Unified Structure of Basic UWB Waveforms

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Abstract—In this brief, a generalized expression for the popular ultra wideband waveforms is derived. We show that all three pulse shapes used in ultra wideband (Gaussian, Modified Hermite and Prolate Spheroidal waveforms) fulfill the Sturm-Liouville differential equation. By using this unified structure, characteristics of the pulses such as orthogonality, finite duration in time and frequency spectrum are explained.

Index Terms—ultra wideband signals, Gaussian pulses, Hermite polynomials, prolate spheroidal waveforms.

I. INTRODUCTION

ONE of the essential functions in communication systems is the representation of a digital symbol by an analog waveform for transmission through a channel. In Ultra WideBand (UWB) systems, the conventional analog waveform is a simple pulse that is directly radiated to the air. These short pulses have typical widths of less than 1 ns and thus a bandwidth of over 1 GHz. In this brief, we will examine how to generate pulse waveforms for UWB systems for simple cases of Gaussian, modified Hermite and prolate spheroidal wave shapes. We will show a unified structure for the three mentioned pulses which is useful for two reasons: 1) All three shapes could be generated using almost the same hardware; i.e., it gives instructions how to design an UWB pulse generator which supports all the basic shapes. 2) Using this unified approach, new pulse shapes could be found which may outperform the previous ones.

The paper is organized as follows. In Section II, basic UWB waveforms are defined. Section III investigates the unified structure of these wave shapes. Orthogonality, finite duration in time and frequency domain characteristics are considered in sections IV, V and VI, respectively. Finally, section VII concludes the paper.

II. BASIC UWB WAVEFORMS

The most common waveforms proposed for UWB communications are derived from the Gaussian, Hermitean and prolate functions.

A. Gaussian Waveforms

A class of waveforms are called Gaussian waveforms because their mathematical definition is similar to the Gauss function [1]. The zero-mean Gauss function is described by

\[ G(x) = \frac{1}{\sqrt{2\pi} \sigma^2} e^{-x^2/2\sigma^2} \]  

where \( \sigma \) is standard deviation. The basis of Gaussian waveforms is a Gaussian pulse represented by the following equation

\[ g_1(t) = Ke^{-\left(t/\tau\right)^2} \]  

where \(-\infty < t < \infty\), \( \tau \) is the time-scaling factor, and \( K \) is a constant. More waveforms can be created by high-pass filtering of this Gaussian pulse; filtering acts in a manner similar to taking the derivative of (2). For example, a Gaussian monocycle, the first derivative of a Gaussian pulse, has the form

\[ g_2(t) = \dot{g}_1(t) = K \frac{-2t}{\tau^2} e^{-\left(t/\tau\right)^2} \]  

By comparing (2) and (3) it can be shown that

\[ \dot{g}_1(t) = A \tau g_1(t) \]  

where \( A = -2/\tau^2 \), A Gaussian monocycle has a single zero crossing. Further derivatives yield additional zero crossings, one additional zero crossing for each additional derivative. If the value of \( \tau \) is fixed, by taking an additional derivative, the fractional bandwidth decreases, while the centre frequency increases.

A Gaussian doublet which is a common and practical UWB pulse shape [2], [3], is the second derivative of (2) and is defined by

\[ g_3(t) = \ddot{g}_1(t) = A \tau \dot{g}_1(t) + A^2 g_1(t) \]  

Differentiation of (5) yields:

\[ \ddot{g}_1(t) = \frac{d\ddot{g}_1(t)}{dt} = A \dot{g}_1(t) + A^2 \ddot{g}_1(t) = A \tau \ddot{g}_1(t) + 2A \dot{g}_1(t) \]

Since \( g_2(t) = \dot{g}_1(t) \), we can write:

\[ \ddot{g}_2(t) = A \tau \ddot{g}_2(t) + 2A \dot{g}_2(t) \]

Similarly we can generalize that

\[ \dddot{g}_n(t) = A \tau \dddot{g}_n(t) - A^2 \ddot{g}_n(t) = 0 \]  

where \( n = 1, 2, 3, \ldots \). The spectrums of the first three Gaussian pulses are shown in Fig. 1. A simple diagram showing the generator circuit of Gaussian pulses is given in Fig. 2.
B. Orthogonal Modified Hermite Waveforms

