Mathematics in Physical Chemistry

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Aim

- Your most valuable asset is your learning ability.
- This course is a practice in learning and specially improves your deduction skills.
- This course provides you with tools applicable in and necessary for modeling many natural phenomena.
- "The fundamental laws necessary for the mathematical treatment of a large part of physics and the whole of chemistry are thus completely known, and the difficulty lies only in the fact that application of these laws leads to equations that are too complex to be solved."
- The first part of the course reviews Linear algebra and calculus while introducing some very useful notations. In the second part of the course, we study ordinary differential equations.
- End of semester objective: Become familiar with mathematical tools needed for understanding physical chemistry.

Course Evaluation

Final exam	11 Bahman 3 PM	40%
Midterm exam	24 Azar 9 AM	40%
Quizz		10%
Class presentation		10%
- Office hours: Due to special situation of corona pandemic office hour is not set, email for an appointment instead.

To be covered in the course

- Complex numbers, Vector analysis and Linear algebra
- Vector rotation, vector multiplication and vector derivatives.
- Series expansion of analytic functions
- Integration and some theorems from calculus
- Dirac delta notation and Fourier transformation
- Curvilinear coordinates.
- Matrices

To be covered in the course

- When we know the relation between change in dependent variable with changes in independent variable we are facing a differential equation.
- The laws of nature are expressed in terms of differential equations. For example, study of chemical kinetics, diffusion and change in a systems state all start with differential equations.
- Analytically solvable ordinary differential equations.
- Due to lack of time a discussion of partial differential equations and a discussion of numerical solutions to differential equations are left to a course in computational chemistry.

References

- Modern calculus and analytic geometry by Richard A. Silverman
- "Mathematical methods for physicists", by George Arfken and Hans Weber
- Ordinary differential equations by D. Shadman and B. Mehri (A thin book in Farsi)
- Linear Algebra, Second Edition, Kenneth Hoffman, Ray Kanze
- The Raman effect by Derek A Long

Set theory

- Set, class or family
- $x \in A$, $y \notin A$
- $A \subset B$ or equivalently $B \supset A$
- $A = B$ iff $A \subset B \wedge B \subset A$, otherewise $A \neq B$.
- Proper subset
- The unique empty set $\emptyset$
- $B - A = \{x | x \in B \wedge x \notin A\}$
- Universal set, U.
- $A^c = U - A$
- Venn diagrams
- $A \cup B$ (A cup B)
- $A \cap B$ (A cap B)
- disjoint set.

Relation

- Symmetric difference, $A \Delta B = (A - B) \cup (B - A)$
- Ordered n-tuples vs. sets.
- Cartesian product of A and B, $A \times B = \{(a, b) | a \in A, b \in B\}$
- $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) | a_1 \in A_1, a_2 \in A_2, \cdots a_n \in A_n\}$
- Real numbers, R
- 2-space $=$ R^2
- 3-space $=$ R^3
- A relationship is defined as a set of ordered pairs (x,y), i.e., a relation S from the set X to the set Y is a subset of the cartesian product $X \times Y$.
- Where the sets X and Y are the same there are 3 types of relations.
- A many to one or one to one relation from a set X to a set Y is called a function from X to Y.

Function

- A function from the set of positive integers to an arbitrary set is called an infinite sequence or sequence, $y = y_n$ or $y_1, y_2, \cdots, y_n, \cdots$ or $y_n \quad (n = 1, 2, \cdots)$ or $\{y_n\}$
- Dom f $= \{$x$|(x, y) \in f$ for some y$\}$, Rng f $= \{$y$|(x, y) \in f$ for some x$\}$
- $f \subset Dom\ f \times Rng\ f \subset X \times Y$
- Arbitrary set X to R, real valued function
- R to R, real function of one real variable
- R to R^n, point valued function or vector function of one real variable.
- R^n to R, real function of sevral real variables
- R^n to R^p, coordinate transformation, each coordinate of the point $y = (y_1, \cdots, y_p) \in R^p$ is a dependent variable
- Domain as the largest set of values for which the formula makes sense.

Complex numbers

- "Imaginary numbers are a fine and wonderful refuge of the divine spirit, almost an amphibian between being and non-being." Gottfried Leibnitz
- Fundamental theorem of algebra: "Every non-constant single-variable polynomial has at least one complex root."
- $X^2 + 1 = 0$ defines $x = i = \sqrt{-1}$. Complex number $x = a + bi = (a, b) = ce^{\theta i}$.
- Complex conjugate, Complex plane, summation, multiplication, division, and logarithm.
- Euler formula, "our jewel", $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$ for real α
- Proof by Taylor expansion
- $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, $\quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$.
- $\cosh(y) = \cos(iy) = \frac{e^{y}+e^{-y}}{2}$,
  $i\sinh(y) = \sin(iy) \to \sinh y = \frac{e^{y}-e^{-y}}{2}$.
- $\cos(x)\cdot\cos(y) = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$,
- $\cos(x+y) = \cos x\cos y - \sin x\sin y$,
  $\sin(x+y) = \sin x\cos y + \cos x\sin y$.

Coordinate System

- Rectangular cartesian coordinate system is a one to one correspondence between ordered sets of numbers and points of space.
- Ordinate (vertical) vs. abscissa (horizontal) axes.
- Round or curvilinear coordinate system
- Curvilinear coordinates are a coordinate system for Euclidean space in which the coordinate lines may be curved, e.g., rectangular, spherical, and cylindrical coordinate systems.
- Coordinate surfaces of the curvilinear systems are curved.
- Plane polar coordinate system,
  $x = r\cos\theta, \quad y = r\sin\theta, \qquad dS = rdrd\theta,$
- Spherical polar coordinates
- $x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \quad dV = r^2\sin\theta drd\phi d\theta$
- Rectangular coordinates

Coordinate System

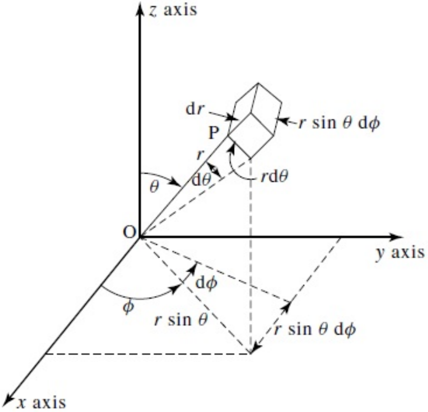
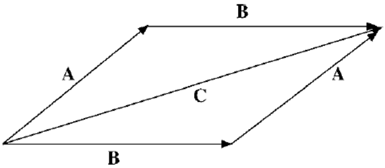
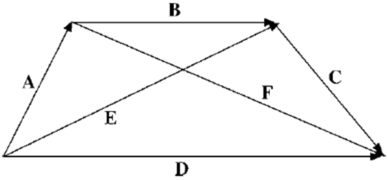


Figure: Polar coordinates taken from The Raman Effect by DA Long

Vector analysis

- Scalar quantities have magnitude vs. vector quantities which have magnitude and direction.
- Triangle law of vector addition.
- Parallelogram law of vector addition (Allows for vector subtraction), further it shows commutativity and associativity.





Vector analysis: direction cosines

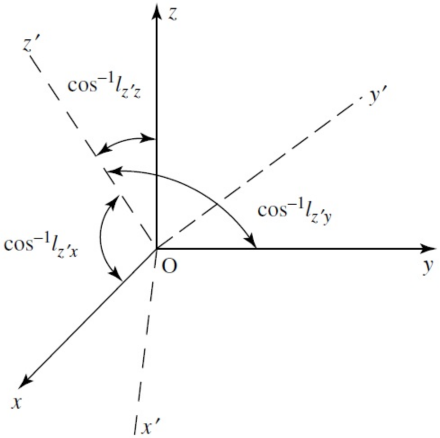


Figure: Direction Cosines taken from The Raman Effect by DA Long

- Direction cosines, projections of $\vec{A}$.
- Geometric or algebraic representation.

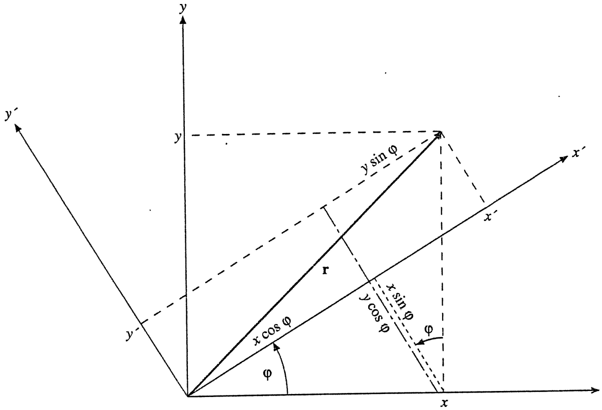
Vector analysis: direction cosines

- Consider two vectors of unit length
  $r_1 = (x_1, y_1, z_1) = (l_{r_1 x}, l_{r_1 y}, l_{r_1 z})$ and
  $r_2 = (x_2, y_2, z_2) = (l_{r_2 x}, l_{r_2 y}, l_{r_2 z})$
- For the angle between r_1 and r_2,
  $\cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2 = l_{r_1 x} l_{r_2 x} + l_{r_1 y} l_{r_2 y} + l_{r_1 z} l_{r_2 z}$
- For orthogonal unit vectors r and r', $l_{rx} l_{r'x} + l_{ry} l_{r'y} + l_{rz} l_{r'z} = 0$
- Also, $l_{rx}^2 + l_{ry}^2 + l_{rz}^2 = 1$
- Thus, $l_{x'x}^2 + l_{x'y}^2 + l_{x'z}^2 = 1 \qquad l_{y'x}^2 + l_{y'y}^2 + l_{y'z}^2 = 1 \qquad l_{z'x}^2 + l_{z'y}^2 + l_{z'z}^2 = 1$.
- Further,
  $l_{x'x} l_{x'y} + l_{y'x} l_{y'y} + l_{z'x} l_{z'y} = 0 \qquad l_{x'Y} l_{x'Z} + l_{y'y} l_{y'z} + l_{z'y} l_{z'z} = 0 \qquad l_{x'z} l_{x'x} + l_{y'z} l_{y'x} + l_{z'z} l_{z'x} = 0$.
- In summary, $l_{\rho'\rho} l_{\sigma'\rho} = \delta_{\rho'\sigma'} \qquad l_{\rho'\rho} l_{\rho'\sigma} = \delta_{\rho\sigma}$

Vector analysis

- Unit vectors, $\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$.
- Expansion of vectors in terms of a set of linearly independent basis allow algebraic definition of vector addition and subtraction, i.e.,
  $\vec{A} \pm \vec{B} = \hat{x}(A_x \pm B_x) + \hat{y}(A_y \pm B_y) + \hat{z}(A_z \pm B_z)$.
- $|A|$, Norm for scalars and vectors.
- $A_x = |A| \cos \alpha, \quad A_y = |A| \cos \beta, \quad A_z = |A| \cos \gamma$
- Pythagorean theorem,
  $|A|^2 = A_x^2 + A_y^2 + A_z^2, \qquad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$
- Vector field: An space to each point of which a vector is associated.
- Direction of vector r is coordinate system independent.

Rotation of the coordinate axes



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- $x' = x\cos\phi + y\sin\phi \qquad y' = -x\sin\phi + y\cos\phi$
- Since each vector can be represented by a point in space a vector field A is defined as an association of vectors to points of space such that
  $A'_x = A_x\cos\phi + A_y\sin\phi \qquad A'_y = -A_x\sin\phi + A_y\cos\phi$

N-dimensional vectors

- $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
- $x \to x_1, \quad y \to x_2, \quad z \to x_3$
- $x_i' = \sum_{j=1}^{N} a_{ij} x_j; \qquad i = 1, 2, \cdots, N; \qquad a_{ij} = \cos(x_i', x_j)$.
- In Cartesian coordinates, $x_i' = \cos(x_i', x_1)x_1 + \cos(x_i', x_2)x_2 + \cdots = \sum_j a_{ij} x_j$ thus $a_{ij} = \frac{\partial x_i'}{\partial x_j}$.
- By considering primed coordinate axis to rotate by $-\phi$, $x_j = \sum_i \cos(x_j, x_i')x_i' = \sum_i \cos(x_i', x_j)x_i' = \sum_i a_{ij} x_i'$ resulting in $a_{ji} = \frac{\partial x_j}{\partial x_i'} = a_{ij}$.
- A is the matrix whose effect is the same as rotating the coordinate axis, whose elements are a_{ij}.

Vectors and vector space

- Orthogonality condition for A: $A^T A = I$ or

$$\sum_i a_{ij} a_{ik} = \sum_i \frac{\partial x_i'}{\partial x_j} \frac{\partial x_i'}{\partial x_k} = \sum_i \frac{\partial x_j}{\partial x_i'} \frac{\partial x_i'}{\partial x_k} = \frac{\partial x_j}{\partial x_k} = \delta_{jk}$$

- By depicting a vector as an n-tuple, $B = (B_1, B_2, \cdots, B_n)$, define:
- Vector equality.
- Vector addition
- Scalar multiplication
- Unique Null vector
- Unique Negative of vector
- Addition is commutative and associative. Scalar multiplication is distributive and associative.

- A group is a set, G, together with an operation * (called the group law of G) that combines any two elements a and b to form another element, denoted a*b or ab.
- Closure: For all a, b in G, the result of the operation, a*b, is also in G.
- Associativity: For all a, b and c in G, (a*b)*c = a*(b*c).
- Identity element: There exists an element e in G such that, for every element a in G, the equation e*a = a*e = a holds. Such an element is unique, and thus one speaks of the identity element.

Group

- Inverse element: For each a in G, there exists an element b in G, commonly denoted a^{-1} (or -a, if the operation is denoted "+"), such that a*b = b*a = e, where e is the identity element.
- Groups for which the commutativity equation, a*b = b*a, always holds are called abelian groups.
- The identity element of a group G is often written as 1 or 1_G a notation inherited from the multiplicative identity.
- If a group is abelian, then one may choose to denote the group operation by + and the identity element by 0; in that case, the group is called an additive group.

Group

- There can be only one identity element in a group, and each element in a group has exactly one inverse element.
- The existence of inverse elements implies that division is possible
- E.g., the set of integers together with the addition operation, but groups are encountered in numerous areas, and help focusing on essential structural aspects.
- Point groups are used to help understand symmetry phenomena in molecular chemistry.
- The symmetry group is an example of a group that is not abelian.

Rings

- A ring consists of a set equipped with two binary operations that generalize the arithmetic operations of addition and multiplication.
- A ring is an abelian group with a second binary operation that is associative, is distributive over the abelian group operation, and has an identity element
- Examples of commutative rings include the set of integers equipped with the addition and multiplication operations, the set of polynomials equipped with their addition and multiplication
- Examples of noncommutative rings include the ring of $n \times n$ real square matrices with $n \geq 2$.

Rings

- A ring is a set R equipped with two binary operations $+$ and $\cdot$
  R is an abelian group under addition:
  $(a + b) + c = a + (b + c)$ for all a, b, c in R .
  $a + b = b + a$ for all a, b in R.
  There is an element 0 in R such that $a + 0 = a$ for all a in R.
  For each a in R there exists -a in R such that $a + (-a) = 0$.
  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in R.
  There is an element 1 in R such that $a \cdot 1 = a$ and $1 \cdot a = a$ for all a in R
  Multiplication is distributive with respect to addition:
  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all a, b, c in R.
  $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all a, b, c in R.

Field

- Field is a set on which addition, subtraction, multiplication, and division are defined, and behave as the corresponding operations on rational and real numbers do.
- There exist an additive inverse -a for all elements a, and a multiplicative inverse b^{-1} for every nonzero element b.
- An operation is a mapping that associates an element of the set to every pair of its elements.
- Associativity of addition and multiplication
- Commutativity of addition and multiplication
- Additive and multiplicative identity
- Additive inverses
- Multiplicative inverses
- Distributivity of multiplication over addition
- The best known fields are the field of rational numbers, the field of real numbers and the field of complex numbers.

Linear vector spaces

- A vector space over a field F is a set V together with two operations that satisfy axioms listed below.
- Vector addition $+ : V \times V \to V$, takes any two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ and assigns to them a third vector commonly written as $\overrightarrow{v} + \overrightarrow{w}$.
- Scalar multiplication $\cdot : F \times V \to V$, takes any scalar a and any vector $\overrightarrow{v}$ and gives another vector $\overrightarrow{av}$. (The vector $\overrightarrow{av}$ is an element of the set V). Elements of V are called vectors. Elements of F are called scalars.
- Phase space in which classical mechanics occur and Hilbert space in which quantum mechanics occur are most important examples.

Linear vector spaces

Axiom	Meaning
Associativity of addition	$\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
Commutativity of addition	$\vec{u}+\vec{v}=\vec{v}+\vec{u}$
Identity element of addition	$\exists\vec{0}\in V$, called the zero vector, such that $\vec{v}+\vec{0}=\vec{v}\ \forall\vec{v}\in V$.
Inverse elements of addition for every $\vec{v}\in V$,	$\exists\overrightarrow{-v}\in V$, called the additive inverse of $\vec{v}$, such that $\vec{v}+(\overrightarrow{-v})=\vec{0}$
Compatibility of scalar multiplication with field multiplication	a$\overrightarrow{(bv)}=(ab)\vec{v}$
Identity element of scalar multiplication $1\vec{v}=\vec{v}$,	1 denotes the multiplicative identity in F
Distributivity of scalar multiplication with respect to vector addition	$a(\vec{u}+\vec{v})=a\vec{u}+a\vec{v}$
Distributivity of scalar multiplication with respect to field addition	$(a+b)\vec{v}=a\vec{v}+b\vec{v}$

Algebra over a field

- a vector space equipped with a bilinear product.
- an algebra is an algebraic structure, which consists of a set, together with operations of multiplication, addition, and scalar multiplication by elements of the underlying field, and satisfies the axioms implied by "vector space" and "bilinear".
- The multiplication operation in an algebra may or may not be associative, leading to the notions of associative algebras and nonassociative algebras.
- Given an integer n, the ring of real square matrices of order n is an example of an associative algebra over the field of real numbers under matrix addition and matrix multiplication since matrix multiplication is associative.

