A New Set of Uniquely Decodable Codes for Overloaded Synchronous CDMA

Amiya Singh, Arash Amini, Poonam Singh, Farokh Marvasti

Abstract

In this paper, we consider the designing of a new set of Uniquely Decodable Codes (UDC) for uncoded synchronous overloaded Code Division Multiple Access (CDMA) for the number of codes exceeding the assigned code length. For the construction, the proposed recursive method at iteration-k generates a matrix that can be classified into k orthogonal subsets of different dimensions. Out of them, all besides the largest (binary Hadamard) one are ternary in nature. There resides an inbuilt twin tree structured cross-correlation hierarchy that facilitates an advantageous balance between the auto and intergroup cross-correlation for the signatures in a subset. This opportunity is further leveraged by the proposed multi-stage detector to maintain the uniquely decodable (errorless) nature of the matrices for noiseless transmission. The simple logic of matched filtering serving as the basic designing block of the decoder provides an enormous saving over the complexity of optimum Maximum Likelihood Decoder (MLD). For the noisy channel, we derive the theoretical expression of the average Bit Error Rate (BER) for the individual subset. Also, we explain the role of the two factors (cardinality of the subset, and net level of interference) in being responsible for the non-uniformity in the order of their error performance.

Keywords

Code Division Multiple Access (CDMA), Overloaded CDMA, Uniquely Decodable Codes, Orthogonality, Multiple Access Interference (MAI), Multi User Detector (MUD), Complexity, Bit Error Rate (BER).

A. Singh and P. Singh are with Department of Electronics and Communication Engg., National Institute of Technology Rourkela, India. ($amiyasingh87@gmail.com, psingh@nitrkl.ac.in$)

A. Amini and F. Marvasti are with Advanced Communication Research Institute (ACRI), Department of Electrical Engg., Sharif University of Technology, Tehran, Iran. ($aamini@sharif.edu, marvasti@sharif.edu$)

I. Introduction

The errorless transmission of the synchronous Code Division Multiple Access (CDMA) using the linear decoders like Matched Filter (MF) can be guaranteed for the noise free channel if we consider the orthogonal matrix (set) like Hadamard (also Uniquely Decodable (UD)) for the purpose of encoding. This aspect further features for its Bit Error Rate (BER) close to that of the optimum when the channel becomes noisy. However, one of the major limitation of the orthogonal matrices is its square dimension that restricts its value to the code length or spreading gain of the signature.

To overcome this capacity hard limit, numerous UD [1]–[14], and non-UD [15]–[21] construction based systems have already been proposed. For the non-UD based approaches like O/O CDMA, PN/O CDMA [15], improved O/O CDMA [16], the maximization in system capacity is achieved by collaborating two different sets of code (signature). But, it strictly disapproves for the errorless decoding. As the reason, the significant level of the Multiple Access Interference (MAI) induced due to the non-zero level of the peak cross-correlation among the signatures is to be held responsible, which hardly abides any purposeful hierarchy or pattern. Therefore, several non-linear iterative decoders [22]–[25] utilizing the technique of serial, and parallel interference cancellation [26]–[28] exist which aims to improve the error performance by partial minimization of the MAI. Nevertheless, their participation also leads to the rise in complexity, and deprives the feasibility of its practical realization. Later, the technique of reusing [29], [30] the Welch Bound Equality (WBE) [31] sequences for different users with different wave forms in [32] leads to the significant maximization in capacity, and simplification of the receiver too. However, its implementation becomes doubtful due to the non-scalable [33] nature of the WBE sequences.

Under the above constraints, there still exists feasible alternative towards the designing of the errorless matrices for overloaded CDMA in terms of the UD Codes (UDC). A matrix $\mathbf{C}$ is considered as UD over $\mathbf{x}$, if for $\mathbf{x}_1 \neq \mathbf{x}_2$, the inequality $\mathbf{C}\mathbf{x}_1 \neq \mathbf{C}\mathbf{x}_2$ is true, where $\mathbf{x}_1$ and $\mathbf{x}_2$ denote two different input vectors. In other words, a UD matrix is injective in nature or there exists one-to-one mapping between the input, and output. Several binary [1]–[3], [5], [7], [13], [14], and ternary [4], [9]–[12] UD matrices with different approaches of construction can be found with significant overloading factor (β), which for the code matrix $\mathbf{C}_{N\times M}$ is usually defined as the ratio: M/N for $M > N$. Moreover, the use of the ternary UD matrices in context

of multi-user coding [4], [9]–[12] has drawn significant attention in literature even if they are of less importance in the coin-weighing problem [34]. In Section II, we discuss them with more details.

The noisy transmission being a common scenario in practical communication applications destroys the errorless attribute of the UDC and the overall error performance of the system is controlled by the cross-correlation property of the code matrices, which is usually best defined by the term Total Squared Correlation (TSC). For a code matrix $\mathbf{C}$, the TSC can be defined as the sum of the squared magnitudes of all inner products of the elemental codes i.e., $\text{TSC}\left(\mathbf{C}\right) \triangleq \sum_{i=1}^{M}\sum_{j=1}^{M}\left|\mathbf{c}_i^H\mathbf{c}_j\right|^2$, where H denotes the conjugate transpose operator. For the real or complex valued matrices, TSC is lower bounded by $\text{TSC}\left(\mathbf{C}\right) \geq \left(\frac{MK}{N}\right)$, where $K = \max\left\{M, N\right\}$ for M and N representing the number of signatures and spreading gain of the codes respectively. For underloaded ($M \leq N$) CDMA, although TSC is a suitable parameter to evaluate the performance, it does not hold valid for the overloaded ($M > N$) CDMA system. It is because TSC being a total measure of correlation lags attention to the users, individually. While, within a particular code set, the net level of peak cross-correlation among the signatures shows random variation, their decoding also gets affected non-uniformly. Therefore, the study of the correlation level of signatures individually or group-wise is preferable than that of TSC. One of such approach is presented in [20] where a new set of binary matrices (not UD) following a linear combination approach are proposed with a tree structured cross-correlation. Further, this hierarchy of cross-correlation becomes useful to propose an optimal tree detector, where a conditional weight estimate table guides for the process of decoding. To note, the complexity of this algorithm is bounded to the low-order-polynomial subjected to the condition that the tree must follow a uniform hierarchy in construction i.e.; the number of child nodes emanating from each node is constant for each level of the tree.

In this paper, we consider the designing of a new set of UDC for overloaded CDMA that addresses two crucial problems. The first one concerns, if unlike the code sets in [1], [5]–[7], [9]–[11], [13], [14] the existence of *multiple orthogonal subsets* within a single UD set is feasible, and if so, what general criterion is to be followed for its recursive construction. The second one is about the designing of the *low complex decoder*. In response to the first, we identified the unique fundamental matrix ($\mathbf{C}^1$ i.e., for $k = 1$) of construction from the existing literature [4],

[9]–[11] and develop a whole new perspective for the recurrent design of the proposed matrices. The proposed construction being recursive results in a UD matrix of dimension (M_k, N_k) for $M_k > N_k$ that can further be split into $k = \log_2 N_k$ number of orthogonal subsets (one binary (Hadamard) and $(k-1)$ ternary) of varied dimension. In response to the second, a uniform *twin-tree structured* cross-correlation is realized for the whole matrix due to the *linear dependency* lying in the formation of the larger subsets from the smaller ones. Later, this becomes thoroughly advantageous in designing the layout of the multi-user detector where the simple logic of matched filtering serves as the basic block of designing and recovers each subset with no error in the absence of noise, given that the detector has a *priori knowledge of the user's status (active or inactive)*.

Rest of the paper is organized as follows. Section II carries the background study on UD matrices with special attention towards role of the basis (fundamental) matrix in the construction. Section III emphasizes on the recursive construction of the proposed matrices followed by designing of the multi-user decoder in Section IV. In Section V, the analytical error performance of the system is discussed focusing more towards the explanation of the error performance of the individual subset. Section VI presents the overview of simulation results. Finally, the conclusion is presented in Section VII.

II. UD Matrices: Background

A. System Model

The synchronous CDMA system (synchronization corresponding to both bit and chip) using the ternary UDC matrix with index-k ($\mathbf{C}^k$ or $\mathbf{C}_{N_k \times M_k}$) for $k \in \mathbf{Z}^+$ can be modeled as

$$\mathbf{y} = \mathbf{r} + \mathbf{n} \tag{1}$$

where $\mathbf{r} = \mathbf{C}^k\mathbf{R}\mathbf{x}$ is the noiseless received vector with $\mathbf{R} = \mathbf{I}_{M_k \times M_k}$ = Identity Matrix with diagonal elements representing the amplitudes assuming the system to be perfectly power controlled. $\mathbf{x} \in \{\pm 1, 0\}^{M_k}$ is the input column vector, and $\mathbf{n}$ denotes the vector corresponding to the AWGN channel with zero mean and variance σ^2. In an effort to provide an unified approach of analysis, let us concentrate on the recursive constructions (mentioned in Table I), for which $\mathbf{C}^1 = \mathbf{B} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{H}_2 & | & \begin{matrix} + \\ 0 \end{matrix} \end{bmatrix}$ = basis matrix. Here, we intend to study the varied

TABLE I: Recursive Consruction of UDC Sets (Binary or Ternary) with $\mathbf{C}^1 = \mathbf{B}$ in (2) , where $B(2^k) = 2^{k-1}k$ (Eq. 3, [12]), Y, N, NL, and NY denote "Yes", "No", "Noiseless", and "Noisy" respectively.