The functions defined by

\[ h_n(t) = (-\tau)^n e^{-t^2/2\tau^2} \left( \frac{d^n}{dt^n} \left( e^{-t^2/2\tau^2} \right) \right) \]  

(9)

where \( n = 0, 1, 2, \ldots \) and \( -\infty < t < \infty \), are called Hermite polynomials. The parameter \( \tau \) is the time-scaling factor. It should be mentioned that the definition of (9) is one of many forms of Hermite polynomials used in the literature. The spectrums of the first three Hermite pulses are shown in Fig. 3. An advantage of hermite pulses is that with their linear combinations, efficient pulse spectrums could be achieved [4].

Hermite and Gaussian polynomials are not orthogonal; however, they can be modified to become orthogonal as follows [5]

\[ h_n(t) = e^{-t^2/4\tau^2} h_n^{(\tau)}(t) \]  

(10)

where \( n = 0, 1, 2, \ldots \) and \( -\infty < t < \infty \). The result is a set of orthogonal functions \( h_n(t) \) which can be easily derived for all values of \( n \). As examples we can write

\[ h_0(t) = 1 \]

\[ h_1(t) = \frac{t}{\tau} e^{-t^2/4\tau^2} \]  

(11)

C. Orthogonal Prolate Spheroidal Wave Functions

One of the waveforms that are practically time and bandlimited is referred to as a Prolate Spheroidal Wave Function (PSWF). This function is the solution of \[ \int_{-T/2}^{T/2} p_n(x) \sin \Omega(t-x) dx = \lambda_n p_n(t) \]  

(14)

or, alternatively, the solution of the differential equation

\[ \frac{d}{dt} \left( 1-t^2 \right) \frac{dp_n(t)}{dt} + (\lambda_n - c^2 t^2) p_n(t) = 0 \]  

(15)
The effective time duration of the pulses are scaled to $0.2^{n}$ where $p_n(t)$ is the prolate spheroidal wave function of order $n$ and $\chi_n$ is the eigenvalue of $p_n(t)$. The constant $c$ is

$$c = \frac{\Omega T}{2}$$

where $\Omega$ is the bandwidth and $T$ is the pulse duration.

In (14), $\lambda$ is the concentration of energy in the interval $[-T/2, T/2]$ given by

$$\lambda = \frac{\int_{-T/2}^{T/2} |p_n(t)|^2 dt}{\int_{-\infty}^{\infty} |p_n(t)|^2 dt}$$

whose values range from 0 to 1. If we solve the differential equation (15) for the highest derivative, we get

$$(1 - t^2) \frac{d^2 p_n(t)}{dt^2} - 2t \frac{dp_n(t)}{dt} + (\chi_n - c^2 t^2) p_n(t) = 0$$

and consequently

$$\ddot{p}_n(t) - \frac{2t}{1 - t^2} \dot{p}_n(t) + \frac{\chi_n - c^2 t^2}{1 - t^2} p_n(t) = 0$$

As can be seen, different orders of the pulses can be simply obtained by changing the values of $\chi_n$; hence, (19) is the basis of a multi-pulse generator. The spectrum of a few prolate functions are shown in Fig. 5.

A simple diagram showing the generator PSWF pulses is given in Fig. 6.

III. Unified Structure of Basic UWB waveforms

By comparing the expressions for the above three basic pulses, it can be observed that all three satisfy a unified structure

$$\frac{d}{dt} [r(t)\dot{y}(t)] + [q(t) + \lambda s(t)] y(t) = 0$$

or

$$\dot{y}(t) + \frac{\dot{r}(t)}{r(t)} \dot{y}(t) + \frac{q(t) + \lambda s(t)}{r(t)} y(t) = 0$$

This is a well-known differential equation called Sturm-Liouville boundary value problem where $r(t) > 0$. Since the above differential equation is of the second order, it should be accompanied with two boundary conditions, these conditions are usually chosen as:

$$\begin{cases} a_1 y(a) + a_2 \dot{y}(a) = 0 \\ b_1 y(b) + b_2 \dot{y}(b) = 0 \end{cases}$$