Algebra over a field

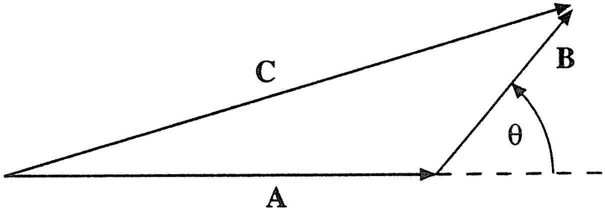
- Three-dimensional Euclidean space with multiplication given by the vector cross product is an example of a nonassociative algebra over the field of real numbers since the vector cross product is nonassociative, satisfying the Jacobi identity instead.
- An algebra is unital or unitary if it has an identity element with respect to the multiplication.
- The ring of real square matrices of order n forms a unital algebra since the identity matrix of order n is the identity element with respect to matrix multiplication.
- Algebras are not to be confused with vector spaces equipped with a bilinear form, like inner product spaces, as, for such a space, the result of a product is not in the space, but rather in the field of coefficients.

Scalar or dot product

- Real n-tuples labeled $\mathbb{R}^n$, complex n-tuples are labeled $\mathbb{C}^n$.
- Inner product should be distributive and associative.
  $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C} \qquad \vec{A}\cdot(y\vec{B}) = (y\vec{A})\cdot\vec{B} = y\vec{A}\cdot\vec{B}$
- Algebraic definition: $\vec{A},\vec{B}\in\mathbb{R}^n \quad \vec{A}\cdot\vec{B}\equiv\sum_i A_i B_i$
- $\vec{A},\vec{B}\in\mathbb{C}^n \quad \vec{A}\cdot\vec{B}\equiv\sum_i A_i^* B_i$
- Dot product of A by a unit vector is the length of A's projection into unit vectors direction.
- $A_x = |A|\cos\alpha \equiv \vec{A}\cdot\hat{x}, \quad A_y = |A|\cos\beta \equiv \vec{A}\cdot\hat{y}, \quad A_z = |A|\cos\gamma \equiv \vec{A}\cdot\hat{z}$.
- Geometric definition: $\vec{A}\cdot\vec{B} = A_B B = A B_A = AB\cos\theta$
- $\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$
- $\hat{x}\cdot\hat{y} = \hat{x}\cdot\hat{z} = \hat{z}\cdot\hat{y} = 0$
- Perpendicular or orthogonal vectors.
- $\hat{x} = e_1, \hat{y} = e_2, \hat{z} = e_3; \qquad e_m\cdot e_n = \delta_{mn}$

Invariance of Scalar or dot product under rotation

- $\vec{B}' \cdot \vec{C}' = \sum_l B'_l C'_l = \sum_l \sum_i \sum_j a_{li} B_i a_{lj} C_j = \sum_{ij} (\sum_l a_{li} a_{lj}) B_i C_j = \sum_{ij} \delta_{ij} B_i C_j = \sum_i B_i C_i = \vec{B} \cdot \vec{C}$; thus dot product is scalar.
- $\vec{C} = \vec{A} + \vec{B}, \qquad \vec{C} \cdot \vec{C} = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) = \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} + 2\vec{A} \cdot \vec{B} \rightarrow \overrightarrow{A} \cdot \overrightarrow{B} = \frac{1}{2}(C^2 - A^2 - B^2)$. Therefore, $\overrightarrow{A} \cdot \overrightarrow{B}$ is a scalar.
- Another derivation for cosine law, $C^2 = A^2 + B^2 + 2AB \cos\theta$



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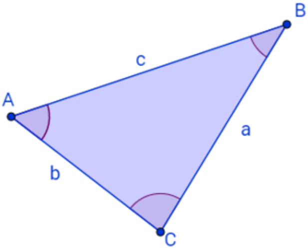
- This reminds us of the sine law: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = d$



Law of sines

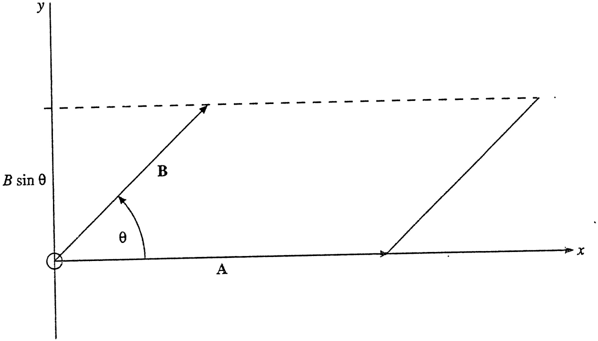
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

- Triangle area,
  $S = \frac{1}{2}ah_a = \frac{1}{2}a(b\sin C) = \frac{1}{2}a(c\sin B) = \frac{1}{2}ch_c = \frac{1}{2}c(b\sin A)$.
- $\frac{1}{2}a(b\sin C) = \frac{1}{2}a(c\sin B) = \frac{1}{2}c(b\sin A)$

Vector or cross product

- Geometric definition: $\overrightarrow{C} = \overrightarrow{A} \times \overrightarrow{B}$ $\quad C = AB\sin\theta$, $\overrightarrow{C}$ is a vector perpendicular to the plane of $\overrightarrow{A}$ and $\overrightarrow{B}$ such that $\overrightarrow{A}$ and $\overrightarrow{B}$ and $\overrightarrow{C}$ form a right-handed system.
- Cross product is non-commutative. $\overrightarrow{A} \times \overrightarrow{B} = -\overrightarrow{B} \times \overrightarrow{A}$
- $\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0$
- $\hat{x} \times \hat{y} = \hat{z}, \quad \hat{x} \times \hat{z} = -\hat{y}, \quad \hat{z} \times \hat{y} = -\hat{x}$
- Angular momentum, $\overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p}$; torque, $\overrightarrow{\tau} = \overrightarrow{r} \times \overrightarrow{F}$ and magnetic force, $\overrightarrow{F}_M = q\overrightarrow{v} \times \overrightarrow{B}$.
- Treating area as a vector quantity.

Vector or cross product



- $\overrightarrow{A} \times \overrightarrow{B} \equiv \overrightarrow{C} = (C_x, C_y, C_z) =$
  $(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) =$
  $(A_xB_y - A_yB_x)\hat{x}\times\hat{y} + (A_xB_z - A_zB_x)\hat{x}\times\hat{z} + (A_yB_z - A_zB_y)\hat{y}\times\hat{z}$
- $C_x = A_yB_z - A_zB_y, \quad C_y = A_zB_x - A_xB_z, \quad C_z = A_xB_y - A_yB_x.$
- $C_i = A_jB_k - A_kB_j$, i,j and k are different.

Vector or cross product

- $\overrightarrow{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$
- $\overrightarrow{A} \cdot \overrightarrow{C} = \overrightarrow{A} \cdot (\overrightarrow{A} \times \overrightarrow{B}) = A_x(A_y B_z - A_z B_y) + A_y(A_z B_x - A_x B_z) + A_z(A_x B_y - A_y B_x) = 0.$
- $\overrightarrow{B} \cdot \overrightarrow{C} = \overrightarrow{B} \cdot (\overrightarrow{A} \times \overrightarrow{B}) = 0.$
- $(\overrightarrow{A} \times \overrightarrow{B}) \cdot (\overrightarrow{A} \times \overrightarrow{B}) = A^2 B^2 \sin^2\theta.$

Levi-Civita symbol

- Levi-Civita symbol, permutation symbol, antisymmetric symbol, or alternating symbol. $\epsilon_{\cdots i_p \cdots i_q \cdots} = -\epsilon_{\cdots i_q \cdots i_p \cdots}$
- $\epsilon_{i_1 i_2 \cdots i_n} = (-1)^p \epsilon_{12\cdots n}$.

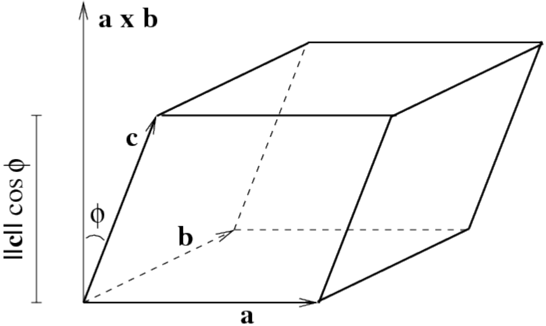
$$\epsilon_{i_1 i_2 \cdots i_n} = \begin{cases} +1 & \text{if } (i_1, i_2, \cdots, i_n) \text{ is an even permutation of } (1, 2, \cdots, n) \\ -1 & \text{if } (i_1, i_2, \cdots, i_n) \text{ is an odd permutation of } (1, 2, \cdots, n) \\ 0 & \text{otherwise (no permutation, repeated index)} \end{cases}$$

- $\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl}$
- $\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{imn} = \sum_{i=1}^{3}(\delta_{ii}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{ki} + \delta_{in}\delta_{ji}\delta_{km} - \delta_{im}\delta_{ji}\delta_{kn} - \delta_{ii}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{ki}) = \delta_{kn}\delta_{jm} - \delta_{jn}\delta_{km}$

Levi-Civita symbol—applications

- Determinant: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \epsilon_{ijk} a_{1i} a_{2j} a_{3k}$
- $C_i = \sum_{jk} \epsilon_{ijk} A_j B_k, \qquad \vec{C} = \sum_{ijk} \epsilon_{ijk} A_j B_k \hat{e}_i = \epsilon_{ijk} A_j B_k \hat{e}_i$
- $(\overrightarrow{A} \times \overrightarrow{B}) \cdot (\overrightarrow{A} \times \overrightarrow{B}) = (\sum_{ijk} \epsilon_{ijk} A_j B_k \hat{e}_i) \cdot (\sum_{lmn} \epsilon_{lmn} A_m B_n \hat{e}_l) = \sum_{ijklmn} \epsilon_{ijk} \epsilon_{lmn} A_j B_k A_m B_n \delta_{il} = \sum_{ijkmn} \epsilon_{ijk} \epsilon_{imn} A_j B_k A_m B_n = \sum_{jkmn} (\delta_{kn}\delta_{jm} - \delta_{jn}\delta_{km}) A_j B_k A_m B_n = \sum_{jkmn} \delta_{kn}\delta_{jm} A_j B_k A_m B_n - \sum_{jkmn} \delta_{jn}\delta_{km} A_j B_k A_m B_n = \sum_{jk} A_j B_k A_j B_k - \sum_{jk} A_j B_k A_k B_j = (\sum_j A_j^2)(\sum_k B_K^2) - (\sum_j A_j B_j)(\sum_k A_k B_k) = |A|^2 |B|^2 (1 - \cos^2\theta)$
- $(\overrightarrow{A} \times \overrightarrow{B})^2 = (\overrightarrow{A})^2 (\overrightarrow{B})^2 - (\overrightarrow{A} \cdot \overrightarrow{B})^2$

Triple scalar product

- $\overrightarrow{A} \cdot \overrightarrow{B} \times \overrightarrow{C} = \overrightarrow{A} \cdot (\sum_{ijk} \epsilon_{ijk} B_j C_k \hat{e}_i) = \sum_{ijk} \epsilon_{ijk} A_i B_j C_k = \sum_{jki} \epsilon_{ijk} B_i C_j A_k = \overrightarrow{B} \cdot \overrightarrow{C} \times \overrightarrow{A} = \overrightarrow{C} \cdot \overrightarrow{A} \times \overrightarrow{B} = -\overrightarrow{A} \cdot \overrightarrow{C} \times \overrightarrow{B} = -\vec{C} \cdot \vec{B} \times \vec{A}.$
- 

- $\overrightarrow{A} \cdot \overrightarrow{B} \times \overrightarrow{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$. Volume of the parallelepiped defined by $\overrightarrow{A}$, $\overrightarrow{B}$ and $\overrightarrow{C}$.

Triple vector product

- $\vec{A} \times (\vec{B} \times \vec{C}) = x\vec{B} + y\vec{C}$
- $0 = x\vec{A} \cdot \vec{B} + y\vec{A} \cdot \vec{C} \rightarrow x = z\vec{A} \cdot \vec{C} \qquad y = -z\vec{A} \cdot \vec{B}$
- $\vec{A} \times (\vec{B} \times \vec{C}) = z(\vec{B}\vec{A} \cdot \vec{C} - \vec{C}\vec{A} \cdot \vec{B})$
- z is magnitude independent.

$$\begin{aligned}
[\hat{A} \times (\hat{B} \times \hat{C})]^2 &= \hat{A}^2(\hat{B} \times \hat{C})^2 - [\hat{A} \cdot (\hat{B} \times \hat{C})]^2 \\
&= 1 - \cos^2\alpha - [\hat{A} \cdot (\hat{B} \times \hat{C})]^2 \\
&= z^2[(\hat{A} \cdot \hat{C})^2 + (\hat{A} \cdot \hat{B})^2 - 2\hat{A} \cdot \hat{B}\hat{A} \cdot \hat{C}\hat{B} \cdot \hat{C}] \\
&= z^2(\cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma)
\end{aligned}$$

- $1 - \cos^2\alpha - [\hat{A} \cdot (\hat{B} \times \hat{C})]^2 = z^2(\cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma)$
- $[\hat{A} \cdot (\hat{B} \times \hat{C})]^2 = 1 - \cos^2\alpha - z^2(\cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma)$

BAC-CAB

- The volume spanned by three vectors is independent of their order, thus $z^2 = 1$.
- $\hat{x} \times (\hat{x} \times \hat{y}) = z((\hat{x} \cdot \hat{y})\hat{x} - (\hat{x} \cdot \hat{x})\hat{y}) = -z\hat{y}$, also, $\hat{x} \times (\hat{x} \times \hat{y}) = \hat{x} \times \hat{z} = -\hat{y}$ thus $z = 1$.
- Lemma: $\vec{A} \times \hat{e}_i = \sum_{mno} \epsilon_{mno} \hat{e}_m A_n \delta_{io} = \sum_{mn} \epsilon_{mni} \hat{e}_m A_n$
- $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{A} \times (\sum_{ijk} \epsilon_{ijk} \hat{e}_i B_j C_k) = \sum_{ijk} \epsilon_{ijk} (\vec{A} \times \hat{e}_i) B_j C_k = \sum_{ijkmn} \epsilon_{ijk} \epsilon_{imn} B_j C_k A_n \hat{e}_m = \sum_{jkmn} (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) B_j C_k A_n \hat{e}_m = \sum_{jkmn} (\delta_{jm}\delta_{kn}) B_j C_k A_n \hat{e}_m - \sum_{jkmn} (\delta_{jn}\delta_{km}) B_j C_k A_n \hat{e}_m = \sum_{jk} B_j C_k A_k \hat{e}_j - \sum_{jk} B_j C_k A_j \hat{e}_k = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

Series

- $\sum_{k=o}^{\infty} a_k(x-a)^k = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$ where $a, a_k \in \mathbb{R}, k \in \mathbb{N}$
- The sequence $\{s_n(x)\}$ where $s_n(x) = \sum_{k=o}^{n} a_k(x-a)^k$ is a partial sum sequence for the above series.
- The above power series is convergent at point x_0 if the partial sum sequence $\{s_n(x)\}$ is convergent at point x_0. I.e., $\lim_{n\to\infty} s_n(x_0) = s(x_0)$
- $s(x_0)$ is the sum of the above series at point x_0.
- $\lim_{n\to\infty} \sum_{k=o}^{n} a_k(x_0-a)^k = \sum_{k=o}^{\infty} a_k(x_0-a)^k = s(x_0)$
- Set $a=0$, $\sum_{k=o}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$. This is absolutely convergent iff $\sum |a_k x^k|$ is convergent.
- Convergence radius, convergence interval or region of convergence.
- Ratio test, $\lim_{n\to\infty} |\frac{a_{n+1}x^{n+1}}{a_n x^n}| < 1$

Taylor series

- Assume f(x) can be represented as a power series around the point a.
- Taylor series of a real or complex valued function f(x) that is infinitely differentiable at a number a: $f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. When $a = 0$, the series is also called a Maclaurin series.
- The Taylor series for any polynomial is the polynomial itself.
- The Maclaurin series for 1/(1- x) is the geometric series $1 + x + x^2 + x^3 + \cdots$ so the Taylor series for 1/x at a = 1 is $1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$.
- Integrate the above Maclaurin series, to find $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$ and the corresponding Taylor series for ln x at a = 1 is $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$.