Year	Author(s) and publication	Type	N_k	M_k	$\beta(k) \sim S_{sum}(k)$	Decoding
1979	Chang and Weldon [35]	Ternary	2^k	$2^{k-1}(k+1)$	Y	NL
1982	Ferguson [10]	Ternary	2^k	$2^{k-1}(k+2)$	Y	NL
1984	Chang [11]	Ternary	k	$\leq B(k)+k$	Y	NL
1995	Khachatrian and Martirossian [13]	Ternary	2^k	$k2^{k-1}+1$	Y	NL
1997	Wu and Chang [14]	Binary	k	$\geq k\left(\frac{\log_2 k-2}{2\log_2 3}\right)+\frac{1}{\log_2 3}$	Y	NL
1998	Khachatrian and Martirossian [36]	Binary	2^k	$2^{k-1}(k+1)+i$	Y	NL
2012	Masayekhi and Marvasti [4]	Ternary	2^k	$2^{k+1}-1$	N	NY

interpretations of **B** existing in different approaches towards the formation of the matrices with larger dimension.

In the early 1960s, the study of UD matrices in the context of the coin-weighing problem [34] was introduced. Later, the construction of the binary UD matrices suggested by many researchers in [35], [37] revealed better scope for the multi-user coding applications. Then, several explicit constructions were proposed satisfying the asymptotic equality between the β (or $\beta(k)$), and maximal achievable sum capacity (S_{sum}) for increasing the value of M_k. Note that a UD construction is said to have the asymptotic equality between β, and $S_{sum}(k)$, if and only if

$$\lim_{N_k\to\infty} \frac{\beta(k)}{S_{sum}(k)} = 1 \text{ (Eq. 4.5, Corollary in [9])}$$

where

$$S_{sum}(k) = \sum_{f=0}^{M_k} \frac{\binom{M_k}{f}}{2^{M_k}} \log_2 \frac{2^{M_k}}{\binom{M_k}{f}} \text{ (Theorem 2.1 in [9]).}$$

In the late 1970, construction of the ternary matrices in [9], being equivalent to that of [35] offers asymptotic equality. Further, the approach of construction in [9] is extrapolated to another generalized form in [10]. In [11], the designing of the proposed matrix remains valid for arbitrary values of N_k. In overall, all the matrices presented in Table I [9]–[11], [13], [14], [36] settles in asymptotic sense (shown by "Y" in column $\beta(k) \sim S_{sum}(k)$). However, their decoding

methods are suitable for the noisefree transmission only. The performance of their decoder if subjected to the noisy transmission fails to present an acceptable level of error performance. So, for these matrices, the use of the optimum Maximum Likelihood Decoder (MLD) [38] is the only solicited option. However, the sheer impracticality associated with its implementation, due to the catastrophic rise in complexity over the linear decoders, convey to look for the suitable substitutes. The designing of such optimal or sub-optimal decoder, to meet an acceptable level of BER and simultaneously gain the substantial reduction in complexity is highly challenging and has drawn the attention of many researchers in the past [20], [39], [40].

Recently, a new recurrent ternary construction [4] has been proposed, where the priority drives straight towards the designing of simplified decoder for the noisy channel, costing the sacrifice in asymptotic equality. In fact, the logic of asymptotic equality theoretically explaining the large capacity of the UD matrices is an inappropriate metric for evaluating the overall performance of the system for noisy transmission. It is because, unlike the noisefree transmission, the actual capacity of the system is directly affected by the decoder's design specifics, when the channel gets noisy. There also exists few non-recursive methods of construction, which are not listed in Table I. Out of them, the tensor product based matrix construction introduces a Simplified MLD (SMLD) [5], [41], where the additional users are usually kept to a suitable minimal value to ensure better error performance. It is due to the fact that for the noisy channel, the effect of MAI becomes more prominent, even if its impact remains unnoticed for the noiseless case. In a way to overcome this challenge, a hierarchy criterion is proposed in [1] that minimizes the impact of MAI to a suitable extent and brings improvement in the level of BER.

For most of the group based overloaded CDMA models in the literature [5], [15], [16], [32], the quality of the recovery of any signature is shown to be directly influenced by the extent to which the intra and inter-group cross-correlations are *balanced*. This metric being a crucial factor towards the efficient detection is usually supervised by the method of construction of the signature matrix. Under such provisions, where sufficient attention also needs to be focused towards the designing of the improved algorithms for construction, most of the works in the literature emphasizes the decoder design only. As a result, their systems [1], [2], [5]–[7], [9]–[16], [24], [25], [36], [42], [43] lag in terms of having a deterministic pattern among the levels of the elements of the associated cross-correlation (interference) matrix. Subsequently, for most of them, the final design of the decoder to overcome the effect of the MAI and offer errorless

detection in the absence of noise, becomes highly complex and loses its feasibility of practical implementation for noisy transmission.

In contrast, our approach to the construction of the matrices carries a well-defined hierarchy towards providing an appropriate balance over the intra and inter-group MAI of a signature in a group or subset. Where for the systems in [5], [15], [16], [23], [32], [42] the complete elimination of the intra-group MAI appears impossible, and becomes a serious concern towards the complexity rise of the decoder, we easily achieve it due to the orthogonal nature of the subsets. Likewise, the adverse impact of the inter-group MAI is balanced by strictly keeping the net level of the cross-correlation on a signature lower than its auto-correlation. In fact, this logic related to the inter-group MAI also governs the fundamental principle of the Matched Filter (MF) detector in conventional CDMA, and therefore, can be considered as a scope for making the overall process of decoding highly simplified. In overall, the proposed model stands unique by providing a system architecture, where the method of construction and decoding fully complements each other to serve a common objective, i.e.; efficient recovery of the input vector with the least complexity. Alternatively, our approach of designing aims at removing the prevalent gap persisting in all other overloaded CDMA systems, in the form of the absence of a balanced and deterministic correlation structure to support the errorless detection. Interestingly, in the present work, we achieve it by leveraging the ternary nature of the matrix to build a uniform twin-tree structured hierarchy in cross-correlation.

B. Root of Construction: the Basis Matrix

Before proceeding further, this is important to focus on the fundamental matrix that controls the overall layout of proposed construction. On interesting note, the fundamental (basis) matrix of our construction ($\mathbf{B}$ in (2)) also delivers the same requisites for the construction methods presented in Table I. More appropriately, for the recursive construction of the matrices in [4], [9]–[11], [13], $\mathbf{C}^1 = \mathbf{B}$ and the researchers, based on the structure of $\mathbf{B}$, have established their unique perspective to form the matrices of larger dimensions. In this work, we also propose a new approach towards the interpretation of $\mathbf{B}$, not only in the method of construction but also to simplify the process of decoding. For our proposed construction, it is easy to trace that $\mathbf{C}^1 \neq \mathbf{B}$. However, the so-called fundamental matrix $\mathbf{B}$ appears in the intermediate stage during the recursive formation of $\mathbf{C}^2$ from $\mathbf{C}^1$ and hence serves the purpose of providing a fundamental

structure for designing, where

$$\mathbf{B} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{H}_2 & | & \begin{matrix} + \\ 0 \end{matrix} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} + & + & + \\ + & - & 0 \end{bmatrix}. \tag{2}$$

a) Correlation Pattern of the Basis Matrix: First, we analyze the pattern of the cross-correlation matrix ($\boldsymbol{\rho}$), associated i.e.;

$$\boldsymbol{\rho} = \mathbf{B}^T\mathbf{B} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \tag{3}$$

In (3), the first, second, and third row of $\boldsymbol{\rho}$ denote the cross-correlation coefficient corresponding to the signature in first, second and third column of $\mathbf{B}$ respectively. From (3), we can easily decipher $\mathbf{Det}(\boldsymbol{\rho}) = 0$, where $\mathbf{Det}$ denotes the determinant of the matrix $\boldsymbol{\rho}$. Equivalently, the first two columns of $\mathbf{B}$ are correlated to the third one, since the last row of $\boldsymbol{\rho}$ can be interpreted as the linear combination of the scalar-multiplied version of the first two rows.

b) Decoding of the Basis Matrix: Decoding of $\mathbf{B}$ from the noisefree received vector $\mathbf{r}$ in (1), if subjected to the logic of MF is not fully errorless. This is due to the presence of an additional user beyond $\mathbf{H}_2$. So, let us split $\mathbf{B}$ into two subsets so that the first and second subset are equal to $\mathbf{H}_2$ and $(1\ 0)$ respectively. Correspondingly, the input vector $\mathbf{x}$ can be defined as $\mathbf{x} = [\mathbf{x}_1\mathbf{x}_2]$, where $\mathbf{x}_1 = [x_{11}x_{12}]$ and $\mathbf{x}_2 = [x_{21}]$. According to the logic of MF, the decoding of $\hat{\mathbf{x}}_1$ can be achieved with no error ($\hat{\mathbf{x}}_1 = \mathbf{x}_1$), where as that of $\hat{\mathbf{x}}_2$ gets erroneous ($\hat{\mathbf{x}}_2 \neq \mathbf{x}_2$). This can be verified from the expression of $\mathbf{z} = [z_{11}z_{12}z_{21}]$ for $\mathbf{x} = [x_{11}x_{12}x_{21}]$, where $\hat{\mathbf{x}} = \text{sign}(\mathbf{z}) = \text{sign}(\boldsymbol{\rho}\mathbf{x})$ e.g.;

$$z_{11} = 2x_{11} + x_{21},\ z_{12} = 2x_{12} + x_{21},\ z_{21} = x_{11} + x_{12} + x_{21}.$$

As evident from the above expression,

$$\hat{x}_{11} = \text{sign}(z_{11}) = x_{11}, \quad \hat{x}_{12} = \text{sign}(z_{12}) = x_{12}$$

holds true irrespective of the value of x_{21} (interfering user of second subset). In contrast, for all 2^3 combinations of $\mathbf{x}$, the similar outcome is not observed while decoding $\hat{x}_{21}$ i.e., $\hat{x}_{21} \neq x_{21}$.