One or both of the endpoints $a, b$ can be $\pm \infty$ as well. The values of $\lambda$ in (21) which lead to a nontrivial solution (the solution should fulfill the boundary conditions) are called eigen-values and the respective nontrivial solutions are called eigen-functions of the equation. The following theorem, which is referred as the main principle of the Sturm-Liouville theory, briefly describes the proper characteristics of the eigen-values and eigen-functions [7]:

**Theorem 1:** Assuming the following properties in the differential equation of (21):

1) All the three functions $r(t), q(t)$ and $s(t)$ are continuous on the closed interval $[a, b]$; in addition $r(t)$ is differentiable and $i(t)$ is continuous on the same interval.
2) Both functions $r(t)$ and $s(t)$ are positive on $[a, b]$.

we have the following relations:

**i** Equation (21) with conditions (22) has countable infinite number of eigen-values which are real and monotonically increasing:

$$\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \in \mathbb{R}$$

**ii** For the eigen-value $\lambda_n$, except for a constant coefficient, there exists a unique eigen-function $(y_n(t))$ and it has exactly $n$ zeros in the interval $(a, b)$. Moreover, eigen-functions are mutually orthogonal with respect to the weight function $s(t)$:

$$\int_a^b y_n(t) y_m(t) s(t) dt = 0$$

Now it is demonstrated that different UWB pulse shapes can be derived from (21) in a straightforward way.

A. Generation of Gaussian Waveforms

By comparing (21) and (8) we can write

$$\frac{\dot{r}(t)}{r(t)} = -At$$
and

$$\frac{q(t) + \lambda_n s(t)}{r(t)} = -\lambda t$$  \hspace{1cm} (26)$$

Equation (25) has a solution in the form of

$$r(t) = k_1 e^{-\frac{t^2}{2\tau^2}}$$  \hspace{1cm} (27)$$

where $k_1$ is a constant. As a solution for (26), we consider $q(t) = 0$, $\lambda_n = n$ and

$$s(t) = -\lambda t = -k_1 A e^{-\frac{t^2}{2\tau^2}}$$  \hspace{1cm} (28)

**B. Generation of Orthogonal Modified Hermite Waveforms**

By comparing (21) and (13) we can write

$$\frac{\dot{r}(t)}{r(t)} = 0$$  \hspace{1cm} (29)$$

and

$$\frac{q(t) + \lambda_n s(t)}{r(t)} = \frac{n}{\tau^2} + \frac{1}{2\tau^2} - t^2$$  \hspace{1cm} (30)$$

From (29) it is clear that $r(t) = k_2$, where $k_2$ is a constant. Substituting in (30), we get

$$q(t) + \lambda_n s(t) = \frac{k_2 m}{\tau^2} + \frac{k_2}{2\tau^2} - \frac{k_2 t^2}{4\tau^4}$$  \hspace{1cm} (31)$$

Consequently, $\lambda_n = n$,

$$s(t) = \frac{k_2}{\tau^2}$$  \hspace{1cm} (32)$$

and

$$q(t) = \frac{k_2}{2\tau^2} - \frac{k_2 t^2}{4\tau^4}$$  \hspace{1cm} (33)$$

**C. Generation of Orthogonal Prolate Spheroidal Wave Functions**

By comparing (21) and (19) we can write

$$\frac{\dot{r}(t)}{r(t)} = -\frac{2t}{1 - t^2}$$  \hspace{1cm} (34)$$

and

$$\frac{q(t) + \lambda_n s(t)}{r(t)} = \frac{\chi_n - c^2 t^2}{1 - t^2}$$  \hspace{1cm} (35)$$

Equation (34) can be solved to derive

$$r(t) = k_3 (1 - t^2)$$  \hspace{1cm} (36)$$

where $k_3$ is a constant. Using (36) in (35) we derive

$$q(t) + \lambda_n s(t) = k_3 \chi_n - k_3 c^2 t^2$$  \hspace{1cm} (37)$$

Hence $\lambda_n = \chi_n$, $s(t) = 1$, and $q(t) = -k_3 c^2 t^2$. To prevent confusion, it should be mentioned that although $\lambda_n$ and $\lambda_n$ are similar in notation, they are different in concept and value; however, the value of each could be generated by knowing the other one.