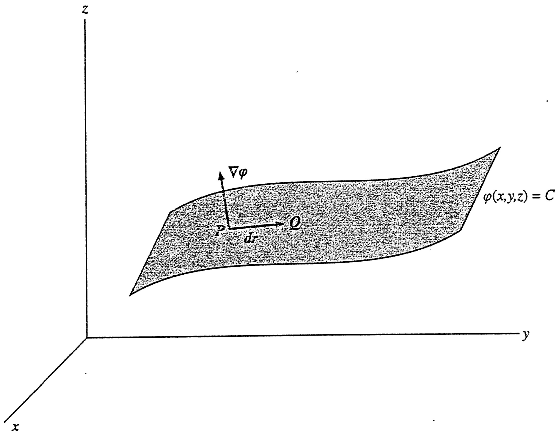
Taylor series

- Taylor series for log x at some $a = x_0$ is: $\log(x_0) + \frac{1}{x_0}(x - x_0) - \frac{1}{x_0^2}\frac{(x-x_0)^2}{2} + \cdots$.
- The Taylor series for the exponential function e^x at $a = 0$ is $\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots =$ $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
- If f(x) is given by a convergent power series in an open disc centered at b in the complex plane, it is analytic in this disc. For x in this disc, f is given by a convergent power series $f(x) = \sum_{n=0}^{\infty} a_n(x-b)^n$.
- Differentiating by x the above formula n times, then setting x = b gives: $\frac{f^{(n)}(b)}{n!} = a_n$ and so the power series expansion agrees with the Taylor series.
- Thus a function is analytic in an open disc centered at b if and only if its Taylor series converges to the value of the function at each point of the disc.

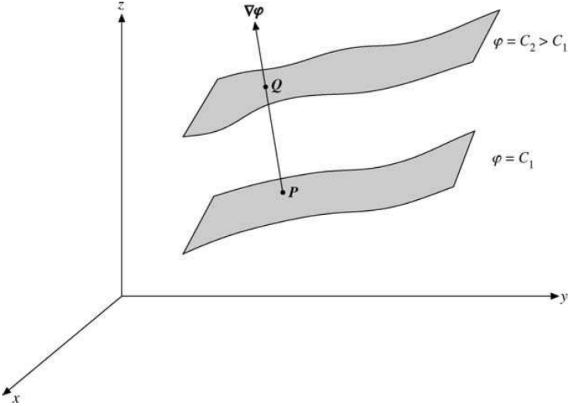
Gradient, ∇

- For a scaler field $\phi'(x_1', x_2', x_3') = \phi(x_1, x_2, x_3)$.
- $\frac{\partial \phi'(x_1',x_2',x_3')}{\partial x_i'} = \frac{\partial \phi(x_1,x_2,x_3)}{\partial x_i'} = \sum_j \frac{\partial \phi}{\partial x_j}\frac{\partial x_j}{\partial x_i'} = \sum_j a_{ij}\frac{\partial \phi}{\partial x_j}$
- $\frac{\partial \phi}{\partial x_j}$ is behaving as a vector component.
- Del $= \nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$
- Thus $\nabla\phi$ is a vector and is called gradient of phi.
- Calculate $\nabla f(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$, result is $\hat{r}\frac{df}{dr}$
- $\nabla\phi \cdot d\vec{r} = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi$
- Over a constant ϕ surface $d\phi = \nabla\phi \cdot d\vec{r} = 0$.

Gradient, ∇

- 

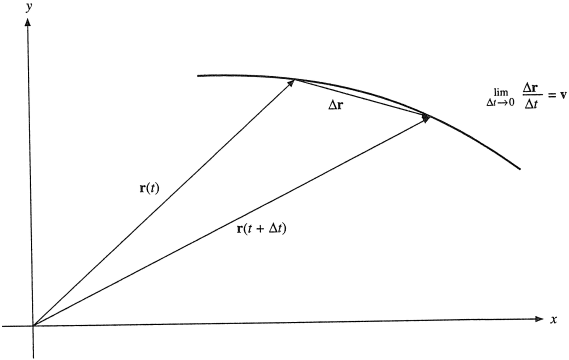
- $d\phi = C_1 - C_2 = \Delta C = (\nabla\phi) \cdot dr$

- 

- Consider $\phi(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$, find $\nabla\phi$ and direction cosines of $\nabla\phi$ at (3,2,1).
- $\int \vec{A}(r) \cdot \nabla f(r) d^3r = -\int f(r) \nabla \cdot \vec{A}(r) d^3r$ where A or f vanish at infinity.
- $\vec{F} = -\nabla U$
- Prove $\nabla(uv) = v\nabla u + u\nabla v$.

Divergence, ∇

- $\frac{d\vec{r}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \vec{v}$



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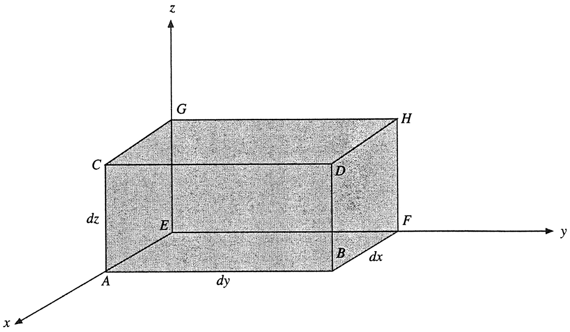
- $\nabla \cdot \vec{r} = (\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}) \cdot (\hat{x}x + \hat{y}y + \hat{z}z) = 3,$
  $\nabla \cdot (\vec{r}f(r)) =?, \qquad \nabla \cdot (\vec{r}r^{n-1}) =?.$
- $\int \vec{A}(r) \cdot \nabla f(r) d^3r = -\int f(r) \nabla \cdot \vec{A}(r) d^3r$ where A or f vanish at infinity.

Divergence, ∇

- Divergence of $\vec{V}$, $\quad \nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$
- $\nabla \cdot (\rho \vec{V})$ for a compressible fluid.
- The flow going through a differential volume per unit time is:



- (rate of flow in)$_{EFGH} = (\rho v_x)|_{x=0} dydz$
- (rate of flow out)$_{ABCD} = (\rho v_x)|_{x=dx} dydz = [\rho v_x + \frac{\partial}{\partial x}(\rho v_x) dx]_{x=0} dydz$

Divergence, ∇

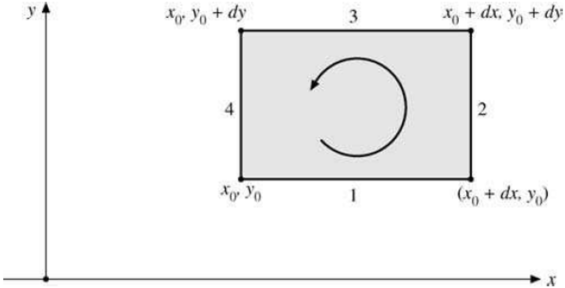
- Net rate of flow out$|_x = \frac{\partial}{\partial x}(\rho v_x)|_{(0,0,0)} dxdydz$
- $\lim_{\Delta x \to 0} \frac{\rho v_x(\Delta x,0,0) - \rho v_x(0,0,0)}{\Delta x} \equiv \frac{\partial[\rho v_x(x,y,z)]}{\partial x}|_{(0,0,0)}$
- Net flow out (per unit time) $= \nabla \cdot (\rho \vec{v})\, dxdydz$.
- Continuity equation: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$.
- $\nabla \cdot (f\vec{V}) = \nabla f \cdot \vec{V} + f \nabla \cdot \vec{V}$
- $\vec{B}$ is solenoidal if and only if $\nabla \cdot B = 0$
- A circular orbit can be represented by $\vec{r} = \hat{x} r \cos \omega t + \hat{y} r \sin \omega t$. Evaluate $r \times \dot{\vec{r}}$ and $\ddot{\vec{r}} + \omega^2 \vec{r} =$
- Divergence of electrostatic field due to a point charge, $\nabla \cdot \vec{E} = \nabla \cdot \frac{q\hat{r}}{4\pi\epsilon_0 r^2} = \frac{q}{4\pi\epsilon_0} \nabla \cdot \frac{\hat{r}}{r^2} = \frac{q}{4\pi\epsilon_0}[\frac{\nabla \cdot \vec{r}}{r^3} + \vec{r} \cdot \nabla \frac{1}{r^3}]$.

Curl, $\nabla\times$

- $\nabla \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \sum_{ijk} \epsilon_{ijk} \hat{e}_i \frac{\partial V_k}{\partial x_j}.$
- $\nabla \times (f\vec{V}) = f\nabla \times \vec{V} + (\nabla f) \times \vec{V}$
- $\nabla \times (\vec{r}F(r)) = 0$
- Show that electrostatic and gravitational forces are irrotational.
- Show that the curl of a vector field is a vector field.
- Curl can be measured by inserting a paddle wheel inside the vector field.

Circulation

- Circulation of a fluid around a differential loop in the xy-plane.



- $\int \vec{V} \cdot d\lambda = \int_1 V_x(x,y)d\lambda_x + \int_2 V_y(x,y)d\lambda_y + \int_3 V_x(x,y)d\lambda_x + \int_4 V_y(x,y)d\lambda_y = V_x(x_0,y_0)dx + [V_y(x_0,y_0) + \frac{\partial V_y}{\partial x}dx]dy + [V_x(x_0,y_0) + \frac{\partial V_x}{\partial y}dy](-dx) + V_y(x_0,y_0)(-dy) = (\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y})dxdy = \nabla \times \vec{V}|_z dxdy$

Successive applications of ∇

- Show that $\vec{u} \times \vec{v}$ is solenoidal if u and v are each irrotational.
- If $\vec{A}$ is irrotational show that $\vec{A} \times \vec{r}$ is solenoidal
- $\nabla \cdot \nabla \phi = \nabla^2 \phi = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})\phi$.
- $\nabla \times \nabla \phi = 0$.
- $\nabla \cdot \nabla \times \vec{V} = 0$
- $\nabla \cdot \nabla \vec{V} = \hat{i} \nabla \cdot \nabla V_x + \hat{j} \nabla \cdot \nabla V_y + \hat{k} \nabla \cdot \nabla V_z$
- $\nabla \times (\nabla \times \vec{V}) = \nabla \nabla \cdot \vec{V} - \nabla \cdot \nabla \vec{V}$

Electromagnetic wave equation

- The set of Maxwell equations:
- $\nabla \cdot \vec{B} = 0$
- $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
- $\nabla \times \vec{B} = \mu_0(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t})$
- $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Electromagnetic wave equation

- The set of Maxwell equations:
- $\nabla \cdot \vec{B} = 0$
- $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
- $\nabla \times \vec{B} = \mu_0(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t})$
- $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- Eliminating B between the last two equations, by noting that $\frac{\partial}{\partial t} \nabla \times \vec{B} = \nabla \times \frac{\partial \vec{B}}{\partial t}$ and assuming no charge flux.
- $\nabla \times (\nabla \times \vec{E}) = -\epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$

Review: Integrals

- $\int x(x+a)^n dx =$
- $\int \frac{1}{a^2+x^2} dx =$
- $\int \frac{x}{a^2+x^2} dx$

- $\int \frac{x^2}{a^2+x^2} dx =$
- $\int \frac{x^3}{a^2+x^2} dx =$
- $\int \tan(ax+b) dx =$

- $\int cotan(ax+b) dx =$

Review: Integrals

- $\int x(x+a)^n dx =$
- $\int \frac{1}{a^2+x^2} dx =$
- $\int \frac{x}{a^2+x^2} dx$
- $= \frac{1}{2} \ln |a^2 + x^2|$
- $\int \frac{x^2}{a^2+x^2} dx =$
- $\int \frac{x^3}{a^2+x^2} dx =$
- $\int \tan(ax+b) dx =$
- $-\frac{1}{a} \ln |\cos(ax+b)|$
- $\int cotan(ax+b) dx =$
- $\frac{1}{a} \ln |\sin(ax+b)|$

Review: Integrals

- $\int \sec(ax+b)dx =$
- $\int cosec(ax+b)dx =$
- $\int \sec^2(x)dx =$
- $\int cosec^2(x)dx =$
- $\int \tan(x)\sec(x)dx =$
- $\int cotan(x)cosec(x)dx =$

Review: Integrals

- $\int \sec(ax+b)dx =$
- $\frac{1}{a}\ln|\sec(ax+b)+\tan(ax+b)|$
- $\int cosec(ax+b)dx =$
- $-\frac{1}{a}\ln|cosec(ax+b)+cotan(ax+b)|$
- $\int \sec^2(x)dx =$
- $\tan(x)$
- $\int cosec^2(x)dx =$
- $cotan(x)$
- $\int \tan(x)\sec(x)dx =$
- $\sec(x)$
- $\int cotan(x)cosec(x)dx =$
- $cosec(x)$

Review: Integrals

- $\int \frac{1}{ax^2+bx+c}dx = \int \frac{dx}{a(x+\frac{b}{2a})^2+c-\frac{b^2}{4a}} = \frac{1}{a}\int \frac{dx}{(x+\frac{b}{2a})^2+c/a-\frac{b^2}{4a^2}} =$
  $\frac{1}{a}\int \frac{du}{u^2+(c/a-\frac{b^2}{4a^2})} = \frac{1}{a}\tan^{-1}(\frac{u}{\sqrt{c/a-\frac{b^2}{4a^2}}}) = \frac{1}{a}\tan^{-1}(\frac{x+\frac{b}{2a}}{\sqrt{c/a-\frac{b^2}{4a^2}}})$
- $\int \frac{1}{(x+a)(x+b)}dx =$
- $\int \frac{x}{ax^2+bx+c}dx =$
- $\int \frac{1}{\sqrt{x\pm a}}dx =$
- $\int x\sqrt{x-a}dx$
- $\int \sqrt{ax+b}dx =$
- $\int \frac{x}{\sqrt{x\pm a}}dx =$
- $\int \sqrt{\frac{x}{a-x}}dx$

Vector integration over a contour

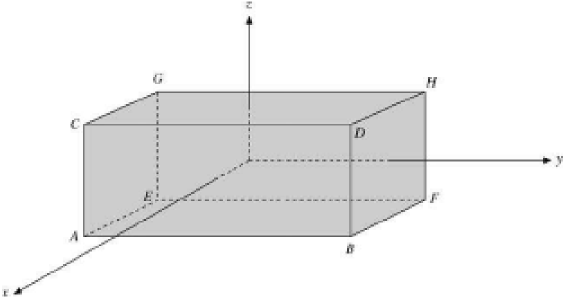
- $\int_C \phi d\vec{r} =$
  $\hat{x} \int_C \phi(x,y,z)dx + \hat{y} \int_C \phi(x,y,z)dy + \hat{z} \int_C \phi(x,y,z)dz$
- $\int_C \vec{V} \cdot d\vec{r}$, e.g., $w = \int F \cdot d\vec{r} =$
  $\int_C \vec{F}_x(x,y,z)dx + \int_C F_y(x,y,z)dy + \int_C F_z(x,y,z)dz$
- $\int_C \vec{V} \times d\vec{r} =$
  $\hat{x} \int_C (V_y dz - V_z dy) - \hat{y} \int_C (V_x dz - V_z dx) + \hat{z} \int_C (V_x dy - V_y dx)$
- Reduce each vector integral to scalar integrals.
- E.g., $\int_{0,0}^{1,1} r^2 dr = \int_{0,0}^{1,1} (x^2 + y^2) dr = \int_{0,0}^{1,1} (x^2 + y^2)(\hat{x} dx + \hat{y} dy)$
- E.g., Calculate W for $F = -\hat{x} y + \hat{y} x$

Surface and volume integration

- $\int_S \phi d\vec{\sigma}$
- $\int_S \vec{V} \cdot d\vec{\sigma}$ (flow or flux through a given surface),
- $\int_S \vec{V} \times d\vec{\sigma}$
- Convention for the direction of surface normal: Outward from a closed surface. In the direction of thumb when contiguous right hand fingers are traversing the perimeter of the surface.
- Volume integrals:
  $\int_v \vec{V} d\tau = \hat{x} \int_v V_x d\tau + \hat{y} \int_v V_y d\tau + \hat{z} \int_v V_z d\tau$

Integral definition of gradient

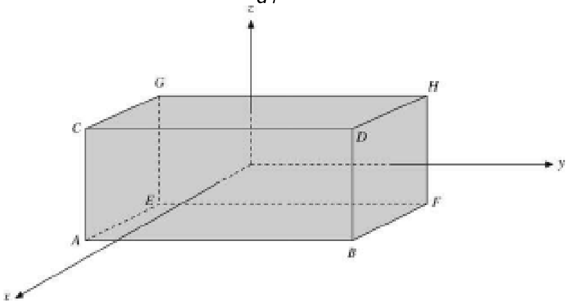
- $\nabla\phi = \lim_{d\tau\to 0} \frac{\int_{S_{d\tau}} \phi d\vec{\sigma}}{d\tau}$
- $d\tau = dxdydz$. Place origin at the center of the differential



  volume, $d\tau$.
- $\int_{S_{d\tau}} \phi d\vec{\sigma} = -i\int_{EFHG}(\phi - \frac{\partial\phi}{\partial x}\frac{dx}{2})dydz + i\int_{ABDC}(\phi + \frac{\partial\phi}{\partial x}\frac{dx}{2})dydz - j\int_{AEGC}(\phi - \frac{\partial\phi}{\partial y}\frac{dy}{2})dxdz + j\int_{BFHD}(\phi + \frac{\partial\phi}{\partial y}\frac{dy}{2})dxdz - k\int_{ABFE}(\phi - \frac{\partial\phi}{\partial z}\frac{dz}{2})dydx + k\int_{CDHG}(\phi + \frac{\partial\phi}{\partial z}\frac{dz}{2})dydx$
- $\int \phi d\vec{\sigma} = (i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z})dxdydz$