So, as a possible approach towards the errorless recovery of $\mathbf{B}$, the whole process of decoding can be split into two stages. First, the orthogonal matrix $\mathbf{H}_2$ is decoded. Second, we estimate its

TABLE II: Construction of SMOS (Type I) Matrices for $k = 1, 2, 3$ e.g., $\mathbf{C}^1 = \left[\mathbf{C}_1^1\right]$, $\mathbf{C}^2 = \left[\mathbf{C}_1^2|\mathbf{C}_2^2\right]$, $\mathbf{C}^3 = \left[\mathbf{C}_1^3|\mathbf{C}_2^3|\mathbf{C}_3^3\right]$

$$\mathbf{C}^1 = \frac{1}{\sqrt{2}}\begin{bmatrix} + & + \\ + & - \end{bmatrix}$$

$$\mathbf{C}_1^2 = \frac{1}{\sqrt{4}}\begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}, \mathbf{C}_2^2 = \frac{1}{\sqrt{4}}\begin{bmatrix} + & + \\ 0 & 0 \\ + & - \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{C}_1^3 = \frac{1}{\sqrt{8}}\begin{bmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{bmatrix}, \mathbf{C}_2^3 = \frac{1}{\sqrt{8}}\begin{bmatrix} + & + & + & + \\ 0 & 0 & 0 & 0 \\ + & - & + & - \\ 0 & 0 & 0 & 0 \\ + & + & - & - \\ 0 & 0 & 0 & 0 \\ + & - & - & + \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{C}_3^3 = \frac{1}{\sqrt{8}}\begin{bmatrix} + & + \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ + & - \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

interference on the additional user and remove it completely so that decoding of $\hat{x}_{21}$ can be set with no error. In Section IV, we exploit the similar logic to devise a multi-stage decoder for the errorless extraction of multiple subsets within the proposed matrices of larger dimension.

III. Signature Matrix with Orthogonal Subsets (SMOS)

A. Recursive Construction: Type I, Type II, Type III

For SMOS (Type I):

First, we introduce the following recursive mechanism for the construction of SMOS (Type-I) $\mathbf{C}_{N_k \times M_k}$ for $N_k = 2^k$ where $k \in \mathbf{Z}^+$.

- Initialize $\mathbf{C}^0 = [1]$ and find $\mathbf{C}^1 = \left(\frac{1}{\sqrt{2}}\mathbf{H}_{2\times 2} \otimes \mathbf{C}^0\right) = \frac{1}{\sqrt{2}}\mathbf{H}_2$
- For $k > 1$, $\mathbf{C}^k = \left(\frac{1}{\sqrt{2}}\mathbf{H}_{2\times 2} \otimes \mathbf{A}\right)$, where $\mathbf{A} = \left[\mathbf{C}^{k-1}|[100\cdots 0]^T{}_{1\times 2^{k-1}}\right]$.

According to the above approach, for $k = 2$, we have $\mathbf{A} = \mathbf{B}$. In [9], for the recursive design of the matrices with higher dimension, the authors interpret the element '1' in the vector $\{\,1 \quad 0\,\}$ in $\mathbf{B}$ as the simple (1×1) *Identity Matrix* and continue the same for the next iterations too. On the contrary, for the proposed construction, we consider the sequence $\{\,1 \quad 0\,\}$ as a one dimensional vector *with one element as 1 and the rest as* 0. From Table II, it is worth realizable that the

proposed matrices with index-k has k orthogonal subsets e.g., $\mathbf{C}^k = \left[\mathbf{C}_1^k|\mathbf{C}_2^k|\cdots|\mathbf{C}_k^k\right]$, where the value of k is in the logarithmic with the spreading gain i.e.; $k = \log_2 N_k$. Furthermore, the subset $\mathbf{C}_i^k$ for $i = 1, 2, \ldots, k$ owns 2^{k-i+1} number of signatures with the *effective spreading gain* (N_{ef}) of $N_{ef_i} = \frac{N_k}{2^{i-1}}$. In the present context, the effective spreading gain of a signature can be defined as the total number of non-zero elements (1,-1) present in it. Now, we present Theorem 1 to prove the UD nature of SMOS (Type I).

Theorem 1: The kth indexed version of the SMOS (Type-I): $\mathbf{C}^k$ with M_k signatures and spreading gain of N_k being classified into k orthogonal subsets is uniquely decodable.

$Proof$: To prove $\mathbf{C}^k$, in general, to be UD over $\mathbf{x}_k \in \{1, 0, -1\}^{M_k}$, let us start with $\mathbf{C}^1$. Following Table II, $\mathbf{C}^1 = \mathbf{H}_2$ is orthogonal, which by default also assures for its UD nature. For $\mathbf{C}^1$ being UD, $\mathbf{B} = \left[\mathbf{C}^1|[10]^T\right]$ can be shown to be of UD type (Theorem 1 in [36]). Now, let us extend the above logic to the generalized sense.

If $\mathbf{C}^{k-1}$ is UD over $\mathbf{x}_{k-1} \in \{1, 0, -1\}^{M_{k-1}}$ e.g., $\mathbf{C}^{k-1}\mathbf{x}_{k-1(1)} \neq \mathbf{C}^{k-1}\mathbf{x}_{k-1(2)}$, then for $\mathbf{A} = \left[\mathbf{C}^{k-1}|[100\cdots0]^T{}_{1\times 2^{k-1}}\right]$, $\mathbf{A}\left(\mathbf{x}_{k-1(1)}\ x_1\right) \neq \mathbf{A}\left(\mathbf{x}_{k-1(2)}\ x_2\right)$ always holds true for $x_1 \neq x_2$, where $\mathbf{x}_{k-1(2)}, \mathbf{x}_{k-1(2)} \in \{1, 0, -1\}^{M_{k-1}}$, and $x_1, x_2 \in \{1, 0, -1\}$. Following the steps of construction of SMOS (Type I), as the remaining part, now, all we need to prove is the UD nature of $\mathbf{C}^k = \left(\frac{1}{\sqrt{2}}\mathbf{H}_{2\times 2} \otimes \mathbf{A}\right)$ that is explained below.

For $\mathbf{C}^k = \left(\frac{1}{\sqrt{2}}\mathbf{H}_2 \otimes\ \mathbf{A}\right) = \begin{bmatrix} \mathbf{A} & \mathbf{A} \\ \mathbf{A} & -\mathbf{A} \end{bmatrix}$ to be injective over $\{1, 0, -1\}^{M_k}$, all sums of the term $\mathbf{C}^k\mathbf{x}_k$ need to be distinct. In other words, $\mathbf{C}^k\mathbf{x}_{k(1)} = \mathbf{C}^k\mathbf{x}_{k(2)}$ will be true, if and only if $\mathbf{x}_{k(1)} = \mathbf{x}_{k(2)}$. To prove this, let us split the input vectors e.g. $\mathbf{x}_{k(1)} = \left[\mathbf{x}_{k(11)}\mathbf{x}_{k(12)}\right]^T$ and $\mathbf{x}_{k(2)} = \left[\mathbf{x}_{k(21)}\mathbf{x}_{k(22)}\right]^T$, such that

$$\mathbf{A}(\mathbf{x}_{k(11)} + \mathbf{x}_{k(12)}) = \mathbf{A}(\mathbf{x}_{k(21)} + \mathbf{x}_{k(22)}) \text{ and } \mathbf{A}(\mathbf{x}_{k(11)} - \mathbf{x}_{k(12)}) = \mathbf{A}(\mathbf{x}_{k(21)} - \mathbf{x}_{k(22)})$$

Further, addition and subtraction of the above two equations results in $\mathbf{A}\mathbf{x}_{k(11)} = \mathbf{A}\mathbf{x}_{k(21)}$ and $\mathbf{A}\mathbf{x}_{k(12)} = \mathbf{A}\mathbf{x}_{k(22)}$ respectively. This in turn implies $\mathbf{x}_{k(1)} = \mathbf{x}_{k(2)}$ and proves $\mathbf{C}^k$ to be uniquely decodable. ■

Under these developments in analysis, let us define the SMOS.

Definition 1: The matrix $\mathbf{C}_{N_k\times M_k}$ is said to be SMOS over the input $\{0, 1, -1\}$, if the following conditions are satisfied.

- $\mathbf{C}^k$ is uniquely decodable over $\{0, 1, -1\}^{M_k}$.