The result of this section is summarized in Table I. In the next sections, we explain the similar properties of the pulses using the above unified structure.

<table>
<thead>
<tr>
<th>Pulse shape</th>
<th>$r(t)$</th>
<th>$q(t)$</th>
<th>$\lambda_n$</th>
<th>$s(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>$k_1 e^{-\frac{t^2}{2\tau^2}}$</td>
<td>0</td>
<td>$n$</td>
<td>$-k_1 A e^{-\frac{t^2}{2\tau^2}}$</td>
</tr>
<tr>
<td>MHP</td>
<td>$k_2$</td>
<td>$\frac{k_2}{\tau^2} - \frac{k_2 t^2}{4\tau^4}$</td>
<td>$n$</td>
<td>$\frac{k_2}{\tau^2}$</td>
</tr>
<tr>
<td>PSWF</td>
<td>$k_3 (1 - t^2)$</td>
<td>$-k_3 c^2 t^2$</td>
<td>$\chi_n$</td>
<td>1</td>
</tr>
</tbody>
</table>

**IV. ORTHOGONALITY**

The eigen-functions of a Sturm-Liouville equation are orthogonal when a weight function is considered (part ii in theorem 1); however, in the engineering literature, orthogonality is defined as:

$$\int_a^b y_n(t) y_m(t).dt = 0$$  \hspace{1cm} (38)$$

Recalling that $s(t)$ is positive, the non-orthogonal eigenfunctions can be made orthogonal using the following change:

$$z_n(t) = \sqrt{s(t)} y_n(t)$$  \hspace{1cm} (39)$$

Now it is easy to check:

$$\int_a^b z_n(t).z_m(t).dt = 0$$  \hspace{1cm} (40)$$

The only thing left to prove is that the produced set $\{z_n(t)\}$ is itself the set of eigen-functions of a Sturn-Liouville equation:

$$\begin{cases}
    y_n(t) = s(t)^{-0.5} z_n(t) \\
    y_0(t) = s(t)^{-0.5} z_0(t) - \frac{k}{2}(s(t)^{0.5} z_0(t))^{1.5} z_0(t) \\
    y_n(t) = s(t)^{-0.5} z_n(t) - \frac{k}{2}(s(t)^{0.5} z_n(t))^{1.5} z_n(t) \\
    y_n(t) = s(t)^{-0.5} z_n(t) - \frac{k}{2}(s(t)^{0.5} z_n(t))^{1.5} z_n(t)
\end{cases}$$  \hspace{1cm} (41)$$

Combining the above results with (21), we get:

$$p(t).z_n(t) + p(t).z_m(t) + \left(k(t) + \frac{q(t)}{s(t)} + \lambda(t)\right) z_n(t) = 0$$  \hspace{1cm} (42)$$

where:

$$\begin{cases}
p(t) = r(t) \\
k(t) = (r(t), z_n(t))^{1.5} z_n(t)
\end{cases}$$  \hspace{1cm} (43)$$

Therefore, the new function set $\{z_n(t)\}$ is the set of eigen-functions of a Sturm-Liouville equation with $s(t) = 1$; consequently, these functions are mutually orthogonal.

It can be shown [8] that the Hermite polynomials introduced in (9), for $r = 1$, satisfy the following differential equation:

$$\ddot{h}_{\epsilon_n}(t) - \dot{h}_{\epsilon_n}(t) + nh_{\epsilon_n}(t) = 0$$  \hspace{1cm} (44)$$

The equivalent Sturm-Liouville form is obtained by:

$$\begin{cases}
r(t) = e^{-\frac{t^2}{2}} \\
q(t) = 0 \\
s(t) = e^{-\frac{t^2}{2}}
\end{cases}$$  \hspace{1cm} (45)$$
which suggests the orthogonality with respect to a Gaussian weight function. Now it is apparent why \( \sqrt{s(t)} = e^{-\frac{t^2}{2}} \) is multiplied in (10) to form the modified Hermite waveforms. Similarly, we can introduce modified Gaussian waveforms. Comparing (3) and (4), we see \( A = -\frac{1}{2} \). By recalling Table I, the weight function for Gaussian waveforms is:

\[
s(t) = -k_1 A e^{-\frac{t^2}{2}} = \frac{2k_1}{\tau} \left( e^{-\frac{t^2}{2}} \right)^2
\]

(46)

Thus the modified Gaussian waveforms are:

\[
g_{n}^{\text{mod}}(t) = e^{\frac{t^2}{2\tau}} g_n(t) = e^{\frac{t^2}{2\tau}} \frac{d^n}{dt^n} \left( e^{-\frac{t^2}{2\tau}} \right)
\]

(47)

which are exactly the same as modified Hermite functions (10). For the case of prolate spherical wave functions, since \( s(t) = 1 \), no modification is required.

