained by considering (15) to yield

$$(c^2 - \omega^2)\ddot{P}_n(\omega) - 2\omega\dot{P}_n(\omega) + (\chi_n - \omega^2)P_n(\omega) = 0. \quad (50)$$

Both of the above differential equations are automatically written in the Sturm-Liouvile form. With respect to the weight functions in above equations, we see that these pulses, even in the Fourier domain, are orthogonal

$$m \neq n: \quad \int_{-\infty}^{\infty} H_n(\omega)H_m(\omega)d\omega = 0$$
$$\int_{-\infty}^{\infty} P_n(\omega)P_m(\omega)d\omega = 0. \quad (51)$$

This result could also be confirmed using Parseval's theorem.

VI. Conclusion

In this paper, three common waveforms for ultra wideband transmission are shown to have the same structure. It is shown that Gaussian, modified Hermite, and prolate spheroidal waveforms are eigenfunctions of different Sturm–Liouville boundary condition problems. Using the orthogonality property of eigenfunctions, modification of Hermite polynomials and orthogonality of the prolate spheroidal waveforms are demonstrated. It is also shown that modified Gaussians are the same as modified Hermites. Modifying the pulse, in addition to validating the orthogonality, limits the spread of the pulse over time. The Fourier transforms of the modified Hermite and prolate spheroidal waveforms are also shown to form another set of Sturm–Liouville eigenfunctions.

References

[1] M. Ghavami, L. B. Michael, and R. Kohno, *Ultra Wideband Signals and Systems in Communication Engineering*, 2nd ed. Hoboken, NJ: Wiley, 2007.

[2] W. T. Ang, C. Jie, and L. Tiejun, "High-order monocycle design and its waveform-generating circuit for UWB communications," *IEEE Trans. Circuits Systems I: Reg. Papers*, vol. 54, no. 8, pp. 1657–1665, Aug. 2007.

[3] Q. Wang and J. Yao, "UWB doublet generation using nonlinearly-biased electro-optic intensity modulator," *Electron. Lett.*, vol. 42, no. 22, pp. 1304–1305, Oct. 2006.

[4] F. Zeng and J. Yao, "An approach to ultrawideband pulse generation and distribution over optical fiber," *IEEE Photon. Technol. Lett.*, vol. 18, no. 7, pp. 823–825, Apr. 2006.

[5] J. A. Silva and M. L. Campos, "Spectrally efficient UWB pulse shaping with application in orthogonal PSM," *IEEE Trans. Commun.*, vol. 55, no. 2, pp. 313–322, Feb. 2007.

[6] M. Ghavami, L. B. Michael, S. Haruyama, and R. Kohno, "A novel UWB pulse shape modulation system," *Wireless Pers. Commun. J.*, vol. 23, pp. 105–120, Oct. 2002.

[7] R. S. Dilmaghani, M. Ghavami, B. Allen, and H. Aghvami, "Novel UWB pulse shaping using prolate spheroidal wave functions," in *Proc. PIMRC2003*, Beijing, China, 2003, vol. 1, pp. 602–606.

[8] D. Slepian, "Prolate spheroidal wave functions, Fourier analysis and uncertainty V: The discrete case," *Bell Syst. Tech. J.*, vol. 57, no. 5, pp. 1371–1430, 1978.

[9] A. Zettl, *Sturm–Liouville Theory*. New York: Amer. Math. Soc., 2005.

[10] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*. New York: Dover, 1972, pp. 771–802.

[11] X. Luo, L. Yang, and G. B. Giannakis, "Designing optimal pulse-shapers for ultra-wideband radios," *J. Commun. Netw.*, vol. 5, no. 4, pp. 344–353, Dec. 2003.

[12] P. Rulikowski and J. Barrett, "Adaptive arbitrary pulse shaper," *IEEE Microw. Wireless Compon. Lett.*, vol. 18, no. 5, pp. 356–358, May 2008.