TABLE III: Construction of SMOS (Type II and III) Matrices for $k = 3$ e.g., $\mathbf{C}^3 = \left[\mathbf{C}_1^3|\mathbf{C}_2^3|\mathbf{C}_3^3\right]$

Type II

$$\mathbf{C}_1^3 = \frac{1}{\sqrt{8}}\begin{bmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{bmatrix}, \mathbf{C}_2^3 = \frac{1}{\sqrt{8}}\begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{C}_3^3 = \frac{1}{\sqrt{8}}\begin{bmatrix} + & + \\ + & - \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Type III

$$\mathbf{C}_1^3 = \frac{1}{\sqrt{8}}\begin{bmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{bmatrix}, \mathbf{C}_2^3 = \frac{1}{\sqrt{8}}\begin{bmatrix} + & + & + & + \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ + & - & + & - \\ + & + & - & - \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ + & - & - & + \end{bmatrix}, \mathbf{C}_3^3 = \frac{1}{\sqrt{8}}\begin{bmatrix} + & + \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ + & - \end{bmatrix}$$

- $\mathbf{C}^k$ comprises of k orthogonal subsets, such that $\mathbf{C}^k = \left[\mathbf{C}_1^k|\mathbf{C}_2^k|\cdots|\mathbf{C}_k^k\right]$, where the number of signatures in $\mathbf{C}_i^k$ are $\frac{N_k}{2^{i-1}}$
- $\mathbf{Det}(\boldsymbol{\rho}) = 0$, for $\boldsymbol{\rho} = (\mathbf{C}^k)^T\mathbf{C}^k$
- The level of peak auto-correlation of an arbitrary signature in subset-$\mathbf{C}_i^k$ must be greater than the net level of peak cross-correlation, due to the $(k-i)$ successive subsets: $\mathbf{C}_{i+1}^k, \mathbf{C}_{i+2}^k, \ldots, \mathbf{C}_k^k$. Mathematically, this can be described by

$$\rho_{ii}(u,u) > \sum_{j=i+1}^{k}\sum_{v=1}^{\frac{N_k}{2^{j-1}}} \rho_{ij}\left(u,v\right) \tag{4}$$

where

$$\rho_{ii} = \left(\mathbf{C}_i^k\right)^T\mathbf{C}_i^k = \begin{bmatrix} \rho_{ii}(1,1) & \rho_{ii}(1,2) & \cdots & \rho_{ii}(1,2^{k-i+1}) \\ \rho_{ii}(2,1) & \rho_{ii}(2,2) & \cdots & \rho_{ii}(2,2^{k-i+1}) \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{ii}(2^{k-i+1},1) & \rho_{ii}(2^{k-i+1},2) & \cdots & \rho_{ii}(2^{k-i+1},2^{k-i+1}) \end{bmatrix},$$

and

$$\rho_{ij} = \left(\mathbf{C}_i^k\right)^T \mathbf{C}_j = \begin{bmatrix} \rho_{ij}(1,1) & \rho_{ij}(1,2) & \cdots & \rho_{ij}(1,2^{k-j+1}) \\ \rho_{ij}(2,1) & \rho_{ij}(2,2) & \cdots & \rho_{ij}(2,2^{k-j+1}) \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{ij}(2^{k-i+1},1) & \rho_{ij}(2^{k-i+1},2) & \cdots & \rho_{ij}(2^{k-i+1},2^{k-j+1}) \end{bmatrix}$$

We can also construct Two other sets of SMOS, abiding the same definition cited above such that $\mathbf{C}^0 = [1]$, $\mathbf{C}^1 = \left(\frac{1}{\sqrt{2}}\mathbf{H}_{2\times 2} \otimes \mathbf{C}^0\right) = \frac{1}{\sqrt{2}}\mathbf{H}_2$. For $k > 1$, $\mathbf{C}^k = \left[\mathbf{C}_1^k|\mathbf{C}_2^k|\cdots|\mathbf{C}_k^k\right]$ can be recursively derived from $\mathbf{C}^{k-1} = \left[\mathbf{C}_1^{k-1}|\mathbf{C}_2^{k-1}|\cdots|\mathbf{C}_{k-1}^{k-1}\right]$, e.g.; for $i = 1$, $\mathbf{C}_i^k = \left(\frac{1}{\sqrt{2}}\mathbf{H}_2 \otimes \mathbf{C}_i^{k-1}\right)$ and for $2 \leq i \leq k$, the expressions for $\mathbf{C}_i^k$ are presented below.

For SMOS (Type II):

$$\mathbf{C}_i^k = \left(\frac{1}{\sqrt{2}}\mathbf{H}_2 \otimes \mathbf{C}_{iZ}^{k-1}|\frac{1}{\sqrt{2}}\mathbf{H}_2 \otimes \mathbf{C}_{iNZ}^{k-1}\right) \text{ for } 2 \leq i \leq k-1$$
$$\mathbf{C}_i^k = \left(\frac{1}{\sqrt{2}}\mathbf{H}_2 \otimes 1|\frac{1}{\sqrt{2}}\mathbf{H}_2 \otimes \mathbf{0}_{1\times 2^k-1}\right) \text{ for } i = k.$$

where $\mathbf{C}_i^{k-1} = \left[\mathbf{C}_{iNZ}^{k-1}|\mathbf{C}_{iZ}^{k-1}\right]$ and $\mathbf{C}_{iNZ}^{k-1}$ and $\mathbf{C}_{iZ}^{k-1}$ denote the sub-matrices of $\mathbf{C}_i^{k-1}$ having all non-zero (1,-1) and zero elements respectively.

For SMOS (Type III):

$$\mathbf{C}_i^k = \begin{bmatrix} \mathbf{C}_i^{k-1} & \widetilde{\mathbf{C}_i^{k-1}} \\ \mathbf{C}_i^{k-1} & -\widetilde{\mathbf{C}_i^{k-1}} \end{bmatrix} \text{ for } 2 \leq i \leq k-1$$
$$\mathbf{C}_i^k = \frac{1}{\sqrt{2^k}} \begin{bmatrix} [10\cdots 0]_{1\times 2^{k-1}} & [0\cdots 01]_{1\times 2^{k-1}} \\ [10\cdots 0]_{1\times 2^{k-1}} & -[0\cdots 01]_{1\times 2^{k-1}} \end{bmatrix} \text{ for } i = k.$$

In the above expression of $\mathbf{C}_i^k$, the notation $\widetilde{\mathbf{C}_i^{k-1}}$ denotes the matrix, such that its mth element is formed from the mth element of $\mathbf{C}_i^{k-1}$ where $\widetilde{\mathbf{C}_{im}^{k-1}} = \mathbf{C}_{im}^{k-1}(t - N_{k-1}T_c)$ with T_c denoting the chip duration of the signature sequence.

B. Twin Tree Structured Cross-correlation

The designing and performance of any detection algorithm in CDMA is driven by the level of MAI on its users. For further explanation, the system model in (1) corresponding to $\mathbf{C}^k = \left[\mathbf{C}_1^k|\mathbf{C}_2^k|\cdots|\mathbf{C}_k^k\right]$ can be redefined as

$$\mathbf{y} = \sum_{i=1}^{k} \mathbf{C}_i^k \mathbf{x}_i + \boldsymbol{n} \tag{5}$$

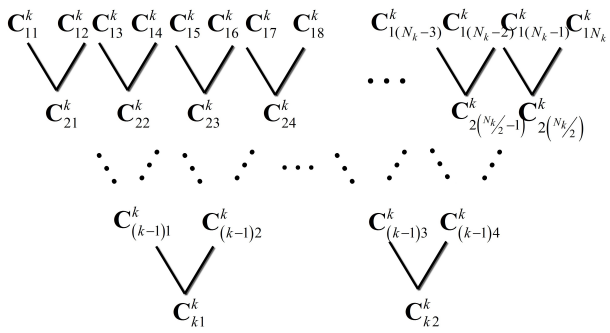

Fig. 1: Uniform Twin Tree Structure for $\mathbf{C}^k = \left[\mathbf{C}^k_1 | \mathbf{C}^k_2 | \cdots | \mathbf{C}^k_k\right]$

In this section, our attention is on the correlation structure of SMOS (Type-I) only. However, the outcome can also be applied to other similar classes (Type II, and III).

The geometric pattern of the subsets in $\mathbf{C}^k$ can be shown to have a *uniform twin tree structured cross-correlation* as shown in Fig. 1. The nodes of the tree at a particular level (depth) $l = 1, 2, \ldots, k$ (i.e., $l = k - i + 1$) collectively represent a specific subset. The following facts can be summarized from the pictorial representation.

- There exist two identical (twin) trees, each of which has its origin or root from the smallest orthogonal subset (i.e.; $\mathbf{C}^k_k$ at the lowest level of the tree, $l = 1$). The nodes at the highest level of the tree (i.e., $l = k$) represent the largest subset: $\mathbf{C}^k_1$.
- Each node (parent) at a level-l generates two nodes (child) for its next higher level $(l + 1)$.
- All the 2^l nodes at level-l collectively form an orthogonal set and each node at a particular level is correlated to its child and parent nodes only.

For node-j at level-l (i.e.; $\mathbf{c}^k_{lj}$), the two child nodes emanated at level-$(l + 1)$ (i.e.; $\mathbf{c}^k_{(l+1)(2j-1)}$ and $\mathbf{c}^k_{(l+1)(2j)}$) can be expressed as the *linear combination* on $\mathbf{c}^k_{lj}$ e.g.,

$$\mathbf{c}^k_{(l+1)(2j-1)} = \mathbf{c}^k_{lj} + \mathbf{c}^k_{lj}\left(t \pm \left({}^{N}/_{2^{l+1}}\right) T_c\right) \tag{6}$$

$$\mathbf{c}^k_{(l+1)\ (2j)} = \mathbf{c}^k_{lj} - \mathbf{c}^k_{lj}\left(t \pm \left({}^{N}/_{2^{l+1}}\right) T_c\right) \tag{7}$$

IV. Multi User Detection (MUD)

From the tree hierarchy in Fig. 1, considering the existence of k multiple subsets in $\mathbf{C}^k$, a signature in subset $\mathbf{C}^k_i$ (or $\mathbf{C}^k_{(k-l+1)}$) is correlated to each of its root sequence in previous $(k - i)$

(or $l-1$) subsets: $\mathbf{C}_{i+1}^k, \mathbf{C}_{i+1}^k, \cdots, \mathbf{C}_k^k$. Along with, the correlation also prevails with (2^i-2) number of child signatures existing in next $(i-1)$ orthogonal subsets. In particular, corresponding to each of the code sequence in $\mathbf{C}_i^k$, the subsets $\mathbf{C}_{i-1}^k, \mathbf{C}_{i-2}^k, \cdots, \mathbf{C}_1^k$ carry $2^1, 2^2, \ldots, 2^{i-1}$ number of child signatures respectively. Therefore, major part of the MAI (or intergroup MAI) on a particular signature in a subset is due to the child signatures. Since the number of child signatures is usually decided by the level of the subset, a subset with lower level (low value of l or high value of i) has comparatively more number of child signatures and hence, is subjected to the higher level of MAI. Equivalently, the following relation summarizes the effect of MAI on different subsets.

$$\text{MAI}_{\mathbf{C}_1^k} < \text{MAI}_{\mathbf{C}_2^k} < \cdots < \text{MAI}_{\mathbf{C}_k^k} \tag{8}$$

According to the expression in (8), each signature in the orthogonal subset at the highest level (of largest size) is subjected to the least level of MAI. Hence, the corresponding subset validates its candidature to be decoded first.