Integral definitions of divergence

- $\nabla \cdot \vec{V} = \lim_{d\tau \to 0} \frac{\int_{S_{d\tau}} \vec{V} \cdot d\vec{\sigma}}{d\tau}$



- $\int_{S_{d\tau}} \vec{V} \cdot d\vec{\sigma} = \int_{EFHG} \vec{V} \cdot d\vec{\sigma} + \int_{ABDC} \vec{V} \cdot d\vec{\sigma} + \int_{AEGC} \vec{V} \cdot d\vec{\sigma} + \int_{BFHD} \vec{V} \cdot d\vec{\sigma} + \int_{ABFE} \vec{V} \cdot d\vec{\sigma} + \int_{CDHG} \vec{V} \cdot d\vec{\sigma} = -\int_{EFHG}(V_x - \frac{\partial V_x}{\partial x}\frac{dx}{2})dydz + \int_{ABDC}(V_x + \frac{\partial V_x}{\partial x}\frac{dx}{2})dydz - \int_{AEGC}(V_y - \frac{\partial V_y}{\partial y}\frac{dy}{2})dxdz + \int_{BFHD}(V_y + \frac{\partial V_y}{\partial y}\frac{dy}{2})dxdz - \int_{ABFE}(V_z - \frac{\partial V_z}{\partial z}\frac{dz}{2})dydx + \int_{CDHG}(V_z + \frac{\partial V_z}{\partial z}\frac{dz}{2})dydx = (\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z})dxdydz$

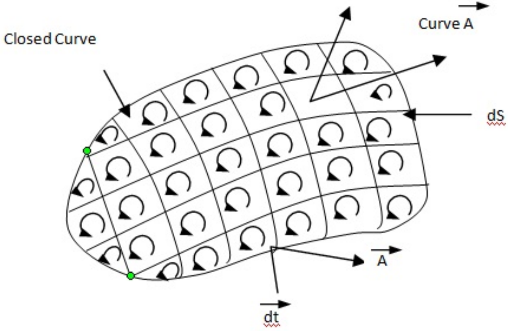
Integral definitions of curl

- $\nabla \times \vec{V} = \lim_{d\tau \to 0} \frac{\int_{S_{d\tau}} d\vec{\sigma} \times \vec{V}}{d\tau}$
- $\int_{S_{d\tau}} \vec{V} \times d\vec{\sigma} = \int_{EFHG} \vec{V} \times d\vec{\sigma} + \int_{ABDC} \vec{V} \times d\vec{\sigma} + \int_{AEGC} \vec{V} \times d\vec{\sigma} + \int_{BFHD} \vec{V} \times d\vec{\sigma} + \int_{ABFE} \vec{V} \times d\vec{\sigma} + \int_{CDHG} \vec{V} \times d\vec{\sigma} = -dydz\vec{V}(-dx/2, 0, 0) \times \hat{x} + dydz\vec{V}(dx/2, 0, 0) \times \hat{x} - dxdz\vec{V}(0, -dy/2, 0) \times \hat{y} + dxdz\vec{V}(0, dy/2, 0) \times \hat{y} - dxdy\vec{V}(0, 0, -dz/2) \times \hat{z} + dxdy\vec{V}(0, 0, dz/2) \times \hat{z} = -dydz(V_z(-dx/2, 0, 0)\hat{y} - V_y(-dx/2, 0, 0)\hat{z}) + dydz(V_z(dx/2, 0, 0)\hat{y} - V_y(dx/2, 0, 0)\hat{z}) - dxdz(-V_z(0, -dy/2, 0)\hat{x} + V_x(0, -dy/2, 0)\hat{z}) + dxdz(-V_z(0, dy/2, 0)\hat{x} + V_x(0, dy/2, 0)\hat{z}) - dxdy(V_y(0, 0, -dz/2)\hat{x} - V_x(0, 0, -dz/2)\hat{y}) + dxdy(V_y(0, 0, dz/2)\hat{x} - V_x(0, 0, dz/2)\hat{y}) = -dydz((V_z(0, 0, 0) - \frac{dx}{2}\frac{\partial V_z(0,0,0)}{\partial x})\hat{y} - (V_y(0, 0, 0) - \frac{dx}{2}\frac{\partial V_y(0,0,0)}{\partial x})\hat{z}) + \cdots$

Theorems

- Gauss's theorem (divergence theorem), $\int_S \vec{V} \cdot d\vec{\sigma} = \int_V \nabla \cdot \vec{V} d\tau$, equates the flow out of a surface S with the sources inside the volume enclosed by it.
- Result: $\int_S \phi d\vec{\sigma} = \int_V \nabla \phi d\tau$ using $\vec{V} = \phi(x, y, z)\vec{a}$
- Result: $\int_S d\vec{\sigma} \times \vec{P} = \int_V \nabla \times \vec{P} d\tau$ using $\vec{V} = \vec{a} \times \vec{P}$
- Prove Green's theorem, $\int_V (u\nabla^2 v - v\nabla^2 u) d\tau = \int_S (u\nabla v - v\nabla u) \cdot d\vec{\sigma}$, by applying Gauss's theorem to the difference of $\nabla \cdot (u\nabla v) = u\nabla^2 v + \nabla u \cdot \nabla v$ and $\nabla \cdot (v\nabla u)$.
- Alternative form, $\int_S u\nabla v \cdot d\vec{\sigma} = \int_V (u\nabla^2 v + \nabla u \cdot \nabla v) d\tau$

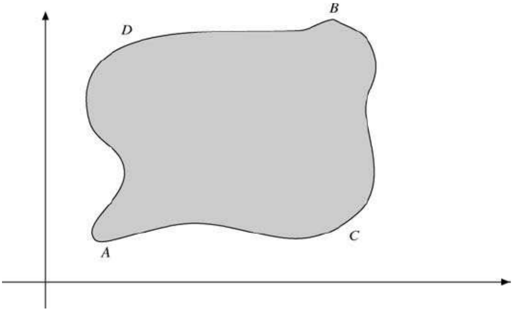
- Stokes theorem: $\oint_{\partial S} \vec{V} \cdot d\vec{\lambda} = \int_S \nabla \times \vec{V} \cdot d\vec{\sigma}$



- Alternate form: $\int_S d\sigma \times \nabla\phi = \oint_{\partial S} \phi d\lambda$ using $\vec{V} = \vec{a}\phi$

Potential theory

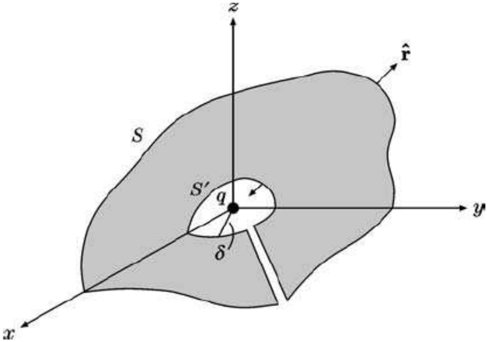
- Scalar potential
- Conservative force
  $\iff \vec{F} = -\nabla\phi \iff \nabla \times \vec{F} = 0 \iff \oint \vec{F} \cdot dr = 0$
- $\nabla \times F = -\nabla \times \nabla\phi = 0$
- $\oint F \cdot dr = -\oint \nabla\phi \cdot dr = -\oint d\phi = 0$
- $\oint_{ACBDA} F \cdot dr = 0 \iff \int_{ACB} F \cdot dr = -\int_{BDA} F \cdot dr = \int_{ADB} F \cdot dr \iff$ the work is path independent.



Potential theory

- $\int_A^B F \cdot dr = \phi(A) - \phi(B) \rightarrow F \cdot dr = -d\phi = -\nabla\phi \cdot dr$. Therefore $(F + \nabla\phi) \cdot dr = 0$
- $\oint F \cdot dr = \int \nabla \times F \cdot d\sigma$ by integrating over the perimeter of an arbitrary differential surface $d\sigma$ we see that $\oint F \cdot dr = 0$ result in $\nabla \times F = 0$.
- Scalar potential for the gravitational force on a unit mass m_1, $F_G = -\frac{Gm_1 m_2 \hat{r}}{r^2} = -\frac{k\hat{r}}{r^2}$?
- Scalar potential for the centrifugal force and simple harmonic oscillator on a unit mass m_1, $\vec{F}_c = \omega^2 \vec{r}$ and $\vec{F}_{SHO} = -k\vec{r}$.
- Exact differentials. How to know if integral of $df = P(x, y)dx + Q(x, y)dy$ is path dependent or independent.
- Vector potential $\vec{B} = \nabla \times \vec{A}$

- Only a point charge at the origin $\vec{E} = \frac{q\hat{r}}{4\pi\epsilon_0 r^2}$
- Gauss's law: $\int_S \vec{E}\cdot d\vec{\sigma} = \begin{cases} 0 & \text{S does not contain the origin,} \\ \frac{q}{\epsilon_0} & \text{S contains the origin.} \end{cases}$
- Closed surface S not including the origin
  $\int_S \frac{\hat{r}\cdot d\vec{\sigma}}{r^2} = \int_V \nabla\cdot(\frac{\hat{r}}{r^2})d\tau$



-
- $\int_S \frac{\hat{r}\cdot d\vec{\sigma}}{r^2} + \int_{S'} \frac{\hat{r}\cdot d\vec{\sigma}'}{\delta^2} = 0$

Gauss's law, Poisson's equation

- $d\sigma' = -\hat{r}\delta^2 d\Omega$
- $\int_S \vec{E} \cdot d\vec{\sigma} = \frac{q}{\epsilon_0} = \int_V \frac{\rho}{\epsilon_0} d\tau$. Further, $\int_S \vec{E} \cdot d\vec{\sigma} = \int_V \nabla \cdot \vec{E} d\tau$
- Maxwell equation: $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
- Poisson's equation: $\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$.
- Laplace's equation $\nabla^2 \phi = 0$
- Substitute ϕ for E into the Gauss's law.
- $\int_v \nabla^2(\frac{1}{r}) d\tau = \begin{cases} -4\pi & 0 \in v, \\ 0 & 0 \notin v. \end{cases}$ Thus
  $\nabla^2(\frac{1}{r}) = -4\pi\delta(\vec{r}) = -4\pi\delta(x)\delta(y)\delta(z)$.

Dirac delta function

- Dirac Delta properties $\begin{cases} \delta(x) = 0 & x \neq 0, \\ f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx. \end{cases}$
- See functions approximating δ in a Mathematica notebook.

$$\delta_n(x) = \begin{cases} 0 & x < -\frac{1}{2n}, \\ n, & -\frac{1}{2n} < x < \frac{1}{2n}, \\ 0 & x > \frac{1}{2n}. \end{cases}$$

- $\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$.
- $\delta_n(x) = \frac{n}{\pi} \frac{1}{1+n^2x^2}$.
- $\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi} \int_{-n}^{n} e^{ixt} dt$.
- $\int_{-\infty}^{\infty} f(x)\delta(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\delta_n(x)dx$

Dirac delta function

- $\delta(x)$ is a distribution defined by the sequences $\delta_n(x)$
- Evenness: $\delta(x) = \delta(-x)$.
- $\int_{-\infty}^{\infty} f(x)\delta(ax)dx = \frac{1}{a}\int_{-\infty}^{\infty} f(\frac{y}{a})\delta(y)dy = \frac{1}{a}f(0)$. Thus $\delta(ax) = \frac{1}{|a|}\delta(x)$.
- $\int_{-\infty}^{\infty} f(x)\delta(g(x))dx = \sum_a \int_{a-\epsilon}^{a+\epsilon} f(x)\delta((x-a)g'(a))dx$. Thus $\delta(g(x)) = \sum_{a, g(a)=0, g'(a)\neq 0} \frac{\delta(x-a)}{|g'(a)|}$.
- Derivative:
  $\int f(x)\delta'(x-x_0)dx = -\int f'(x)\delta(x-x_0)dx = -f'(x_0)$.
- Delta Operator: $\mathcal{L}(x_0) = \int dx\delta(x-x_0)$.
- $\int\int\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \delta(\vec{r})r^2 dr \sin\theta d\theta d\phi$

Representation of Dirac delta by orthogonal functions

- Consider an infinite dimensional vector space where elements of the underlying set are functions.
  $(f+g)(x) = f(x) + g(x) \quad (cf)(x) = cf(x)$.
- Inner product maybe defined as $f(x) \cdot g(x) = \int_a^b f(x)g(x)dx$ where either a, b or both can be ∞.
- No good and natural basis but real orthogonal functions $\{\phi_n(x), n = 0, 1, 2, \cdots\}$ form a basis for this vector space.
- Their orthonormality relation is
  $\phi_m \cdot \phi_n = \int_a^b \phi_m(x)\phi_n(x)dx = \delta_{mn}$
- Around any point x_0 a good basis is the set $\{(x-x_0)^0, (x-x_0), (x-x_0)^2, \cdots\}$ which is not orthonormal.
- Use Gram-Schmidt orthonormalization.
- For square integrable functions use $\{\sin(n\pi x), \cos(n\pi x)\}$
- Expanding delta function in this bases:
  $\delta(x-t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x)$.
- Take the inner product of both sides by $\phi_m(x)$ to derive coefficients.

Fourier transform

- $\delta(x-t) = \sum_{n=0}^{\infty} \phi_n(t)\phi_n(x) = \delta(t-x)$
- $\int F(t)\delta(x-t)dt = \int \sum_{p=0}^{\infty} a_p\phi_p(t) \sum_{n=0}^{\infty} \phi_n(t)\phi_n(x)dt = \sum_{n,p=0}^{\infty} a_p\phi_n(x)\delta_{np} = \sum_{p=0}^{\infty} a_p\phi_p(x) = F(x)$
- Fourier integral translates a function from one domain into another, $\mathcal{F}(f(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt = F(\omega)$.
- Inverse Fourier transform is $\mathcal{F}^{-1}(F(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\exp(-i\omega t)d\omega = f(t)$.
- For the position and momentum conjugate variables: $\mathcal{F}(\psi(x)) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x)e^{ixp/\hbar}dx = \psi(p)$, $\mathcal{F}^{-1}(\psi(p)) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(p)\exp(-ixp/\hbar)dp = \psi(x)$.
- $\mathcal{F}(\delta(x)) = \frac{1}{\sqrt{2\pi}}$, $\delta(x) = \mathcal{F}^{-1}(\mathcal{F}(\delta(x))) = \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ixp/\hbar)dp$
- Calculate the Fourier transform of Gauss distribution.
- Deduce the uncertainty principle.

Random variable

- a random variable, random quantity, aleatory variable, or stochastic variable is described informally as a variable whose values depend on outcomes of a random phenomenon.
- A random variable's possible values might represent the possible outcomes of a yet-to-be-performed experiment, or the possible outcomes of a past experiment whose already-existing value is uncertain
- They may also conceptually represent either the results of an "objectively" random process (such as rolling a die) or the "subjective" randomness that results from incomplete knowledge of a quantity.
- The domain of a random variable is a sample space, which is interpreted as the set of possible outcomes of a random phenomenon.
- A random variable has a probability distribution, which specifies the probability of its values.

Probability distributions

- Random variables can be discrete, taking any of a specified finite or countable list of values, endowed with a probability mass function characteristic of the random variable's probability distribution; or continuous, taking any numerical value in an interval or collection of intervals, via a probability density function that is characteristic of the random variable's probability distribution; or a mixture of both types.
- Any random variable can be described by its cumulative distribution function, which describes the probability that the random variable will be less than or equal to a certain value.
- The term "random variable" in statistics is traditionally limited to the real-valued case ($E = \mathbb{R}$). In this case, the structure of the real numbers makes it possible to define quantities such as the expected value and variance of a random variable, its cumulative distribution function, and the moments of its distribution.

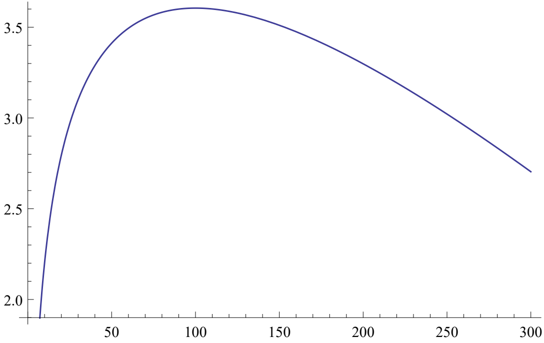
Probability distributions

- Average of a discrete random variable, $\bar{u} = \frac{\sum_{j=1}^{M} u_j p(u_j)}{\sum_{j=1}^{M} p(u_j)}$
- Average of any function of u: $\overline{f(u)} = \sum_{j=1}^{M} f(u_j) p(u_j)$
- m'th moment of distribution $\overline{u^m}$
- m'th central moment of distribution $\overline{(u - \bar{u})^m}$ including variance.
- The Poisson distribution is popular for modeling the number of times an event occurs in an interval of time or space.
- A discrete random variable X is said to have a Poisson distribution with parameter $a > 0$, if, for m = 0, 1, 2, ..., the probability mass function of X is given by: $Pr(X = m) = \frac{a^m e^{-a}}{m!}$
- The positive real number a is equal to the expected value of X and also to its variance

Probability distributions

- The Poisson distribution may be useful to model events such as: The number of meteorites greater than 1 meter diameter that strike Earth in a year; The number of patients arriving in an emergency room between 10 and 11 pm; The number of laser photons hitting a detector in a particular time interval
- $\overline{f(u)} = \int f(u)p(u)du$
- Gauss, Gaussian or normal distribution with the probability density of $p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$.
- Central limit theorem: Average of samples of observations of random variables independently drawn from independent distributions converge in distribution to the normal, that is, they become normally distributed when the number of observations is sufficiently large.