V. Finite duration in time

One of the important requirements of a pulse, to be a candidate for transmission, is its spread along the time. In order to prevent inter symbol interference, the pulse should be limited in time. In our case, since the pulses are eigenfunctions of a differential equation, theoretically they will not vanish. To overcome this undesired effect, one way is to use finite end points \( (a, b \text{ in Theorem 1}) \). In this case, transmitting pulses are formed by considering the eigen-functions between the end points; i.e., if we represent the end points by \( \tau \) and set of eigen-functions by \( \{f_n(t)\} \), then the pulses are:

\[
p_n(t) = \begin{cases} 
  f_n(t) & \text{if } a < t < b \\
  0 & \text{otherwise}
\end{cases}
\]

(48)

Orthogonality is obviously preserved in this way. Prolate spheroidal wave functions are examples of this case; though the eigen-functions are not finite in time, they are only included on \( [-\frac{T}{2}, \frac{T}{2}] \) for transmission. The other option is to use fast decaying pulses. Even though these functions theoretically spread over time, their effective part is limited. The original Hermite functions (9) are a class of polynomials; however, the modifying part \( \sqrt{s(t)} \) decays exponentially. Therefore, the resultant functions have exponential decay rate which makes them good approximates of a time-limited pulse. In fact, the modifying part is useful for both orthogonality and finite duration.

VI. Pulses in Frequency domain

In practice, due to some external conditions on the spectrum of the transmitted signal, pulses should have specific frequency shapes. This task is usually performed using shapers such as filters or modulators [9]. The primary step for designing the shapers is to find the Fourier transform of the basic pulses. Since the modified Gaussian waveforms are the same as modified Hermites, in this section we focus on modified Hermite and prolate spheroidal wave functions. We denote the Fourier transform of \( y(t) \) by \( Y(\omega) \). It is well known that the transforms of \( y(t) \) and \( t \cdot y(t) \) are \( j\omega Y(\omega) \) and \( j\dot{Y}(\omega) \), respectively. Using these simple equalities, if we take the Fourier transform of both sides of (13), after simplification we have:

\[
\tilde{H}_n(\omega) + (4n + 2 - (2\tau \omega)^2) H_n(\omega) = 0
\]

(49)

Hence, \( \{H_n(\omega)\} \) are real valued functions which form a Sturm-Liouville eigen-function set. Similar results are obtained by considering (15):

\[
(\epsilon^2 - \omega^2) \tilde{P}_n(\omega) - 2\epsilon \tilde{P}_n(\omega) + (\chi_n - \omega^2) P_n(\omega) = 0
\]

(50)

Both of the above differential equations are automatically written in the Sturm-Liouville form. With respect to the weight functions in above equations, we see that these pulses, even in the Fourier domain are orthogonal:

\[
\int_{-\infty}^{\infty} H_n(\omega).H_m(\omega).d\omega = 0 \quad (m \neq n)
\]

(51)

This result could also be confirmed using Parseval’s Theorem.

VII. Conclusion

In this paper, three common waveforms for ultra wideband transmission are shown to have the same structure. It is shown that Gaussian, Modified Hermite and prolate spheroidal waveforms are eigen-functions of different Sturm-Liouville boundary condition problems. Using the orthogonality property of eigen-functions, modification of Hermite polynomials and orthogonality of the Prolate Spheroidal waveforms are demonstrated. It is also shown that modified Gaussians are the same as modified Hermites. Modifying the pulse, in addition to validating the orthogonality, limits the spread of the pulse over time. The Fourier transforms of the Modified Hermite and Prolate Spheroidal waveforms are also shown to form another set of Sturm-Liouville eigen-functions.

REFERENCES