Fig. 2 (b) shows the overall cross-correlation matrix associated with that of $\mathbf{C}^3$ i.e., $\boldsymbol{\rho}_3 = \left(\mathbf{C}^3\right)^T \mathbf{C}^3$, where the matrix and its respective tree structure are shown by Table II and Fig. 2 (a) respectively. The row-a or column-a in Fig. 3 presents the correlation coefficients for user-a for $1 \leq a \leq 14$. While the *non-zero entry* in a cell indicates the presence of correlation among the particular signatures, the cells with *no entries* implies the orthogonal nature of the signatures involved.

Now, we attempt to justify the errorless nature of the proposed decoder. So, first, we present Lemma 1 which further guides to the whole proof in Theorem 2.

Lemma 1: For $\mathbf{C}^k \in \{\pm 1, 0\}^{N_k \times M_k}$ denoting the fully loaded SMOS (Type I), the decoding of the subset with the least MAI using MF detection is always errofree.

Proof: According to (8), the subset with the least MAI in SMOS $\mathbf{C}^k$ is $\mathbf{C}_1^k$. Adopting the logic of MF, the decision of $\text{sign}(\mathbf{z}_1)$ corresponding to $\mathbf{x}_1$ will be errorless, if and only if $\hat{\mathbf{x}_1} = \text{sign}(\mathbf{z}_1) = \mathbf{x}_1$ where

$$\mathbf{z}_1 = \mathbf{r}\left(\mathbf{C}_1^k\right)^T = \sum_{i=1}^{k} \rho_{1i}\mathbf{x}_i \tag{9}$$

for $\boldsymbol{\rho}_1 = [\boldsymbol{\rho}_{11}|\boldsymbol{\rho}_{12}|\cdots|\boldsymbol{\rho}_{1k}]$ denoting the matrix representing the cross-correlation coefficients of all the subsets with respect to $\mathbf{C}_1^k$, such that $\boldsymbol{\rho}_{11} = \left(\mathbf{C}_1^k\right)^T \mathbf{C}_1^k$, $\boldsymbol{\rho}_{12} = \left(\mathbf{C}_1^k\right)^T \mathbf{C}_2^k$, $\ldots$, $\boldsymbol{\rho}_{1k} =$

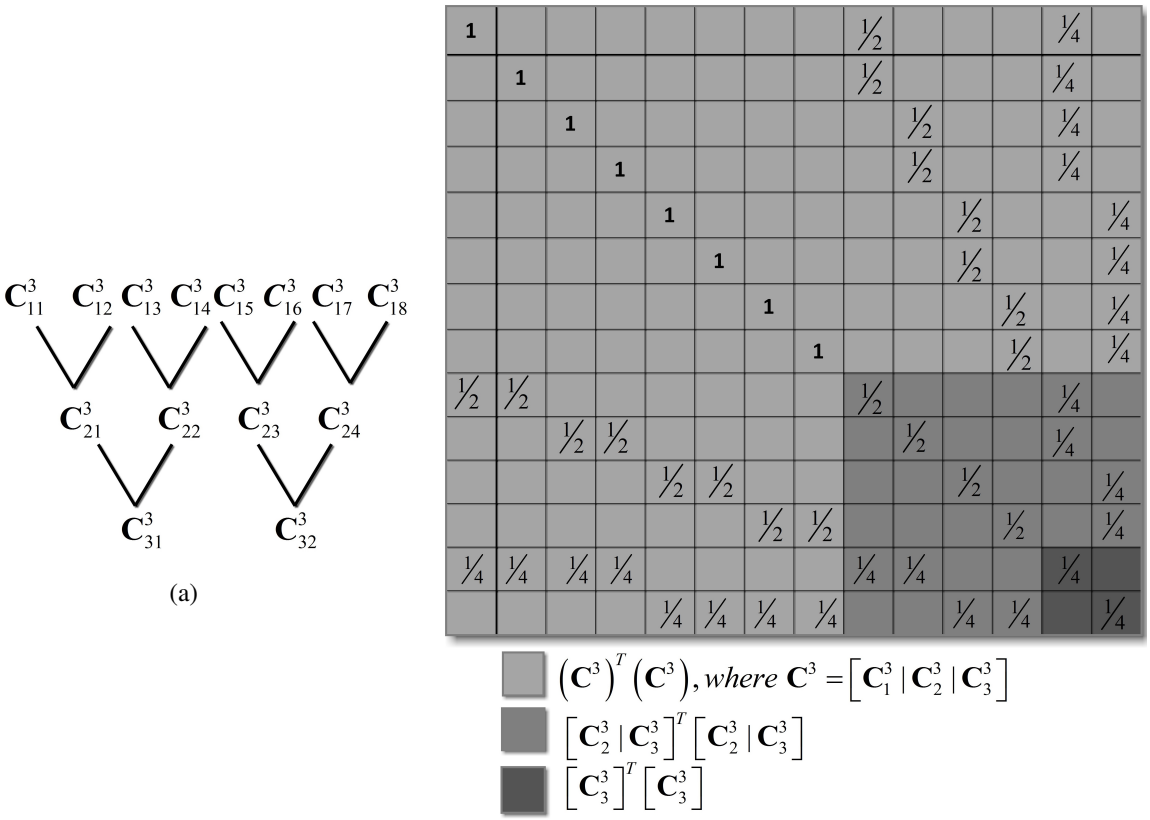

Fig. 2: (a) Twin tree Structure for $\mathbf{C}^3$ (b) Correlation Matrix for $\mathbf{C}^3$: $\boldsymbol{\rho}_3 = \left(\mathbf{C}^3\right)^T\mathbf{C}^3$

$\left(\mathbf{C}^k_1\right)^T\mathbf{C}^k_k$. So, for any signature in $\mathbf{C}^k_1$ (say signature-u), the relation $\hat{x_{1u}} = x_{1u}$, is true, only if the following expression on $\boldsymbol{\rho}_1$ holds valid.

$$\rho_{11}(u,u) > \sum_{i=2}^{k}\sum_{v=1}^{\frac{N_k}{2^{i-1}}} \rho_{1i}\left(u,v\right) \tag{10}$$

Note that the expression in (10) fully complies with the attribute of SMOS cited in (4). Here, the fundamental logic of decoding is governed by the fact that the errorless recovery of the input data in the multi-user environment in CDMA is feasible, *if the level of its auto-correlation exceeds that of the net cross-correlation on it*.

For better realization, let us take a look at the correlation matrix for $\mathbf{C}^3$ ($\boldsymbol{\rho}_3$ in Fig. 3). The first

8 rows or columns indicate the cross-correlation entries for the users of $\mathbf{C}_1^3$, which can be denoted individually as $\mathbf{C}_{1a}^3$ for $1 \leq a \leq 8$. For an arbitrary user in $\mathbf{C}_1^3$, the *correlation vector* presenting the non-zero correlation coefficients is found to be $\{1, 1/2^1, 1/2^2\}$, where the first and the rest indicate the level of auto and peak cross-correlations respectively. Extending the analysis to the general case of $\mathbf{C}^k$, the correlation vector of any user in $\mathbf{C}_1^k$ becomes $\{1/2^0, 1/2^1, 1/2^2, \cdots, 1/2^{k-1}\}$, which also approves the relation in (10), since $1 > (1/2^1 + 1/2^2 + \cdots + 1/2^k)$. Now, if we remove $\mathbf{C}_1^k$ from $\mathbf{C}^k$ indicating the successful detection of $\hat{\mathbf{x}_{k1}}$ then following (8), $\mathbf{C}_2^k$ is to be counted as the subset with the least MAI and the correlation vector for any user in $\mathbf{C}_2^k$ becomes $\{1/2^1, 1/2^2, \cdots, 1/2^{k-1}\}$. This verifies the logic associated in (10) too, as $1/2^1 > (1/2^2 + 1/2^3 + \cdots + 1/2^k)$. Thus, with the similar approach of removing a series of subsets from the higher level of the tree (say p subsets: $\mathbf{C}_1^k$, $\mathbf{C}_2^k$ $\ldots$, $\mathbf{C}_p^k$), we can show the decoding of subset-$(p+1)$ to be errorfree. Therefore, for the proposed code design, it is possible to decoder the subset with the least MAI with no error. ■

Theorem 2: For $\mathbf{C}^k \in \{\pm 1, 0\}^{N_k \times M_k}$ being the fully loaded SMOS (Type I), there exists a feasible model low complex decoder for the error less detection of the input vector $\boldsymbol{x} \in \{1, -1\}^{M_k}$.