Stirling's approximation

- $\ln N! = \sum_{m=1}^{N} \ln m \approx \int_1^N \ln x dx = N \ln N - N$
- $\Gamma(N+1) = N! = \int_0^\infty e^{-x} x^N dx = \int_0^\infty e^{Ng(x)} dx$ where $g(x) = \ln x - x/N$
- 

- $g(x) \approx \ln N - 1 - \frac{(x-N)^2}{2N^2}$
- $N! \approx \int_0^\infty e^{N \ln N - N - \frac{(x-N)^2}{2N}} dx = e^{N \ln N - N} \int_0^\infty e^{-\frac{(x-N)^2}{2N}} dx = e^{N \ln N - N} \sqrt{2\pi N}$

Binomial and multinomial distribution

- r-permutation $\{x_1, x_2, \cdots, x_r\}$ from a set of n elements $\{a_1, a_2, \cdots, a_n\}$ is the number of ways to pick r elements and arrange them in order. This is the product $n(n-1)\cdots(n-r+1)$ which is also called falling factorial.
- The formula for permutation is given by ${}^nP_r = (n)_r = \frac{n!}{(n-r)!}$
- A combination is a way of selecting members from a group, such that the order of selection does not matter.
- A k-combination of a set S is a subset of k distinct elements of S.
- The number of k-combinations is equal to the binomial coefficient $\binom{n}{k} = {}^nC_k = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{k!(n-k)!}$
- Distributing k distinguishable balls into n distinguishable bins $(k < n)$, with exclusion, corresponds to forming a permutation of size k, taken from a set of size n.

Binomial and multinomial distribution

- There are ${}^{n}P_k = (n)_k = n(n-1)\cdots(n-k+1)$ different ways to distribute k distinguishable balls into n distinguishable boxes, with exclusion.
- Distributing k indistinguishable balls into n distinguishable boxes, with exclusion corresponds to forming a combination of size k, taken from a set of size n.
- There are $C(n,k) = {}^{n}C_k = \left(\begin{array}{c} n \\ k \end{array} \right)$ different ways to distribute k indistinguishable balls into n distinguishable boxes, with exclusion.
- Distributing N distinguishable objects over 2 states $\frac{N!}{N_1!(N-N_1)!}$
- $(x+y)^N = \sum_{N_1=0}^{N} \frac{N!}{N_1!(N-N_1)!} x^{N-N_1} y^{N_1} = \sum_{N_1,N_2}^{*} \frac{N!}{N_1!N_2!} x^{N_2} y^{N_1}$
- Distributing N distinguishable objects over r states $\frac{N!}{N_1!N_2!\cdots N_r!} = \frac{N!}{\Pi_{j=1}^{r} N_j!}$
- $(x_1 + x_2 + \cdots + x_r)^N = \sum_{N_1=0}^{N} \sum_{N_2=0}^{N} \cdots \sum_{N_r=0}^{N*} \frac{N!}{\Pi_{j=1}^{r} N_j!} x_1^{N_1} \cdots x_r^{N_r}$

Method of Lagrange multipliers

- To maximize $f(x_1, x_2, \cdots, x_r)$, we have $\delta f = \sum_{j=1}^{r} (\frac{\partial f}{\partial x_j})_0 \delta x_j = 0$; since δx_j are independent of one another the extremum is found by setting $(\frac{\partial f}{\partial x_j})_0 = 0$
- With the constraint $g(x_1, x_2, \cdots, x_r) = 0$, we have $\delta g = \sum_{j=1}^{r} (\frac{\partial g}{\partial x_j})_0 \delta x_j = 0$ which serves as a relation among values of δx_j.
- Assuming $\delta x_\mu = \delta x_\mu(\delta x_1, \cdots, \delta x_{\mu-1}, \delta x_{\mu+1}, \cdots, \delta x_r)$.
- $\delta f - \lambda \delta g = \sum_{j=1}^{r} (\frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j})_0 \delta x_j$
- $\lambda = (\frac{\partial f}{\partial x_\mu})_0 / (\frac{\partial g}{\partial x_\mu})_0$
- $(\frac{\partial f}{\partial x_j})_0 - \lambda (\frac{\partial g}{\partial x_j})_0 = 0$
- With a number of constraints $(\frac{\partial f}{\partial x_j})_0 - \lambda_1 (\frac{\partial g_1}{\partial x_j})_0 - \lambda_2 (\frac{\partial g_2}{\partial x_j})_0 - \cdots = 0$

Binomial distribution for large numbers

- To maximize $f(N_1) = \frac{N!}{N_1!(N-N_1)!}$, we maximize $\ln f(N_1)$.
- $\frac{d \ln f(N_1)}{dN_1} = 0 \to \frac{d}{dN_1}(N \ln N - N - N_1 \ln N_1 + N_1 - (N - N_1)\ln(N - N_1) + (N - N_1)) = -\ln N_1 - 1 + 1 + \ln(N - N_1) + 1 - 1 = 0 \to N_1^* = N/2$
- $\ln f(N_1) = \ln f(N_1^*) + \frac{1}{2}(\frac{d^2 \ln f(N_1)}{dN_1^2})_{N_1=N_1^*}(N_1 - N_1^*)^2 + \cdots$
- $f(N_1) = f(N_1^*) \exp[-\frac{2(N_1-N_1^*)^2}{N}]$. Thus $\sigma_{N_1} \approx N^{1/2}/2$
- Binomial coefficient peaks sharply at $N_1 = N_2 = N/2$
- Multinomial coefficients peaks sharply at $N_1 = N_2 = \cdots = N_s = N/s$

Maximum term method

- $S = \sum_{n=1}^{M} T_n \qquad T_n > 0$
- $T_m \leq S \leq MT_m \rightarrow \ln T_m \leq \ln S \leq \ln T_m + \ln M$
- If $T_m = \exp(M) \rightarrow M \leq \ln S \leq M + \ln M$
- If M is very large $S = T_m$

Important equations in physics

- Laplace's equation: $\nabla^2\phi = 0$ or $\Delta\phi = 0$. Its solutions describe the behaviour of electric, gravitational and fluid potentials. Laplace's equation is also the steady-state heat equation.
- Helmholtz equation represents a time-independent form of the wave equation: $\nabla^2 A + k^2 A = 0$, where k is the wavenumber and A is amplitude. HE commonly results from separation of variables in a PDE involving both time and space varibles. E.g., the wave equation $(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})u(r,t) = 0$
- Diffusion equation: $\frac{\partial\phi(r,t)}{\partial t} = \nabla \cdot [D(\phi, r)\nabla(\phi(r,t))]$, where $\phi(r,t)$ is the density of the diffusing material at location r and time t, $D(\phi, r)$ is the collective diffusion coefficient for density at location r. If D is constant, $\frac{\partial\phi(r,t)}{\partial t} = D\Delta\phi(r,t)$ also called heat equation.
- Schrodinger wave equation: $i\hbar\frac{\partial}{\partial t}|\psi(r,t)\rangle = \hat{H}|\psi(r,t)\rangle$.

Important equations in physics

- For the nonrelativistic relative motion of two particles in the coordinate basis, $i\hbar\frac{\partial}{\partial t}\psi(r,t) = [-\frac{\hbar^2}{2\mu}\nabla^2 + V(r,t)]\psi(r,t)$.
- When Hamiltonian is not explicitly dependent on time, we have the time independent Schrodinger equation: $\hat{H}\psi = E\psi$.
- For the nonrelativistic relative motion of two particle in the coordinate basis, $[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)]\psi(r) = E\psi(r)$.
- All have the form $\nabla^2\psi + k^2\psi = 0$.
- Any coordinate system in which this equation is separable is of great interest.
- Thus finding expressions for gradient, divergence, curl and laplacian in a general coordinate system is of great interest.

Curvilinear coordinates

- A point can be specified as the intersection of the 3 planes x = constant, y = constant and z = constant.
- A point can be desdcribed by the intersection of three curvilinear coordinate surfaces q_1 = constant, q_2 = constant, q_3 = constant.
- Associate a unit vector $\hat{q}_i$ normal to the surface q_i = constant and in the direction of increasing q_i.
- General vector $\vec{V} = \hat{q}_1 V_1 + \hat{q}_2 V_2 + \hat{q}_3 V_3$.
- While coordinate or position vectors can be simpler, e.g., $\vec{r} = r\hat{r}$ in spherical polar coordinates and $\vec{r} = \rho\hat{\rho} + z\hat{z}$ for cylindrical coordinates.
- $\hat{q}_i^2 = 1$, for a right handed coordinate system $\hat{q}_1 \cdot (\hat{q}_2 \times \hat{q}_3) > 0$.

Curvilinear coordinates

- $ds^2 = dx^2 + dy^2 + dz^2 = \sum_{ij} h_{ij}^2 dq_i dq_j$
- h_{ij} are referred to as the metric.
- $dx = (\frac{\partial x}{\partial q_1})dq_1 + (\frac{\partial x}{\partial q_2})dq_2 + (\frac{\partial x}{\partial q_3})dq_3$
- $dy = (\frac{\partial y}{\partial q_1})dq_1 + (\frac{\partial y}{\partial q_2})dq_2 + (\frac{\partial y}{\partial q_3})dq_3$
- $dz = (\frac{\partial z}{\partial q_1})dq_1 + (\frac{\partial z}{\partial q_2})dq_2 + (\frac{\partial z}{\partial q_3})dq_3$
- $ds^2 = d\vec{r} \cdot d\vec{r} = d\vec{r}^2 = \sum_{ij} \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j} dq_i dq_j$.
- Thus: $h_{ij}^2 = \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j} = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j}$, valid in metric or Riemannian spaces.
- For orthogonal coordinate systems: $\hat{q}_i \cdot \hat{q}_j = \delta_{ij}$.

Curvilinear coordinates

- For orthogonal coordinate systems:
  $ds^2 = h_{11}^2 dq_1^2 + h_{22}^2 dq_2^2 + h_{33}^2 dq_3^2$, i.e., $h_{ij} = 0, \quad i \neq j$.
- Setting $h_{ii} = h_i > 0 \quad ds^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$.
- ds_i is the differential length in the direction of increasing q_i.
- Scale factors may be identified as $ds_i = h_i dq_i$ with length dimension. $\frac{\partial \vec{r}}{\partial q_i} = h_i \hat{q}_i$
- The differential distance vector
  $d\vec{r} = h_1 dq_1 \hat{q}_1 + h_2 dq_2 \hat{q}_2 + h_3 dq_3 \hat{q}_3$
- $\int \vec{V} \cdot d\vec{r} = \sum_i \int V_i h_i dq_i$
- For orthogonal coordinates: $d\sigma_{ij} = ds_i ds_j = h_i h_j dq_i dq_j$ and
  $d\tau = ds_1 ds_2 ds_3 = h_1 h_2 h_3 dq_1 dq_2 dq_3$
- $d\vec{\sigma} = ds_2 ds_3 \hat{q}_1 + ds_1 ds_3 \hat{q}_2 + ds_2 ds_1 \hat{q}_3 =$
  $h_2 h_3 dq_2 dq_3 \hat{q}_1 + h_1 h_3 dq_1 dq_3 \hat{q}_2 + h_2 h_1 dq_2 dq_1 \hat{q}_3$

Curvilinear coordinates

- $\int_S \vec{V} \cdot d\vec{\sigma} =$
  $\int V_1 h_2 h_3 dq_2 dq_3 + \int V_2 h_1 h_3 dq_1 dq_3 + \int V_3 h_2 h_1 dq_2 dq_1$
- vector algebra is the same in orthogonal curvilinear coordinates as in Cartesian coordinates.
  $\vec{A} \cdot \vec{B} = \sum_{ik} A_i \hat{q}_i \cdot \hat{q}_k B_k = \sum_{ik} A_i B_k \delta_{ik} = \sum_i A_i B_i$
- $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{q}_1 & \hat{q}_2 & \hat{q}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$
- To perform a double integral in a curvilinear coordinate one needs to express a cartesian surface element in terms of the curvilinear coordinates.
- $d\vec{r}_1 = \vec{r}(q_1 + dq_1, q_2) - \vec{r}(q_1, q_2) = \frac{\partial \vec{r}}{\partial q_1} dq_1 \quad d\vec{r}_2 =$
  $\vec{r}(q_1, q_2 + dq_2) - \vec{r}(q_1, q_2) = \frac{\partial \vec{r}}{\partial q_2} dq_2$
- $dxdy = d\vec{r}_1 \times d\vec{r}_2|_z = [\frac{\partial x}{\partial q_1}\frac{\partial y}{\partial q_2} - \frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_1}] dq_1 dq_2 =$
  $\begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{vmatrix} dq_1 dq_2$

Curvilinear coordinates

- The transformation coefficient in determinant form is called the Jacobian
- Similarly, $dxdydz = dr_1 \cdot (dr_2 \times dr_3)$
- $dxdydz = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{vmatrix} dq_1 dq_2 dq_3$
- Volume Jacobian is $h_1 h_2 h_3 (\hat{q}_1 \times \hat{q}_2) \cdot \hat{q}_3$
- In polar coordinates: $x = \rho \cos\phi \quad y = \rho \sin\phi \quad J = ?$
- In spherical coordinates:
  $x = r \sin\theta \cos\phi \quad y = r \sin\theta \sin\phi \quad z = r \cos\theta \quad J = ?$
- The Jacobian matrix of a vector-valued function in several variables is the matrix of all its first-order partial derivatives.
- When this matrix is square, its determinant is referred to as the Jacobian determinant.

Jacobian

- $\mathbf{J} = \begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}.$
- At each point where a function is differentiable, its Jacobian matrix can also be thought of as describing the amount of "stretching", "rotating" or "transforming" that the function imposes locally near that point.
- If f is differentiable at a point p in R^n, then its differential is represented by $J_f(p)$. The linear transformation represented by $J_f(p)$ is the best linear approximation of f near the point p.
- $\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{p}) = \mathbf{J_f}(\mathbf{p})(\mathbf{x} - \mathbf{p}) + o(\|\mathbf{x} - \mathbf{p}\|) \quad (\text{as } \mathbf{x} \to \mathbf{p}),$
- $o(\|\mathbf{x} - \mathbf{p}\|)$ is a quantity that approaches zero much faster than the distance between x and p does as x approaches p.

Jacobian

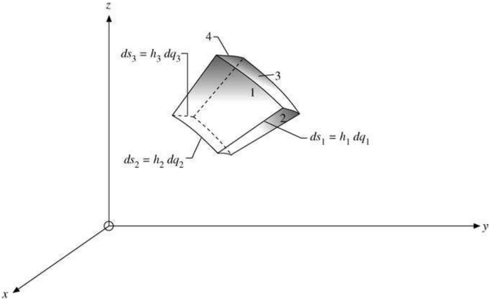
- the Jacobian may be regarded as a kind of "first-order derivative" of a vector-valued function of several variables.
- The Jacobian of the gradient of a scalar function of several variables has a special name: the Hessian matrix, which in a sense is the "second derivative" of the function.
- Inverse function theorem: the continuously differentiable function f is invertible near a point $p \in R^n$ if the Jacobian determinant at p is non-zero.
- The absolute value of the Jacobian determinant at p gives the factor by which the function f expands or shrinks volumes near p.
- the n-dimensional dV element is in general a parallelepiped in the new coordinate system, and the n-volume of a parallelepiped is the determinant of its edge vectors.