Proof: The proof is guided by the outcome from Lemma 1, adopting which the subset of $\mathbf{C}^k$ with the least MAI (i.e., $\mathbf{C}_1^k$) can be decoded with no error. This can be considered as the first stage of the multi stage decoder. After it is correctly decoded (i.e., $\hat{\mathbf{x}}_1 = \mathbf{x}_1$), this is possible to accurately estimate its level of MAI on other $(k-1)$ subsets i.e., $\mathbf{i}_1 = \mathbf{C}_1^k \hat{\mathbf{x}}_1$. Now, subtraction of $\mathbf{i}_1$ from $\mathbf{r}$ (also call $\mathbf{r}_1$) generates $\boldsymbol{r}_2$, such that

$$\mathbf{r}_2 = \mathbf{r}_1 - \mathbf{i}_1 = \sum_{i=2}^{k} \mathbf{C}_i^k \mathbf{x}_i$$

is the summed transmitted signal of the matrix with remaining $(k-1)$ subsets still left to be decoded: $\left[\mathbf{C}_2^k | \mathbf{C}_3^k | \cdots | \mathbf{C}_k^k\right]$. According to (8), $\mathbf{C}_2^k$ then becomes the subset under the least MAI. With reference to Fig. 3, for each of the $(N_k/2)$ users of $\mathbf{C}_2^k$, the correlation vector becomes $\{1/2^1, 1/2^2, \cdots, 1/2^{k-1}\}$ and validates the relation in (10). Hence, the input vector corresponding to $\mathbf{C}_2^k$ is also detectable with no error (i.e., $\hat{\mathbf{x}}_2 = \mathbf{x}_2$) and this becomes the second stage of MUD. On a recurrent mode, similar interpretation of Lemma 1 is to be carried out sequentially till stage-k so as to validate the error free decoding of $\hat{\mathbf{x}}_3, \hat{\mathbf{x}}_4, \ldots, \hat{\mathbf{x}}_k$ corresponding to $\mathbf{C}_3^k, \mathbf{C}_4^k, \ldots, \mathbf{C}_k^k$ from $\mathbf{r}_3$, $\mathbf{r}_4$, $\ldots$, $\mathbf{r}_k$ denoting the summed data vector for $\left[\mathbf{C}_3^k | \mathbf{C}_4^k | \cdots | \mathbf{C}_k^k\right]$, $\left[\mathbf{C}_4^k | \cdots | \mathbf{C}_k^k\right]$, $\ldots$, $\left[\mathbf{C}_k^k\right]$

TABLE IV

MUD for Noisy Channel
For stage-i ($1 \leq i \leq k$), Estimate • $\mathbf{i}_i = \mathbf{C}_{i-1}^k \hat{\mathbf{x}}_{i-1}^k$ • $\mathbf{y}_i = \mathbf{y}_{i-1} - \mathbf{i}_{i-1}$ (where $\mathbf{y}_1 = \mathbf{y}$, and $\mathbf{i}_1 = \mathbf{0}_{N_k \times 1}$) • $\mathbf{z}_i = \mathbf{y}_i \mathbf{C}_i^k$ • $\hat{\mathbf{x}}_i = \text{sign}(\mathbf{z}_i)$ Finally, $\hat{\mathbf{x}} = \{\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \cdots, \hat{\mathbf{x}}_k\}$ is the decoded vector.

respectively. This completes the proof. ■

From the above proofs (Lemma 1 and Theorem 2), the reduced complexity of the decoder's design can be speculated. Now, we present a practically implementable design of the decoder for noisy transmission in Table IV. The only deviation that needs to be noted in the following steps is the substitution of **r** (noiseless received vector) by **y** (the noisy received vector in (1)).

Note 1: Careful observation reveals that the proposed code set $\mathbf{C}^k$ is a subset of the UD matrix generated at the kth iteration of the construction proposed in [11], [36]. This implies that for these matrices, all the features of the proposed system can be realized, if and only if the transmission corresponding to the specific signatures (besides $\mathbf{C}^k$) remain inactive. In other words, the system in [11], [36] can have a low complex decoder for noisy transmission at the cost of sacrificing the asymptotic equality of $\beta(k)$ with $S_{sum}(k)$.

V. Performance Analysis

A. Bit Error Rate

While for the noisefree transmission, the proposed MUD guarantees for unique detectability, existence of error in the recovery of $\mathbf{x}_k$ is inevitable, when the channel gets noisy. From Table IV, the elaborated expression of $\mathbf{z}_i$ corresponding to the subset $\mathbf{C}_i^k$ can be written as

$$\mathbf{z}_i = \left(\frac{N_{ef_i}}{N_k}\right) \mathbf{I}\mathbf{x}_i + \sum_{u=(i+1)}^{k} \boldsymbol{\rho}_{iu} \mathbf{x}_u + \sum_{v=1}^{i-1} \boldsymbol{\rho}_{iv} (\mathbf{x}_v - \hat{\mathbf{x}}_v) + \mathbf{n}_i \tag{11}$$

where $\mathbf{x}_i$ is the desired input vector, $\boldsymbol{\rho}_{iu}$ and $\boldsymbol{\rho}_{iv}$ denote the cross-correlation matrix of the desired subset ($\mathbf{C}_i^k$) with respect to the child subsets (already detected, and denoted as $\mathbf{C}_v^k$) and the root subsets (to be detected, and denoted as $\mathbf{C}_u^k$). In (11), the terms involving $\boldsymbol{\rho}_{iu}$ and $\boldsymbol{\rho}_{iv}$ represent the sources of MAI where $(\mathbf{x}_v - \hat{\mathbf{x}}_v)$ indicates the error introduced during decoding in previous $(i-1)$ iterations and $\mathbf{n}_i = \left(\mathbf{C}_i^k\right)^T \mathbf{n} =$ AWGN vector with zero mean.

To determine the probability of error for the jth user of $\mathbf{C}_i^k$, let us rewrite the expression in (11), such that

$$z_{ij} = \left(\frac{N_{ef_i}}{N_k}\right) x_{ij} + \sum_{u=(i+1)}^{k} \rho_{iu(j)} \mathbf{x}_u + \sum_{v=1}^{i-1} \rho_{iv(j)} (\mathbf{x}_v - \hat{\mathbf{x}}_v) + n_{i(j)} \tag{12}$$

where $\mathbf{x}_i = \left[x_{i1} x_{i2} \cdots x_{i\frac{N_k}{2^{i-1}}}\right]$, $\mathbf{z}_i = \left[z_{i1} z_{i2} \cdots z_{i\frac{N_k}{2^{i-1}}}\right]$, and $\mathbf{n}_i = \left[n_{i(1)} n_{i(2)} \cdots n_{i(N_k)}\right]$. Following the central limit theorem, the terms (second and third) contributing to MAI in (12) can be considered as the secondary source of noise. Hence, this is logical to simplify the expression in (12) as

$$z_{ij} = \left(\frac{N_{ef_i}}{N_k}\right) x_{ij} + \eta_{ij}. \tag{13}$$

With the expression in (13) representing the communication system model for a single user BPSK system, we can write the probability of error of the user-j in subset-i as

$$P_e^{ij} = \frac{1}{\sqrt{2\pi}\sigma} \int_{\frac{N_{ef_i}}{N} x_{ij}}^{\infty} e^{-\left(\eta_{ij}^2/\sigma^2\right)} d\eta_{ij} \tag{14}$$

which on further modification can also be written as

$$P_e^{ij} = \frac{1}{2} erfc\left(\frac{\frac{N_{ef_i}}{N} x_{ij}}{\sqrt{2}\sigma}\right) = Q\left(\sqrt{\frac{\left(\frac{N_{ef_i}}{N}\right)^2 \mathrm{E}\left(x_{ij}^2\right)}{\sigma^2}}\right). \tag{15}$$

For better accuracy, the value of σ^2 in (15) is estimated to be

$$\mathrm{E}\left({\eta_{ij}}^2\right) = \sum_{u=(i+1)}^{k} \rho_{iu(j)}^2 \mathrm{E}\left({\mathbf{x}_u}^2\right) + \sum_{v=1}^{i-1} \rho_{iv(j)}^2 \mathrm{E}\left((\mathbf{x}_v - \hat{\mathbf{x}}_v)^2\right) + \mathrm{E}\left({n_{i(j)}}^2\right) \tag{16}$$

In (16),

$$\mathrm{E}\left((\mathbf{x}_v - \hat{\mathbf{x}}_v)^2\right) = 0{P_c}^v + 4{P_e}^v$$

where P_c^v and P_e^v denote the probability of correctness and error in decoding, corresponding to the binary input. Therefore, the resultant expression for P_e^{ij} is derived to be

$$P_e^{ij} = Q\left(\sqrt{\frac{\left(\frac{N_{ef_i}}{N}\right)^2 \mathrm{E}\left(x_{ij}^2\right)}{\sum\limits_{u=i+1}^{k} \rho_{iu(j)}^2 + 4\sum\limits_{v=1}^{i-1} \rho_{iu(j)}^2 P_e^v + \mathrm{E}\left(n_{i(j)}^2\right)}}\right), \tag{17}$$

where $\mathrm{E}\left({n_{i(j)}}^2\right) = ({}^{N_o}/_2)$.

B. Error Performance of Individual Subsets

For the CDMA system employing the ternary matrices, error performance of a user or group of users using the linear decoder is primarily influenced by two crucial metrics i.e.; the net level of MAI and the available diversity (spreading gain). Therefore, we may expect the hierarchy of MAI in (8) and the order of N_{ef} defined as

$$N_{ef_1} > N_{ef_2} > \cdots > N_{ef_k} \tag{18}$$

to jointly approve for the following order in BER among the k subsets.

$$\mathrm{BER}_{\mathbf{C}_1^k} < \mathrm{BER}_{\mathbf{C}_2^k} < \cdots < \mathrm{BER}_{\mathbf{C}_k^k}, \tag{19}$$

In the above expressions, N_{ef_i} and $\mathrm{BER}_{\mathbf{C}_i^k}$ denote the effective spreading gain and the average BER for subset $\mathbf{C}_i^k$. Here, this is important to realize that the value of N_{ef_i} is directly proportional to the transmitted power associated with a signature in subset-i. Now, it is worthy to follow the derived expression in (17) and explain behavior of the curves representing the average BER of the individual subsets.