Jacobian

- According to the inverse function theorem, the matrix inverse of the Jacobian matrix of an invertible function is the Jacobian matrix of the inverse function.
- E.g., The Jacobian matrix of the function $F : R^3 \to R^4$ with components
$$\begin{aligned} y_1 &= x_1 \\ y_2 &= 5x_3 \\ y_3 &= 4x_2^2 - 2x_3 \\ y_4 &= x_3 \sin x_1 \end{aligned}$$
- E.g., The Jacobian determinant of the function $F : R^3 \to R^3$ with components
$$\begin{aligned} y_1 &= 5x_2 \\ y_2 &= 4x_1^2 - 2\sin(x_2 x_3) \\ y_3 &= x_2 x_3 \end{aligned}$$

Differential vector operations in orthogonal coordinates

- Gradient is the vector of maximum space rate of change
- Since ds_i is the differential length in the direction of increasing q_i, this direction is depicted by the unit vector $\hat{q}_i$.
  $\nabla\psi \cdot \hat{q}_i = \nabla\psi|_i = \frac{\partial\psi}{\partial s_i} = \frac{\partial\psi}{h_i \partial q_i}$.
- $\nabla\psi(q_1, q_2, q_3) = \hat{q}_1 \frac{\partial\psi}{\partial s_1} + \hat{q}_2 \frac{\partial\psi}{\partial s_2} + \hat{q}_3 \frac{\partial\psi}{\partial s_3} = \hat{q}_1 \frac{\partial\psi}{h_1 \partial q_1} + \hat{q}_2 \frac{\partial\psi}{h_2 \partial q_2} + \hat{q}_3 \frac{\partial\psi}{h_3 \partial q_3}$
- $d\psi = \nabla\psi \cdot dr = \sum_i \frac{1}{h_i} \frac{\partial\psi}{\partial q_i} ds_i = \sum_i \frac{\partial\psi}{\partial q_i} dq_i$
- $\nabla \cdot \vec{V}(q_1, q_2, q_3) = \lim_{d\tau \to 0} \frac{\int_{S_{d\tau}} \vec{V} \cdot d\vec{\sigma}}{d\tau}$



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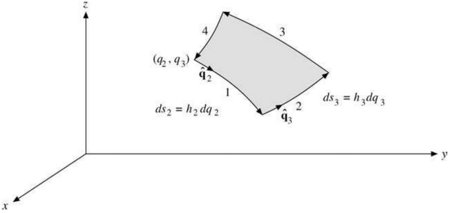
Differential vector operations: Divergence

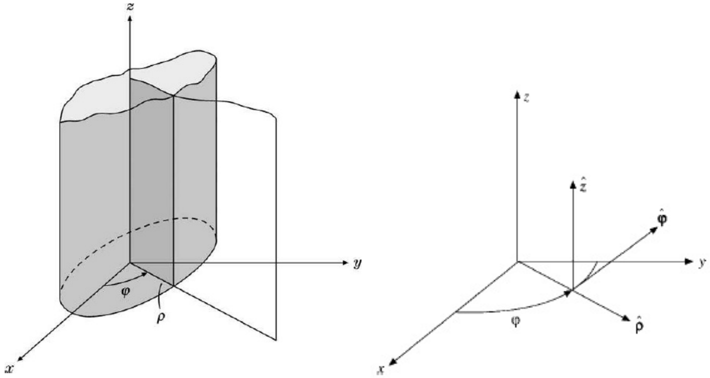
- Area integrals for the two q_1 = constant surfaces are $V_1(q_1+dq_1,q_2,q_3)ds_2ds_3 - V_1(q_1,q_2,q_3)ds_2ds_3 = [V_1h_2h_3 + \frac{\partial}{\partial q_1}(V_1h_2h_3)dq_1]dq_2dq_3 - V_1h_2h_3dq_2dq_3 = \frac{\partial}{\partial q_1}(V_1h_2h_3)dq_1dq_2dq_3$
- $\int \vec{V}\cdot d\sigma = [\frac{\partial}{\partial q_1}(V_1h_2h_3) + \frac{\partial}{\partial q_2}(V_2h_1h_3) + \frac{\partial}{\partial q_3}(V_3h_2h_1)]dq_1dq_2dq_3$ where $V_i = \hat{q}_i\cdot\vec{V}$
- $\nabla\cdot\vec{V}(q_1,q_2,q_3) = \frac{1}{h_1h_2h_3}[\frac{\partial}{\partial q_1}(V_1h_2h_3) + \frac{\partial}{\partial q_2}(V_2h_1h_3) + \frac{\partial}{\partial q_3}(V_3h_2h_1)]$
- Using $\vec{V} = \nabla\psi(q_1,q_2,q_3), \quad \nabla\cdot\vec{V} = \nabla^2\psi = \frac{1}{h_1h_2h_3}[\frac{\partial}{\partial q_1}(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial q_1}) + \frac{\partial}{\partial q_2}(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial q_2}) + \frac{\partial}{\partial q_3}(\frac{h_2h_1}{h_3}\frac{\partial\psi}{\partial q_3})]$

Differential vector operations: Curl

- Assuming the surface s to lay on q_1 = constant surface.
- $\lim_{s\to 0}\int_s \nabla\times\vec{V}\cdot d\vec{\sigma} = \hat{q}_1\cdot(\nabla\times\vec{V})h_2h_3dq_2dq_3 = \oint_{\partial_s}\vec{V}\cdot d\vec{r}$
- 

- $\oint_{\partial_s}\vec{V}\cdot d\vec{r} = V_2h_2dq_2 + [V_3h_3 + \frac{\partial}{\partial q_2}(V_3h_3)dq_2]dq_3 - [V_2h_2 + \frac{\partial}{\partial q_3}(V_2h_2)dq_3]dq_2 - V_3h_3dq_3 = [\frac{\partial}{\partial q_2}(V_3h_3) - \frac{\partial}{\partial q_3}(V_2h_2)]dq_2dq_3$
- $\nabla\times\vec{V}|_1 = \frac{1}{h_2h_3}[\frac{\partial}{\partial q_2}(V_3h_3) - \frac{\partial}{\partial q_3}(V_2h_2)]$
- Permuting the indices $\nabla\times\vec{V}|_2 = \frac{1}{h_3h_1}[\frac{\partial}{\partial q_3}(V_1h_1) - \frac{\partial}{\partial q_1}(V_3h_3)]$
- Thus $\nabla\times\vec{V} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{q}_1 & h_2\hat{q}_2 & h_3\hat{q}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1V_1 & h_2V_2 & h_3V_3 \end{vmatrix}$

Circular cylindrical coordinates

- (ρ, ϕ, z), $\quad 0 \le \rho < \infty$, $\quad 0 \le \phi \le 2\pi$, and $-\infty < z < \infty$
- 

- $x = \rho \cos\phi$, $\quad y = \rho \sin\phi$, $\quad z = z$
- Using: $h_{ij}^2 = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j}$
- $h_1 = h_\rho = 1$, $\quad h_2 = h_\phi = \rho$, $\quad h_3 = h_z = 1$.
- $\vec{r} = \hat{\rho}\rho + \hat{z}z$, $\quad \vec{V} = \hat{\rho}V_\rho + \hat{\phi}V_\phi + \hat{z}V_z$

Circular cylindrical coordinates

- $\nabla\psi(\rho,\phi,z) = \hat{\rho}\frac{\partial\psi}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{k}\frac{\partial\psi}{\partial z}$
- $\nabla\cdot V = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\phi}{\partial\phi} + \frac{\partial V_z}{\partial z}$
- $\nabla^2\psi = \frac{1}{\rho}[\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) + \frac{\partial}{\partial\phi}(\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}) + \frac{\partial}{\partial z}(\rho\frac{\partial\psi}{\partial z})]$
- $\nabla\times V = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\phi & V_z \end{vmatrix}$

Spherical polar coordinates

- $(r, \theta, \phi), \quad 0 \le r < \infty, \quad 0 \le \theta \le \pi$, and $0 < \phi < 2\pi$
- $x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta$
- $h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_\phi = r\sin\theta.$
- $\hat{r} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta, \quad \hat{\theta} = \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta, \quad \hat{\phi} = -\hat{i}\sin\phi + \hat{j}\cos\phi$
- $\nabla\psi = \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$
- $\nabla\cdot\vec{V} = \frac{1}{r^2\sin\theta}[\sin\theta\frac{\partial}{\partial r}(r^2 V_r) + r\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + r\frac{\partial V_\phi}{\partial\phi}]$
- $\nabla\cdot\nabla\psi = \frac{1}{r^2\sin\theta}[\sin\theta\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) + \frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\phi^2}]$
- $\nabla\times\vec{V} =$
- $\nabla f(r) = \qquad , \nabla r^n =$
- $\nabla\cdot\hat{r}f(r) = \qquad \nabla\cdot\hat{r}r^n$
- $\nabla^2 f(r) = \qquad , \nabla^2 r^n$
- $\nabla\times\hat{r}f(r) =$

Separation of variables

- Helmholtz equation: $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0$
- Assume $\psi(x, y, z) = X(x)Y(y)Z(z)$
- $YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} + k^2XYZ = 0$
- $\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 - \frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2}$
- $\frac{1}{X}\frac{d^2X}{dx^2} = -l^2, \qquad -k^2 - \frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2} = -l^2$
- $\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2 + l^2 - \frac{1}{Z}\frac{d^2Z}{dz^2}$
- $\frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2, \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2 + l^2 + m^2 = -n^2$
- $\psi_{lmn}(x, y, z) = X_l(x)Y_m(y)Z_n(z)$
- $\Psi = \sum_{lmn} a_{lmn}\psi_{lmn}(x, y, z)$
- Process would work for $k^2 = f(x) + g(y) + h(z) + k'^2$, e.g.,
  $\frac{1}{X}\frac{d^2X}{dx^2} + f(x) = -l^2, \qquad l^2 = k'^2 + \frac{1}{Y}\frac{d^2Y}{dy^2} + g(y) + \frac{1}{Z}\frac{d^2Z}{dz^2} + h(z)$
- $\frac{1}{Y}\frac{d^2Y}{dy^2} + g(y) = l^2 - k'^2 - \frac{1}{Z}\frac{d^2Z}{dz^2} - h(z)$

Separation of variables

- In spherical polar coordinates Helmholtz equation read: $\frac{1}{r^2\sin\theta}[\sin\theta\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) + \frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\phi^2}] = -k^2\psi$
- Try $\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$
- $\frac{1}{Rr^2}\frac{d}{dr}(r^2\frac{dR}{dr}) + \frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + \frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = -k^2$
- $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = r^2\sin^2\theta[-k^2 - \frac{1}{Rr^2}\frac{d}{dr}(r^2\frac{dR}{dr}) - \frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta})]$
- $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2$
- $\frac{1}{Rr^2}\frac{d}{dr}(r^2\frac{dR}{dr}) + \frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) - \frac{m^2}{r^2\sin^2\theta} = -k^2$
- $\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr}) + r^2k^2 = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + \frac{m^2}{\sin^2\theta}$
- $\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) - \frac{m^2}{\sin^2\theta}\Theta + Q\Theta = 0$
- $\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dR}{dr}) + k^2R - \frac{QR}{r^2} = 0$
- Most general solution: $\psi_{Qm}(r,\theta,\phi) = \sum_{Qm} R_Q(r)\Theta_{Qm}(\theta)\Phi_m(\phi)$

Matrices

- A two dimensional array of elements is called a matrix.
- A matrix with m rows and n columns is called an m by n matrix.
- If number of rows and columns are equal matrix is called square matrix.
- Matrix A is determined by determining its elements a_{ij}.
- A+B = C iff $a_{ij} + b_{ij} = c_{ij}$
- AB = C iff $c_{ij} = \sum_k a_{ik} b_{kj}$
- The identity matrix $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$.
- If matrix A is composed of elements a_{ij}, transpose of A, A^T, is composed of elements a_{ji}.
- Inverse of the sqare matrix A is defined by $AA^{-1} = A^{-1}A = I$.

- Find the inverse of matrix A by placing an identity matrix I on A's side and turning A into I by row operations.
- Find the inverse of matrix A by placing identity matrix I below or on top of A and turning A into I by column operations.
- If A is a square matrix, then the minor of the entry in the i-th row and j-th column (also called the (i,j) minor, or a first minor) is the determinant of the submatrix formed by deleting the i-th row and j-th column. This number is often denoted $\mathrm{M}_{\mathrm{i,j}}$. The (i,j) cofactor is obtained by multiplying the minor by $(-1)^{i+j}$.
- The cofactors feature prominently in Laplace's formula for the expansion of determinants, which is a method of computing larger determinants in terms of smaller ones. Given the $n \times n$ matrix (a_{ij}), det(A) can be written as the sum of the cofactors of any row or column of the matrix multiplied by the entries that generated them.

- The cofactor expansion along the jth column gives: $\det(\mathbf{A}) = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$
- The cofactor expansion along the ith row gives: $\det(\mathbf{A}) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$
- One can write down the inverse of an invertible matrix by computing its cofactors and using Cramer's rule.
- The matrix formed by all of the cofactors of a square matrix A is called the cofactor matrix or comatrix:
$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}.$$
- The transpose of the cofactor matrix is called the adjugate matrix or the classical adjoint of A.
- Then the inverse of A is the transpose of the cofactor matrix times the reciprocal of the determinant of A: $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathsf{T}}$.

Symmetric Matrices

- A is symmetric $\iff A = A^{\mathsf{T}} \iff$ for every $i, j, \quad a_{ji} = a_{ij}$.
- A real symmetric matrix represents a self-adjoint operator over a real inner product space.
- The corresponding object for a complex inner product space is a Hermitian matrix with complex-valued entries, which is equal to its conjugate transpose.
- The sum and difference of two symmetric matrices is again symmetric.
- given symmetric matrices A and B, then AB is symmetric if and only if A and B commute.
- For integer n, A^n is symmetric if A is symmetric.
- If A^{-1} exists, it is symmetric if and only if A is symmetric.
- $X = \frac{1}{2}\left(X + X^{\mathsf{T}}\right) + \frac{1}{2}\left(X - X^{\mathsf{T}}\right)$.
- A symmetric $n \times n$ matrix is determined by $\frac{1}{2}n(n+1)$ scalars.
- A skew-symmetric matrix is determined by $\frac{1}{2}n(n-1)$ scalars.

Symmetric Matrices

- Two square matrices A and B over a field are called congruent if there exists an invertible matrix P over the same field such that $P^T AP = B$.
- Matrix congruence is an equivalence relation.
- Matrix congruence arises when considering the effect of change of basis.
- Any matrix congruent to a symmetric matrix is symmetric: if X is a symmetric matrix then so is AXA^{T} for any matrix A.
- The real $n \times n$ matrix A is symmetric if and only if $\langle Ax, y\rangle = \langle x, Ay\rangle \quad \forall x, y \in \mathbb{R}^n$.

Unitary Matrices

- A complex square matrix U is unitary if its conjugate transpose $U^\dagger$ is also its inverse–that is, if $U^\dagger U = UU^\dagger = I$, where I is the identity matrix.
- The Hermitian conjugate of a matrix is denoted by a dagger (†) and the equation above becomes $U^\dagger U = UU^\dagger = I$.
- The real analogue of a unitary matrix is an orthogonal matrix.
- Unitary matrices have significant importance in quantum mechanics because they preserve norms, and thus, probability amplitudes.

Symmetric Matrices

- For every symmetric real matrix A there exists a real orthogonal matrix Q such that $D = Q^{\mathrm{T}}AQ$ is a diagonal matrix. Every symmetric matrix is thus, up to choice of an orthonormal basis, a diagonal matrix.
- If A and B are $n \times n$ real symmetric matrices that commute, they can be simultaneously diagonalized: there exists a basis of $\mathbb{R}^n$ such that every element of the basis is an eigenvector for both A and B.
- Every real symmetric matrix is Hermitian, and therefore all its eigenvalues are real.
- The property of being symmetric for real matrices corresponds to the property of being Hermitian for complex matrices.
- if A is a complex symmetric matrix, there is a unitary matrix U such that UAU^{T} is a real diagonal matrix with non-negative entries. This result is referred to as the Autonne–Takagi factorization.

Matrices

- Det(A) $= \epsilon_{i_1 \cdots i_n} a_{1i_1} \cdots a_{ni_n}$
- Theorem: $Det(AB) = Det(A)Det(B)$. Thus $Det(A^{-1}A) = Det(I) \to Det(A^{-1})Det(A) = 1 \to Det(A^{-1}) = \frac{1}{Det(A)}$
- Matrix A is invertible iff $Det(A) \neq 0$
- Orthogonal matrix R satisfies $R^{-1} = R^T$ or equivalently $RR^T = I$
- det(R^T)=det(R) thus for an orthogonal matrix det$(R)^2$=1.
- Special orthogonal matrix: A real orthogonal matrix with det(R)=1; it represents a proper rotation.
- Eigenvalues and eigenvectors: $A\vec{V} = \lambda\vec{V}$
- $(A - \lambda I)\vec{V} = 0, \qquad det(A - \lambda I) = 0$ which is the characteristic polynomial.
- A similarity transformation or conjugation is $B = P^{-1}AP$. Similar matrices represent the same linear operator under two different bases, with P being the change of basis matrix.

Matrices

- Consider a general rotation $y = Tx$ using the change of basis matrix P, $y' = Sx' \rightarrow Py = SPx \rightarrow y = (P^{-1}SP)x = Tx$
- Similarity is an equivalence relation on the space of square matrices.
- Similar matrices share properties of their underlying operator:
- Rank: the rank of a matrix A is the dimension of the vector space generated (or spanned) by its columns.
- The column rank of A is the dimension of the column space of A, while the row rank of A is the dimension of the row space of A.
- This number (i.e., the number of linearly independent rows or columns) is simply called the rank of A.
- A matrix is said to have full rank if its rank equals the largest possible for a matrix of the same dimensions, which is the lesser of the number of rows and columns. A matrix is said to be rank deficient if it does not have full rank.

Matrices

- if a linear operator on a vector space has finite-dimensional image, then the rank of the operator is defined as the dimension of the image.
- Characteristic polynomial: the characteristic polynomial of a square matrix is a polynomial which is invariant under matrix similarity transformation and has the eigenvalues as roots. It has the determinant and the trace of the matrix as coefficients. $p_A(t) = det(tI - A)$
- The characteristic equation is the equation obtained by equating the characteristic polynomial to zero.
- $\mathrm{p_A(t)}$ is monic (its leading coefficient is 1) and its degree is n.
- It's constant coefficient $\mathrm{p_A}(0)$ is $\det(-A) = (-1)^n \det(A)$, the coefficient of t^n is one, and the coefficient of t^{n-1} is tr(-A) $=$ -tr(A).
- $t^2 - tr(A)t + \det(A)$.

Matrices

- A complex square matrix A is normal if it commutes with its conjugate transpose A†: A normal $\iff A^\dagger A = AA^\dagger$
- The spectral theorem states that a matrix is normal if and only if it is unitarily similar to a diagonal matrix, and therefore any matrix A satisfying the equation $A^\dagger A = AA^\dagger$ is diagonalizable.
- Among complex matrices, all unitary, Hermitian, and skew-Hermitian matrices are normal.
- Likewise, among real matrices, all orthogonal, symmetric, and skew-symmetric matrices are normal.
- A defective matrix is a square matrix that does not have a complete basis of eigenvectors, and is therefore not diagonalizable.
- An $n \times n$ matrix is defective if and only if it does not have n linearly independent eigenvectors.