For lower values of E_b/N_o (or higher values of $\mathrm{E}\left(n_{i(j)}^2\right)$ in the denominator in (17)), when the level of MAI is constant, the order of available diversity (N_{ef_i} in the numerator) primarily dominates and results in the order of error performance of the subsets, as shown in (19). On a closer investigation of the MUD, the flow of the algorithm appears to be sequential i.e.; the subset with lower value of N_{ef} or at lower level of the tree is recovered only after the decoding of that with higher value of N_{ef} or at higher level. Consequently, the BER performance of the latter being improved than that of the former is perceptible. Nevertheless, for the higher values of E_b/N_o, there exists the possibility of *unusual variation* in the order of their BER. Therefore, the analysis takes a different turn with the following elaboration.

For higher values of E_b/N_o (or lower values of $\mathrm{E}\left(n_{i(j)}^2\right)$ in the denominator in (17)), the factor that crucially controls the quality of recovery of the immediate next subset ($\mathbf{C}_{i+1}^k$) is the net level of MAI, due to the remaining subsets ($\mathbf{C}_{i+2}^k, \ldots \mathbf{C}_k^k$). From Lemma 1, this is already clear that for $\left[\mathbf{C}_{i+1}^k|\mathbf{C}_{i+2}^k|\ldots|\mathbf{C}_k^k\right]$ being the decoding matrix, the net level of MAI on $\mathbf{C}_{i+1}^k$ is always less than that on $\mathbf{C}_i^k$ corresponding to the decoding matrix of $\left[\mathbf{C}_i^k|\mathbf{C}_{i+1}^k|\ldots|\mathbf{C}_k^k\right]$. So, the lowering in level of BER for $\mathbf{C}_{i+1}^k$ (lowering of first term in denominator in (17)), as compared to $\mathbf{C}_i^k$ is expected over a certain higher range of E_b/N_o, even if it carries a low order diversity. This indicates that there exists a high probability that the impact of MAI will dominate over that of the diversity. As a result, the subsets spaced at the bottom level of the tree will perform better than that of the top level. The value of E_b/N_o at which the fall in BER with respect to that of the subsets at higher level will start, is influenced by its level in the tree hierarchy (hence, the total level of intergroup MAI) and the value of N_{ef}.

C. Complexity

The proposed decoder deciphers all the k orthogonal subsets of $\mathbf{C}^k$ in k sequential stages. The detection of a particular subset at a specific stage is achieved by simple logic of matched filtering. Moreover, each stage is followed by an intermediate stage meant for the estimation and cancellation of the interference due the subsets, already detected. However, the complexity rise if compared with that of the iterative cancellation techniques involved in [15]–[17], [23], [25], [42] is found to be highly marginal. Hence, the performance of the decoder as compared to that of the optimum MLD [38] shows a catastrophic saving in complexity. We recall that the method of detection using optimum MLD strictly demands the calculation of 2^{M_k} Euclidean Distance (ED) vectors of length N_k, adopting which the rise in complexity behaves exponentially with the value of N_k.

VI. Simulation Results

In this section, we focus on the BER versus (E_b/N_0) performance of the proposed system, assuming the channel to be AWGN. The system is supposed to be BPSK modulated and perfectly power controlled.

Fig. 3 is meant to offer an insight of the efficiency of SMOS as compared to two other class of codes: binary random and WBE sequences. Where for SMOS (Type- I, II, III), we consider the

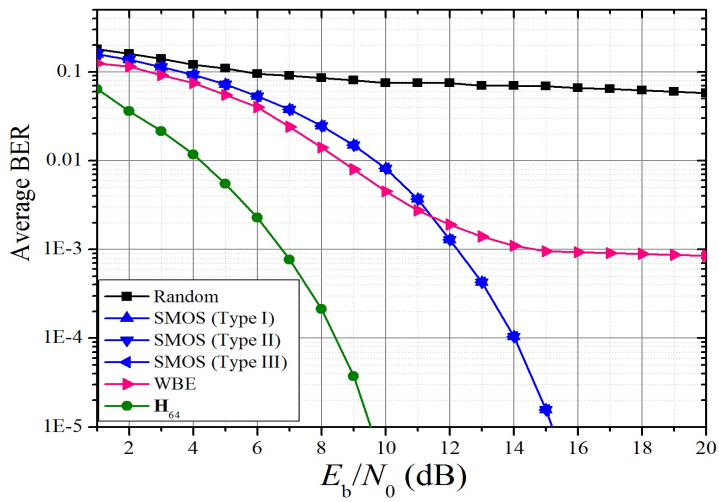

Fig. 3: BER versus (E_b/N_o) performance for three different systems of dimension (64×96): ternary SMOS (Type I, II, III), binary random and BWBE with iterative decoder.

proposed MUD (as shown in Table IV), an iterative decoder with soft limiting [44] is preferred for the random and WBE sequences. To have the uniformity in analysis, the matrix dimension for all is kept at (64×96) leading to $\beta = 1.5$. Simulation of conventional CDMA using Hadamard matrix of dimension (64×64) is also included as the performance benchmark. As evident, for $E_b/N_o < 11$ dB, a marginal improvement in BER of the WBE codes as compared to SMOS is observed. But, for $E_b/N_o > 11$ dB, the level of BER of WBE saturates. This is because the mapping of binary WBE codes unlike the UD matrices lags the invertible characteristic. Therefore, by no means, it is possible to reduce the BER below the error floor, even not by enhancing the E_b/N_o to infinite. In contrast, for SMOS, the gradual approach of the BER level to zero with increase in E_b/N_o is justified. Also, in spite of the varied architectures involved in constructions, the coveted equality in the level of error performance for all the three types of SMOS is shown. Hence, onwards, for all other simulations, we consider the SMOS (Type I) only and this is not uncommon to expect the Type- II and III to carry the similar performance profile.

In Fig. 4, the variation in error performance of SMOS (Type-I), subjected to the increase in value of β (for $k = 6$ or $N_k = 64$) is presented. We start with the largest subset of $\mathbf{C}^6_{64\times126}$

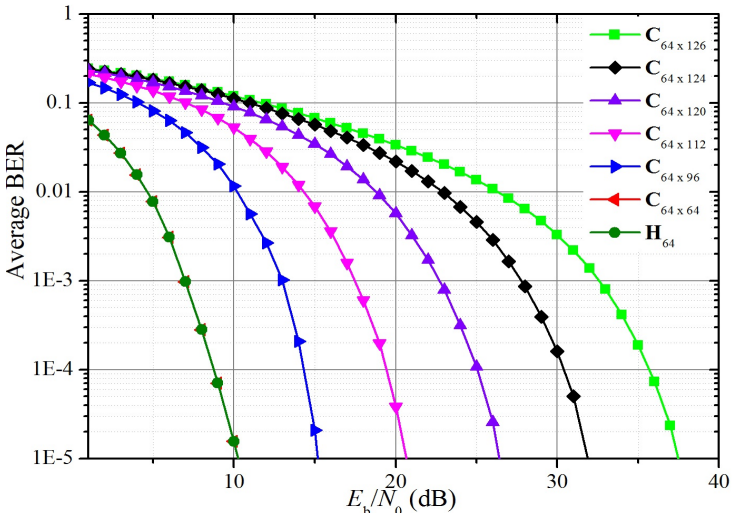

Fig. 4: Variation in BER versus (E_b/N_o) for SMOS (Type I) with increase in matrix dimension for $k = 6$ i.e., $N_k = 64$: $\mathbf{C}_{64\times64} = [\mathbf{C}_1^6]$, $\mathbf{C}_{64\times96} = [\mathbf{C}_1^6|\mathbf{C}_2^6]$, $\mathbf{C}_{64\times112} = [\mathbf{C}_1^6|\mathbf{C}_2^6|\mathbf{C}_3^6]$, $\mathbf{C}_{64\times120} = [\mathbf{C}_1^6|\mathbf{C}_2^6|\mathbf{C}_3^6|\mathbf{C}_4^6]$, $\mathbf{C}_{64\times124} = [\mathbf{C}_1^6|\mathbf{C}_2^6|\mathbf{C}_3^6|\mathbf{C}_4^6|\mathbf{C}_5^6]$, $\mathbf{C}_{64\times126} = [\mathbf{C}_1^6|\mathbf{C}_2^6|\mathbf{C}_3^6|\mathbf{C}_4^6|\mathbf{C}_5^6|\mathbf{C}_6^6]$ corresponding to $\beta = 1, 1.5, 1.75, 1.875, 1.94, 1.97$.

(i.e., $\mathbf{C}_1^6$) and subsequently, smaller subsets ($\mathbf{C}_2^6$, $\mathbf{C}_3^6$, $\mathbf{C}_4^6$, $\mathbf{C}_5^6$, $\mathbf{C}_6^6$) are added one-by-one in order to realize six different enhanced loading conditions: $\beta = 1.5, 1.75, 1.875, 1.94, 1.97$. For $\mathbf{C}_1^6$, the error performance being identical to that of $\mathbf{H}_{64}$ is obvious, since $\mathbf{C}_1^6 = \mathbf{H}_{64}$. The further degradation in the level of BER with the increase in β is due to the hike, both in the level of MAI and number of stages involved in decoding.

Fig. 5 (a) and (b) illustrates the error performance of the individual subsets of $\mathbf{C}^6 = \mathbf{C}_{64\times126} = [\mathbf{C}_1^6|\mathbf{C}_2^6|\mathbf{C}_3^6|\mathbf{C}_4^6|\mathbf{C}_5^6|\mathbf{C}_6^6]$, when the detection is achieved by the proposed MUD and the conventional MF decoder respectively. While the curves in Fig. 5 (b) validates the expression in (8), their behavior in Fig. 5 (a) shows an unprecedented variation over the range of E_b/N_o. This is already explained in detail in Section V (B). In Fig. 5 (b), the dramatic lowering in the level of BER of $\mathbf{C}_1^6$ is best explained by Lemma 1.