- Eigendecomposition of a matrix:
- Let A be a square $n \times n$ matrix with n linearly independent eigenvectors q_i (where $i = 1, \ldots, n$). Then A can be factorized as $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$
- Q is the square $n \times n$ matrix whose ith column is the eigenvector q_i of A, and Λ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, $\Lambda_{ii} = \lambda_i$.
- The decomposition can be derived from the fundamental

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

property of eigenvectors: $\mathbf{A}\mathbf{Q} = \mathbf{Q}\mathbf{\Lambda}$

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}.$$

Matrices

- Consider a system of n first order linear equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

- Such a system can be written in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Matrices

- $AX = B$
- If $Det(A) \neq 0, \quad X = A^{-1}B$ and is uniquely determined.
- If $B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ the above system of linear equations is called homogeneous.
- In order for this system to have any solution other than the trivial $X = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, Det(A) must equal zero.

Determinants

- Slater expansion
- Jacobi's formula: $\frac{d}{dt}\det A(t) = \operatorname{tr}\left(\operatorname{adj}(A(t))\,\frac{dA(t)}{dt}\right)$
- Corollary: $\det e^{tB} = e^{\operatorname{tr}(tB)}$

Tensors

- Every scaler is a tensor of rank 0.
- Every vector is a tensor of rank 1.
- Contravariant vector: $A'_i = \sum_j \frac{\partial x'_i}{\partial x_j} A_j$, e.g., spatial coordinate
- Covariant vector: $A'_i = \sum_j \frac{\partial x_j}{\partial x'_i} A_j$, e.g., gradient vector.
- In cartesian cordinates $\frac{\partial x_j}{\partial x'_i} = \frac{\partial x'_i}{\partial x_j} = a_{ij}$
- Every square matrix is a tensor of rank 2, if its components translate in a coordinate rotation according to:
  $A'^{ij} = \sum_{kl} \frac{\partial x'_i}{\partial x_k} \frac{\partial x'_j}{\partial x_l} A^{kl}$ $\qquad B'^{i}_{j} = \sum_{kl} \frac{\partial x'_i}{\partial x_k} \frac{\partial x_l}{\partial x'_j} B^k_l$ $\qquad C'_{ij} = \sum_{kl} \frac{\partial x_k}{\partial x'_i} \frac{\partial x_l}{\partial x'_j} A^{kl}$
- $\begin{bmatrix} -xy & -y^2 \\ x^2 & xy \end{bmatrix}$ is a tensor, while $\begin{bmatrix} -xy & -y^2 \\ x^2 & -xy \end{bmatrix}$ is not.
- Show that the Kronecker delta δ^k_l is a mixed second rank tensor.

Contraction, direct product

- Kronecker delta is isotropic.
- A symmetric tensor $A^{mn} = A^{nm}$
- An antisymmetric tensor $A^{mn} = -A^{nm}$
- Contraction is a generalisation of trace, it reduces the rank by 2.
- Contraction include equating a contravariant and a covariant index and summing over that common index.
- Direct product: covariant vector a_i and contravariant vector b^j multiplied component by component, give tensor $a_i b^j$
- $a'_i b'^j = \frac{\partial x_k}{\partial x'_i} a_k \frac{\partial x'_j}{\partial x_l} b^l = \frac{\partial x_k}{\partial x'_i} \frac{\partial x'_j}{\partial x_l} (a_k b^l)$
- Contracting direct product $a'_i b'^j = a_k b^k$, giving the regular scaler product.

Contraction, direct product

- Quotient rule: If A and B in the relations $K_iA_i = B \qquad K_{ij}A_j = B_i \qquad K_{ij}A_{jk} = B_{ik} \qquad K_{ijkl}A_{ij} = B_{kl} \qquad K_{ij}A_k = B_{ijk}$ are tensors and these relations are valid in all cartesian coordinates, then K is also a tensor.
- An inversion has transformation coefficients $a_{ij} = \delta_{ij}$
- Polar vectors: An inversion of coordinate axes and a change in the sign of coordinates leaves the vector unchanged after an inversion, e.g., distance vector.
- Pseudo vector or axial vector: For vectors defined as cross product of two polar vectors, component sign does not change by coordinate transformation and thus the vectors inverts by coordinate axes inversion.

Contraction, direct product

- E.g., angular velocity, $\omega = r \times v$; angular momentum, $L = r \times p$; torque, $N = r \times f$; magnetic induction field B, $\frac{\partial B}{\partial t} = -\nabla \times E$.
- Pseudovector and pseudotensor transformations: $C'_i = |a| a_{ij} C_j$, $\qquad A'_{ij} = |a| a_{ik} a_{jl} A_{kl}$, where $|a|$ is the determinant of the array of coefficients.
- Tensor densities: differing from tensors only in reflections or inversions of the coordinates.
- Triple scaler product $S = A \times B \cdot C$ is a pseudoscaler, thus volume is a pseudoscaler.

Euler angles

- The Euler angles are three angles introduced by Leonhard Euler to describe the orientation of a rigid body with respect to a fixed coordinate system.
- They can also represent the orientation of a mobile frame of reference in physics or the orientation of a general basis in 3-dimensional linear algebra.
- The three elemental rotations may be extrinsic (rotations about the axes xyz of the original coordinate system, which is assumed to remain motionless), or intrinsic (rotations about the axes of the rotating coordinate system XYZ, solidary with the moving body, which changes its orientation after each elemental rotation).
- Euler angles are typically denoted as α, β, γ, or ϕ, θ, ψ.
- Proper Euler angles (z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y)
- Tait–Bryan angles (x-y-z, y-z-x, z-x-y, x-z-y, z-y-x, y-x-z)

- The axes of the original frame are denoted as x, y, z and the axes of the rotated frame as X, Y, Z.

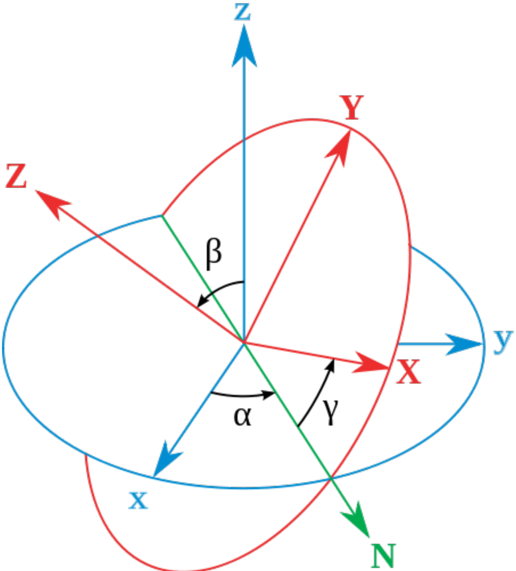


Figure: Proper Euler angles geometrical definition. The xyz (fixed) system is shown in blue, the XYZ (rotated) system is shown in red. The line of nodes (N) is shown in green, taken from `en.wikipedia.org`

Euler angles

- line of nodes as the intersection of the planes xy and XY (it can also be defined as the common perpendicular to the axes z and Z and then written as the vector product $N = z \times Z$).
- α (or φ) is the angle between the x axis and the N axis (x-convention - it could also be defined between y and N, called y-convention)
- β (or θ) is the angle between the z axis and the Z axis
- γ (or ψ) is the angle between the N axis and the X axis (x-convention).
- Euler angles between two reference frames are defined only if both frames have the same handedness.
- Intrinsic rotations are elemental rotations that occur about the axes of a coordinate system XYZ attached to a moving body. Therefore, they change their orientation after each elemental rotation.

Euler angles

- α (or φ) represents a rotation around the z axis, β (or θ) represents a rotation around the x' axis, γ (or ψ) represents a rotation around the z" axis.
- Extrinsic rotations are elemental rotations that occur about the axes of the fixed coordinate system xyz. The XYZ system rotates, while xyz is fixed.
- Starting with XYZ overlapping xyz, a composition of three extrinsic rotations can be used to reach any target orientation for XYZ. The Euler or Tait–Bryan angles (α, β, γ) are the amplitudes of these elemental rotations.
- The XYZ system rotates about the z axis by γ. The X axis is now at angle γ with respect to the x axis.
- The XYZ system rotates again about the x axis by β. The Z axis is now at angle β with respect to the z axis.
- The XYZ system rotates a third time about the z axis by α.

Euler angles

- for α and γ, the range is defined modulo 2π radians.
- for β, the range covers π radians (but can't be said to be modulo π).
- Precession, nutation, and intrinsic rotation (spin) are defined as the movements obtained by changing one of the Euler angles while leaving the other two constant.

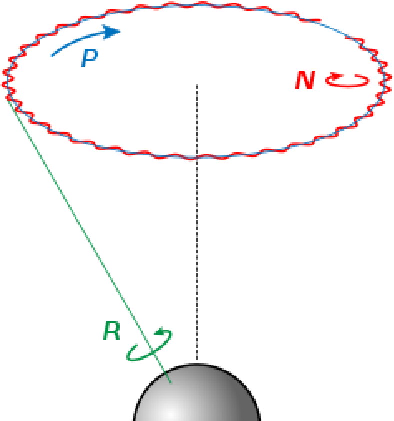


Figure: Euler basic motions of the Earth. Intrinsic (green), Precession (blue) and Nutation (red) taken from en.wikipedia.org

Euler angles

- any rotation matrix R can be decomposed as a product of three elemental rotation matrices
- $R = X(\alpha)Y(\beta)Z(\gamma)$ is a rotation matrix that may be used to represent a composition of extrinsic rotations about axes z, y, x, (in that order), or a composition of intrinsic rotations about axes x-y'-z''
- the number of euler angles in dimension D is quadratic in D; since any one rotation consists of choosing two dimensions to rotate between, the total number of rotations available in dimension D is $N_{rot} = \binom{D}{2} = D(D-1)/2$.
- When studying rigid bodies in general, one calls the xyz system space coordinates, and the XYZ system body coordinates.

Euler angles

- The space coordinates are treated as unmoving, while the body coordinates are considered embedded in the moving body.
- Calculations involving acceleration, angular acceleration, angular velocity, angular momentum, and kinetic energy are often easiest in body coordinates, because then the moment of inertia tensor does not change in time.
- If one also diagonalizes the rigid body's moment of inertia tensor (with nine components, six of which are independent), then one has a set of coordinates (called the principal axes) in which the moment of inertia tensor has only three components.

Leibniz integral rule

- $\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) =$
  $f\left(x, b(x)\right) \cdot \frac{d}{dx} b(x) - f\left(x, a(x)\right) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt,$
- A generalisation of the fundamental theorem of calculus; if $F(x) = \int_a^x f(t)\,dt$ then F'(x)=f(x),
- $F(x_1 + \Delta x) - F(x_1) = \int_a^{x_1+\Delta x} f(t)\,dt - \int_a^{x_1} f(t)\,dt = \int_{x_1}^{x_1+\Delta x} f(t)\,dt.$
- $F(x_1 + \Delta x) - F(x_1) = f(c) \cdot \Delta x.$
- $\lim_{\Delta x \to 0} \frac{F(x_1+\Delta x)-F(x_1)}{\Delta x} = \lim_{\Delta x \to 0} f(c).$

Differential equations

- Ordinary differential equations only contain functions of a single variable.
- Differential equations with partial derivatives include functions of more than one variable.
- The highest order derivative in the differential equation determines the order of the differential equation.
- $(y'')^3 + 2yy' + 5xy = \sin x$ is an ordinary differential equation of order 2.
- $(\frac{dy}{dx})^2 - [\sin(xy) - 4x]^2 = 0$ is an ordinary differential equation of the first order.
- $\frac{\partial^3 u}{\partial x^3} + x\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x \partial t} = 0$ is a differential equation with partial derivatives of the third order.

Ordinary differential equations

- $F(x, y, y', \cdots, y^{(n)}) = 0$ on an interval I.
- F is rewritten as, $y^{(n)} = f(x, y, \cdots, y^{(n-1)})$
- A function ϕ such that $\phi^{(n)} = f(x, \phi, \cdots, \phi^{(n-1)})$ is a solution to this differential equation on I.
- Initial conditions are restrictions on the solution at a single point, while boundary conditions are restrictions on the solution at different points.
- E.g., $y' = 2y - 4x \rightarrow y = ce^{2x} + 2x + 1$
- E.g., $y'' + y = x \rightarrow y = c_1 \cos x + c_2 \sin x + x$
- $a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x)$ is a linear ordinary differential equation which constitutes our focus in this section of the course.

- $y^{(4)} + 4y''' + 3y = x; \qquad y_1 = \frac{x}{3}, \quad y_2 = e^{-x} + \frac{x}{3}$
- $x^2 y'' + 5xy' + 4y = 0,\ x > 0; \quad y_1 = x^{-2}, y_2 = x^{-2} \ln x$
- $y' - 2xy = 1; \quad y = e^{x^2} \int_0^x e^{-t^2} dt + e^{x^2}$
- $u_{xx} + u_{yy} = 0; \quad u_1 = x^2 + y^2, u_2 = xy$
- $u_{tt} - c^2 u_{xx} = 0; \qquad u_1 = \sin(x + ct), u_2 = \sin(x - ct), u_3 = f(x + ct) + g(x - ct)$
- $u_{xx} + u_{yy} + u_{zz} = 0; u = (x^2 + y^2 + z^2)^{-1/2}$
- $x^2 y'' + xy' + y = 0, y(1) = 1, y'(1) = -1; \quad y = \cos(\ln x) - \sin(\ln x)$

First order differential equations

- $y' = f(x, y) \quad y(x_0) = y_0$ there exists a unique solution if f and $\frac{\partial f}{\partial x}$ are continuous around (x_0, y_0).
- First order linear differential equations: $\frac{dy}{dx} + a(x)y = f(x)$
- Assuming $A(x) = \int^x a(t)dt$,
  $\frac{d}{dx}(e^{A(x)}y) = e^{A(x)}(y' + a(x)y) = e^{A(x)}f(x)$
- General solution is: $y = e^{-A(x)} \int^x e^{A(t)} f(t)dt + ce^{-A(x)}$
- Imposing the initial condition, $y(x_0) = y_0$,
  $y = e^{-A(x)} \int_{x_0}^x e^{A(t)} f(t)dt + y_0 e^{-(A(x)-A(x_0))}$
- e.g., $y' = y + \sin x$, $e^{-x}(y' - y) = (e^{-x}y)' = e^{-x}\sin x$
- $e^{-x}y = \int^x e^{-t} \sin t dt + c = \frac{-1}{2}e^{-x}(\sin x + \cos x) + c$

First order differential equations

- Solve $y' = y + \sin x, \quad y(0) = 1$
- $(x \ln x)y' + y = 6x^3, \quad x > 1$, thus $(y \ln x)' = 6x^2$,
  $y = \frac{2x^3+c}{\ln x} \quad x > 1$.
- Assuming $a(x)$ and $f(x)$ to be continuous on the interval (α, β) for every $x_0 \in (\alpha, \beta)$, the initial value problem $y' + a(x)y = f(x) \quad y(x_0) = y_0$, for every value of y_0 has one and only one solution on the interval (α, β).
- $xy' + 2y = 4x^2, \quad x > 0, \qquad y(1) = 2$, result in $y = x^2 + \frac{c}{x^2}$.
- Solve it for y(1)=1.