Fig. 6 illustrates the impact of the detection error introduced at one stage of the MUD in affecting the average error performance of the subsequent stages. In Fig. 6 (a) and (b), we show the average and the minimum level of the BER performance for the individual subsets respectively. To validate the tendency of the curves for the minimum level of BER, we follow

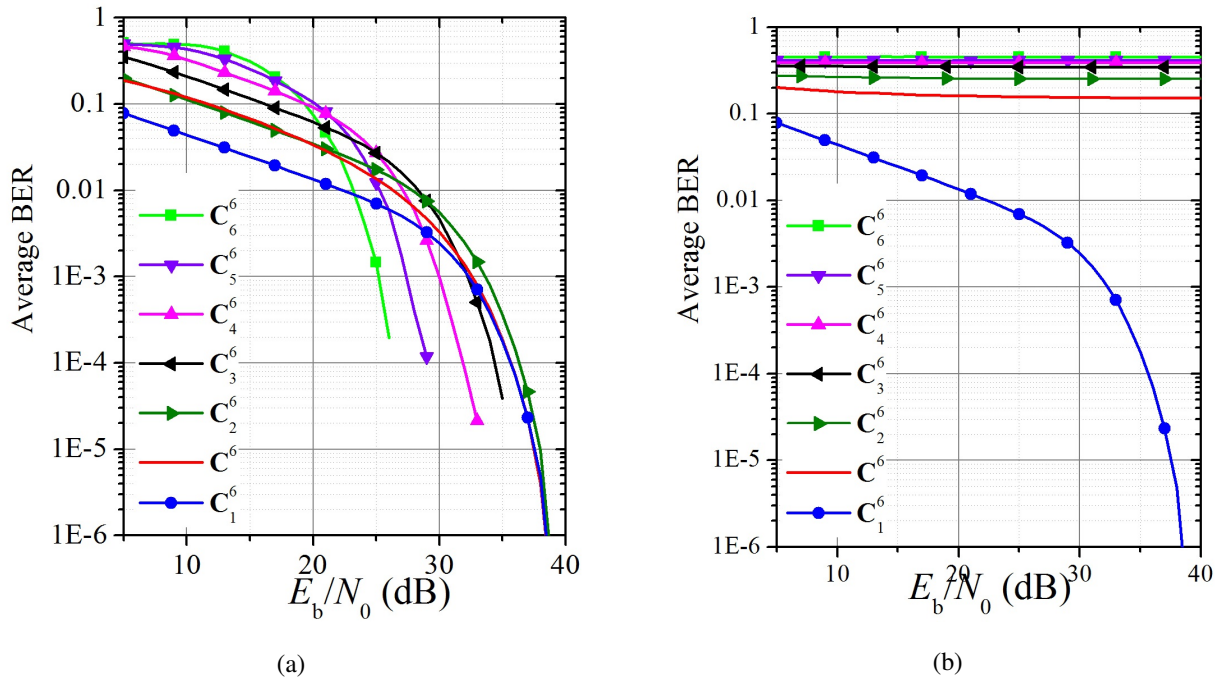

Fig. 5: BER versus E_b/N_o for individual subsets of $\mathbf{C}_{64\times126} = \left[\mathbf{C}_1^6|\mathbf{C}_2^6|\mathbf{C}_3^6|\mathbf{C}_4^6|\mathbf{C}_5^6|\mathbf{C}_6^6\right]$ for synchronous transmission for (a) proposed MUD (b) matched filter decoder

the statement of Lemma 1. It is just because Lemma 1 serves as the fundamental layout towards the designing of the proposed MUD, following whose extrapolation for the noisy transmission, the subset under the least MAI can be efficiently decoded with the minimum probability of error. So, in order to plot for the subsets $\mathbf{C}_1^6$, $\mathbf{C}_2^6$, $\mathbf{C}_3^6$, $\mathbf{C}_4^6$, $\mathbf{C}_5^6$, and $\mathbf{C}_6^6$, we selected the decoding matrices $\left[\mathbf{C}_1^6|\mathbf{C}_2^6|\mathbf{C}_3^6|\mathbf{C}_4^6|\mathbf{C}_5^6|\mathbf{C}_6^6\right]$, $\left[\mathbf{C}_2^6|\mathbf{C}_3^6|\mathbf{C}_4^6|\mathbf{C}_5^6|\mathbf{C}_6^6\right]$, $\left[\mathbf{C}_3^6|\mathbf{C}_4^6|\mathbf{C}_5^6|\boldsymbol{C}_6^6\right]$, $\left[\mathbf{C}_4^6|\mathbf{C}_5^6|\mathbf{C}_6^6\right]$, $\left[\mathbf{C}_5^6|\mathbf{C}_6^6\right]$, $\left[\mathbf{C}_6^6\right]$ respectively, such that each subset following the tree hierarchy of the corresponding matrix is under the least level of MAI. It further implies that in order to plot the minimum level of BER of a particular subset of $\mathbf{C}_{64\times126}$, we assume the subsets at the higher levels of the tree structure (if any) to be inactive (not transmitting). With this approach, our intention is to neglect the effect of MAI on the subsequent stages of MUD due to the detection error induced in previous stages (second term in the denominator in (17)). Still, in both the cases, the behavior of the subsets hardly shows any noticeable deviation as far as their order among the BER is concerned over a

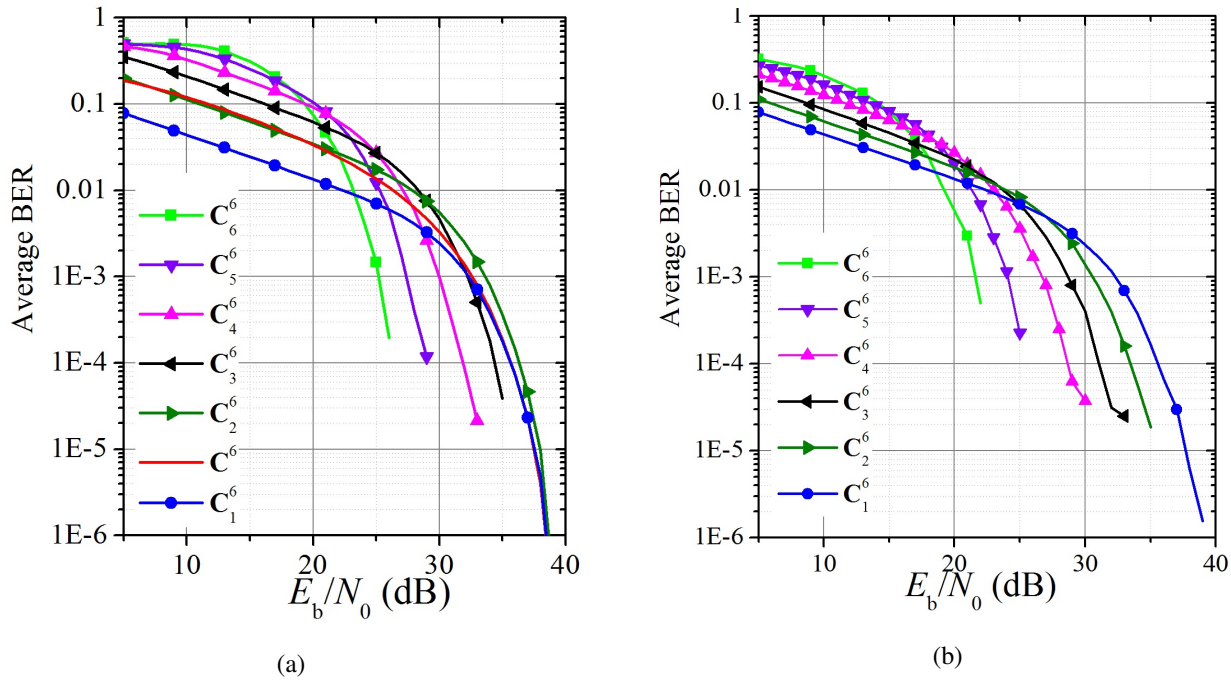

Fig. 6: For individual subsets of $\mathbf{C}_{64\times126}$: (a) average BER versus E_b/N_o for (b) minimum BER versus E_b/N_o

fixed span of E_b/N_o. Indirectly, this confirms that for the lower and higher values E_b/N_o, it is the two factors only: the value of N_{ef} and the net level of MAI on the subset, which controls this order. On the other hand, it also validates the fact the propagation error from one to next stage hardly has any command over this order, even though it has a marginal impact on the overall error performance of the subsets.

VII. Conclusion

We investigated the problem of designing a new set of UD matrices for overloaded synchronous CDMA system that carries multiple orthogonal subsets. Based on the study of literature, we identified the basis matrix and built up an unique perspective towards the construction of the proposed matrix that comprises of multiple orthogonal subsets. While the orthogonality of each subset ensured for the zero intra-group MAI, the existing inter-group MAI among the subsets is balanced by the advantageous pattern of the twin-tree structured cross-correlation hierarchy.

As a result, the linear decoding logic of matched filtering became efficient to support as the fundamental block of the decoder for the errorfree recovery for the noiseless channel and strongly retained its low complexity even for the noisy case. While analyzing the error performance, we emphasized more on the individual subsets than that of the overall matrix and observed the non-uniformity in their order. The impact of the spreading diversity and MAI being jointly responsible for the unusual behavior of the BER curves was clearly explained with reference to the derived expression of the average error performance. Moreover, the superiority in BER of the proposed code set over the binary random and WBE sequences was also shown through simulation. By and large, it may be concluded that imposing an efficient correlation structure on the signature set in CDMA can not only bring a dramatic reduction in complexity of the decoder but also guarantee for the better error performance.

In the future studies, we will investigate on other feasible matrix constructions with the tree-structured hierarchy of correlation. Along with, the system design involving more complicated wireless communication system with multi-path (or fading) channel, higher order modulation will also be considered.

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