First order differential equations

- $y' + \frac{y}{x} = 3\cos 2x, x > 0$
- $y' + 3y = x + e^{-2x}$
- $(x^2+1)y' + y + 1 = 0$
- $y' \sin 2x = y \cos 2x$
- $xy' + y + 4 = 0, x > 0$
- $x^2 y' - xy = x^2 + 4, x > 0$
- $y' + 2y = xe^{-2x}; y(1) = 0$
- $y' + \frac{2}{x}y = \frac{\cos x}{x^2}; y(\pi) = 0$
- $y' + y \cot x = 2x - x^2 \cot x, y(\frac{\pi}{2}) = \frac{\pi^2}{4} + 1$
- $y' - x^3 y = -4x^3; y(0) = 6$
- $y' + y \tan x = \sin 2x; y(0) = 1$
- $\sin x y' + \cos x y = \cos 2x, x \in (0,\pi); y(\frac{\pi}{2}) = 1/2$
- $y' + \frac{y}{x} = e^{x^2},\ x > 0;\ y(1) = 0$
- $y' + y = xe^{-x}; y(0) = 1$

Nonlinear First order DEs

- For nonlinear equations there is no general method for solving the DE.
- Separable differential equations:
  $y' = f(x, y) \to p(x) + q(y)y' = 0$
- $p(x)dx + q(y)dy = 0 \to d[P(x) + Q(y)] = 0 \to$
  $P(x) + Q(y) = c \to y = \phi(x, c)$
- E.g., $y' = \frac{2+\sin x}{3(y-1)^2} \to (2 + \sin x)dx - 3(y-1)^2 dy = 0 \to$
  $2x - \cos x - (y-1)^3 = c \to y = 1 + (2x - \cos x - c)^{1/3}$
- E.g., $y' = \frac{x^3 y - y}{y^4 - y^2 + 1}, y(0) = 1 \to (y^3 - y + 1/y)dy =$
  $(x^3 - 1)dx \to y^4/4 - y^2/2 + \ln|y| = x^4/4 - x + c$

Complete first order DE

- $y' = -\frac{p(x,y)}{q(x,y)} \to p(x,y)dx + q(x,y)dy = 0$ this equation is complete in a region D if and only if there is a g such that $dg(x,y) = p(x,y)dx + q(x,y)dy$
- $\frac{\partial g}{\partial x} = p(x,y), \quad \frac{\partial g}{\partial y} = q(x,y)$
- E.g., For $(4x - y)dx + (2y - x)dy = 0, \quad g(x,y) = 2x^2 - xy + y^2$, g is an integral of the differential equation and the curves $g(x,y) = c$ are its integral curves.
- Theorem: The necessary and sufficient condition for completeness of $p(x,y)dx + q(x,y)dy = 0$ in a region D of the xy plane is to have $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}, \quad (x,y) \in D$

Complete first order DE

- The condition is necessary since $g_{xy} = g_{yx}$, to prove sufficiency consider g such that $g_x(x,y) = p(x,y), \quad g_y(x,y) = q(x,y)$ thus we have $g(x,y) = \int^x p(t,y)dt + h(y) \rightarrow g_y(x,y) = \int^x \frac{\partial p(t,y)}{\partial y} dt + h'(y) = q(x,y)$ thus $h'(y) = q(x,y) - \int^x \frac{\partial p(t,y)}{\partial y} dt$
- If we show that the right hand side is only a function of y, we have an algorithm for evaluating g.
- $\frac{\partial}{\partial x}[q(x,y) - \int^x \frac{\partial p(t,y)}{\partial y} dt] = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 0$
- E.g., $(4x - y)dx + (2y - x)dy = 0$ for which $\frac{\partial p}{\partial y} = -1, \quad \frac{\partial q}{\partial x} = -1$. Thus $dg(x,y) = (4x - y)dx + (2y - x)dy$

Completing a first order DE

- $g(x,y) = 2x^2 - xy + h(y)$ so
  $-x + h'(y) = 2y - x \quad h(y) = y^2 + c,$
- $g(x,y) = 2x^2 - xy + y^2 + c$
- Integration factor
- $\mu(x,y)p(x,y)dx + \mu(x,y)q(x,y)dy = 0$
- $\frac{\partial}{\partial y}(\mu p) = \frac{\partial}{\partial x}(\mu q)$
- $p(x,y)\frac{\partial \mu}{\partial y} - q(x,y)\frac{\partial \mu}{\partial x} + (\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x})\mu = 0.$ This PDE must be solved to find the integrating factor.
- E.g., $x^2 - y^2 + 2xyy' = 0$, Assuming
  $\mu = \mu(x), \quad \mu(x)(x^2 - y^2)dx + \mu(x)(2xy)dy = 0$
- $\frac{\partial}{\partial y}[\mu(x^2 - y^2)] = \frac{\partial}{\partial x}[\mu(2xy)] \rightarrow x\mu' + 2\mu = 0 \rightarrow \mu(x) = x^{-2}$
- $(1 - \frac{y^2}{x^2})dx + (\frac{2y}{x})dy = 0 \rightarrow x + y^2/x = c \rightarrow y^2 + (x-a)^2 = a^2$

Completing a first order DE: excersize

- $y' = x^3 y^{-2}$
- $(1+x^2)^{1/2} y' = 1 + y^2$
- $y' = xy^2 + y^2 + xy + y;\ Y(1) = 1$
- $(x+1)y' + y^2 = 0; y(0) = 1$
- $(2x - y)dx - xdy = 0$
- $(x - 2y)dx + (4y - 2x)dy = 0$
- $\frac{ydx - xdy}{y^2} + xdx = 0$
- $3(x-1)^2 dx - 2ydy = 0$
- $e^{x^2 y}(1 + 2x^2 y)dx + x^3 e^{x^2 y} dy = 0$
- $(x^2 + y^2)^2 (xdx + ydy) + 2dx + 3dy = 0$
- $(x^2 + y^2)dx + 2xydy = 0, y(1) = 1$
- $\frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y^2} = 0, y(2) = 2$
- $(x - y)dx + (2y - x)dy = 0, y(0) = 1$
- If $\mu = \mu(x)$, $\frac{\partial \mu}{\partial y} = 0, \quad \frac{d\mu}{\mu} = \frac{p_y - q_x}{q} dx$

Completing a first order DE

- If $\mu = \mu(y)$, $\quad \frac{d\mu}{\mu} = \frac{q_x - p_y}{p} dy$
- $(x^2 - y^2) - 2xyy' = 0$
- $y + (y^2 - x)y' = 0$
- $(3xy + y^2) + (x^2 + xy)y' = 0$
- $(3xy + y^2)dx + (3xy + x^2)dy = 0$

- Bernoulli equation: $y' + a(X)y = b(x)y^\alpha$ use $z = y^{1-\alpha}$ get $z' + (1-\alpha)a(x)z - (1-\alpha)b(x) = 0$
- $xy' - y = e^x y^3$
- Riccati equation: $y' = a(x)y + b(x)y^2 + c(x)$ assume $y = \phi(x)$ to be a private solution and use $y = \phi(x) + 1/z$ to derive $z' + [a(x) + 2\phi(x)b(x)]z = -b(x)$.
- $y' = 1 + x^2 - 2xy + y^2, \quad \phi(x) = x$
- $y' - xy^2 + (2x-1)y = x - 1, \quad \phi(x) = 1$
- $y' + xy^2 - 2x^2y + x^3 = x + 1, \quad \phi(x) = x - 1$
- $y' + y^2 - (1 + 2e^x)y + e^{2x} = 0, \quad \phi(x) = e^x$
- $y' + y^2 - 2y + 1 = 0$

Completing a first order DE

- If $\mu = \mu(y)$, $\quad \frac{d\mu}{\mu} = \frac{q_x - p_y}{p} dy$
- $(x^2 - y^2) - 2xyy' = 0$
- $y + (y^2 - x)y' = 0$
- $(3xy + y^2) + (x^2 + xy)y' = 0$
- $(3xy + y^2)dx + (3xy + x^2)dy = 0$
- $\mu = x + y$
- Bernoulli equation: $y' + a(X)y = b(x)y^\alpha$ use $z = y^{1-\alpha}$ get $z' + (1-\alpha)a(x)z - (1-\alpha)b(x) = 0$
- $xy' - y = e^x y^3$
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- $y' = 1 + x^2 - 2xy + y^2, \quad \phi(x) = x$
- $y' - xy^2 + (2x-1)y = x - 1, \quad \phi(x) = 1$
- $y' + xy^2 - 2x^2y + x^3 = x + 1, \quad \phi(x) = x - 1$
- $y' + y^2 - (1 + 2e^x)y + e^{2x} = 0, \quad \phi(x) = e^x$
- $y' + y^2 - 2y + 1 = 0$

- $a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x)$
- $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x)$
- $L_n \equiv \frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x)$
- $L_n[y] = f(x)$
- Existence and uniqueness theorem: If $p_1, p_2, \cdots, p_n$ and f are continuous on the interval I, $\forall x_0 \in I$ the above equation has one and only one solution $y = \phi(x)$ satisfying $\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \phi''(x_0) = \alpha_3, \cdots, \phi^{(n-1)}(x_0) = \alpha_n$.
- $y'' + p(x)y' + q(x)y = 0; \quad y(x_0) = 0, y'(x_0) = 0$ only has the trivial solution.

Linear differential equations

- $xy'' + (\cos x)y' + \frac{x}{1+x}y = 2x$ solutions can be determined for each of the intervals $(-\infty, -1), (-1, 0)$ and $(0, \infty)$.
- Homogeneous differential equations have f(x)=0. E.g., $y'' + p(x)y' + q(x)y = 0$.
- Operator L is called linear iff for arbitrary constants $c_1, c_2, c_3, \cdots, c_k$ and functions $\phi_1, \phi_2, \cdots, \phi_k$; $\quad L[c_1\phi_1 + c_2\phi_2 + \cdots + c_k\phi_k] = c_1 L[\phi_1] + c_2 L[\phi_2] + \cdots + c_k L[\phi_k]$.
- $c_1\phi_1 + c_2\phi_2 + \cdots + c_k\phi_k = \sum_i c_i\phi_i$ is a linear combination of the k functions ϕ_i.
- If $\phi_1, \phi_2, \cdots, \phi_k$ are solutions of $L_n[y] = 0$ each linear combination of them is a solution as $L_n[\sum_{i=1}^k c_i\phi_i] = \sum_{i=1}^k c_i L_n[\phi_i] = 0$.

Homogeneous Linear differential equations

- $L_2[y] = y'' - y = 0$
- $y''' + y' = 0$
- m functions $g_1, g_2, \cdots, g_m$ are linearly independent on the interval I iff $c_1 g_1(x) + c_2 g_2(x) + \cdots + c_m g_m(x) = 0$ implies that $c_1 = c_2 = \cdots = c_m = 0$.
- The set of functions $g_1, g_2, \cdots, g_m$ are linearly dependent on the interval I if there is a set of constants $c_1, c_2, \cdots, c_m$ including at least one non zero c_i such that for $\forall x \in I \quad c_1 g_1(x) + c_2 g_2(x) + \cdots + c_m g_m(x) = 0$.
- E.g., $\{e^{r_1 x}, e^{r_2 x}\}$.
- E.g., $\{e^x, e^{-x}, \cosh x\}$.

Wronskian

- Introduced by Polish mathematician Jozef Wronski.
- If $f_1, f_2, \cdots, f_n$ are (n-1) times differentiable functions on I,

$$W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

- E.g., $W(x^2, x^3) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4$
- E.g., $W(1, e^x, e^{-x}) = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 2$

Wronskian

- Theorem: Given $p(x)$ and $q(x)$ continuous on I, two solutions of $L_2[y] = y'' + p(x)y' + q(x)y = 0$ are linearly independent on I iff their Wronskian is non-zero on I.
- If ϕ_1 and ϕ_2 are dependent
  $\exists b_1, b_2 \neq 0| \quad b_1\phi_1 + b_2\phi_2 = 0 \quad b_1\phi_1' + b_2\phi_2' = 0$
- $\left[\begin{array}{cc} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{array} \right] \left[\begin{array}{c} b_1 \\ b_2 \end{array} \right] = 0$
- Nonzero Wronskian implies $b_1 = b_2 = 0$ and that ϕ_1 is linearly independent from ϕ_2.
- Assume $\{\phi_1, \phi_2\}$ are linearly independent and
  $\exists x_0 \quad W(\phi_1, \phi_2)(x_0) = 0$
- $\left[\begin{array}{cc} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{array} \right] \left[\begin{array}{c} b_1 \\ b_2 \end{array} \right] = 0$ has nontrivial solutions b_{10}, b_{20}

Wronskian

- Define $\psi(x) = b_{10}\phi_1(x) + b_{20}\phi_2(x)$
- $\psi(x_0) = b_{10}\phi_1(x_0) + b_{20}\phi_2(x_0) = 0$
- $\psi'(x_0) = b_{10}\phi_1'(x_0) + b_{20}\phi_2'(x_0) = 0$
- $\psi(x)$ is the solution to $L_n[y] = 0, \quad \psi(x_0) = 0, \quad \psi'(x_0) = 0$
  According to the existence and uniqueness theorem $\psi \equiv 0$.
- This implies linear dependence of $\{\phi_1, \phi_2\}$.

Wronskian

- Theorem: Wronskian of the solutions to the $L_2[y] = 0$ on I are either never zero or always zero.
- Proof: $W(\phi_1, \phi_2)(x) = \phi_1\phi_2' - \phi_2\phi_1'$, $\quad \frac{dW}{dx} = \phi_1\phi_2'' - \phi_2\phi_1'' = p(x)(\phi_1'\phi_2 - \phi_2'\phi_1) = -p(x)W$
- Abel relation: $W(\phi_1, \phi_2)(x) = ce^{-\int_{x_0}^{x} p(t)dt}, \quad x \in I$
- $W(\phi_1, \phi_2)(x) = W(\phi_1, \phi_2)(x_0)e^{-\int_{x_0}^{x} p(t)dt}, \quad x \in I$
- If $p_1(x), p_2(x), \cdots, p_n(x)$ are continuous on the interval I, then solutions $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$ of $L_n[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$ are linearly independent iff their Wronskian is nonzero.
- Further, $\frac{dW}{dx} + p_1(x)W = 0$

Wronskian

- $W(\phi_1, \cdots, \phi_n)(x) = W(\phi_1, \cdots, \phi_n)(x_0)e^{-\int_{x_0}^{x} p_1(t)dt}, \quad x \in I$
- $y''' - 4y'' + 5y' - 2y = 0$ has solutions
  $\phi_1 = e^x,\ \phi_2 = xe^x,\ \phi_3 = e^{2x}$, these constitute a fundamental set of solutions.
- Theorem: Linear homogeneous differential equation of order n has n linearly independent solutions.
- Proof: consider

$$
\begin{aligned}
L_n[y] = 0; &\quad y(x_0) = 1, y'(x_0) = 0, y''(x_0) = 0, \cdots, y^{(n-1)}(x_0) = 0 \\
L_n[y] = 0; &\quad y(x_0) = 0, y'(x_0) = 1, y''(x_0) = 0, \cdots, y^{(n-1)}(x_0) = 0 \\
\vdots \qquad &\qquad\qquad\qquad\qquad \vdots \\
L_n[y] = 0; &\quad y(x_0) = 0, y'(x_0) = 0, y''(x_0) = 0, \cdots, y^{(n-1)}(x_0) = 1
\end{aligned}
$$

$\#$ of solutions of a LHDE

- By existence and uniqueness theorem the above equations have solutions $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$
- $$\begin{aligned} c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) &= 0 \\ c_1\phi_1'(x) + c_2\phi_2'(x) + \cdots + c_n\phi_n'(x) &= 0 \\ c_1\phi_1''(x) + c_2\phi_2''(x) + \cdots + c_n\phi_n''(x) &= 0 \\ \vdots &= \vdots \\ c_1\phi_1^{(n-1)}(x) + c_2\phi_2^{(n-1)}(x) + \cdots + c_n\phi_n^{(n-1)}(x) &= 0 \end{aligned}$$
- Substitute $x = x_0$ to derive $c_1 = c_2 = \cdots = c_n = 0$
- n linearly independent solutions of a linear differential equation of order n are called a fundamental set of that equation.

Linear vector space of solutions

- Theorem: If $p_1(x), p_2(x), \cdots, p_n(x)$ are continuous on the interval I, and if solutions $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$ are a fundamental set of $L_n[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$ on I, for every solution $\phi(x)$ there is a unique set $c_1, \cdots, c_n$ such that $\phi(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)$
- Proof: Assume $\phi(x_0) = \alpha_0,\ \phi'(x_0) = \alpha_1,\ \cdots,\ \phi^{(n-1)}(x_0) = \alpha_{n-1}$

$$
\begin{aligned}
c_1\phi_1(x_0) + c_2\phi_2(x_0) + \cdots + c_n\phi_n(x_0) &= \alpha_0 \\
c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \cdots + c_n\phi_n'(x_0) &= \alpha_1 \\
\vdots &= \vdots \\
c_1\phi_1^{(n-1)}(x_0) + c_2\phi_2^{(n-1)}(x_0) + \cdots + c_n\phi_n^{(n-1)}(x_0) &= \alpha_{n-1}
\end{aligned}
$$

Linear vector space of solutions

- if solutions $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$ are a fundamental set of $L_n[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$ on I, $W(\phi_1, \cdots, \phi_n)(x) \neq 0$. Thus the above system has unique solutions $c_1^0, \cdots, c_n^0$. Define $\psi = c_1^0\phi_1(x) + c_2^0\phi_2(x) + \cdots + c_n^0\phi_n(x)$
- According to existence and uniqueness theorem $\psi = \phi$.

Linear nonhomogeneous differential equations

- Consider a private solution $\phi_p(x)$ of $L_n[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x)$ where $p_i(x)$ and $f(x)$ are continuous on I, and $\{\phi_1(x), \phi_2(x), \cdots, \phi_n(x)\}$ is a fundamental set of the corresponding linear homogeneous DE. If $\phi(x)$ is any other solution to the $L_n[y] = f(x)$ then $L_n[\phi - \phi_p] = L_n[\phi] - L_n[\phi_p] = 0$ thus $\phi = c_i\phi_i + \phi_p$

Linear nonhomogeneous differential equations

- Theorem: If $\phi_p(x)$ is a private solution of $L_n[y] = f(x)$, every solution can be written as $\phi(x) = c_k\phi_k(x) + \phi_p(x)$ this is called a general solution.
- Find the general solution to $y^{(4)} + 2y'' + y = x$
- $\phi_p = x, \quad \{\cos x, \sin x, x\cos x, x\sin x\}, \quad \phi(x) =?$
- E.g., $y'' - y = x, \quad y(0) = 0, y'(0) = 1$
- $\phi_p = -x \quad \{e^x, e^{-x}\}$
- E.g.,
  $x^2y'' + 4xy' + 2y = 6x + 1, \quad x > 0, \quad y(1) = 2, \quad y(2) = 1$
- $\phi_p = x + 1/2, \{1/x, 1/x^2\}$

Linear differential equations: Exercise

- If $L[y] = y'' + ay' + by$, find a) $L[\cos x]$, b) $L[x^2]$, c) $L[x^r]$, d) $L[e^{rx}]$
- If $L[y] = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$ determine $L[e^{rx}]$
- $L[y] = x^2 y'' + axy' + by$ determine $L[x^r]$, do the same for $L[y] = x^3 y''' + a_1 x^2 y'' + a_2 xy' + a_3 y$
- Check validity of given solution and determine its validity integral. $xy'' + y' = 0; \quad \phi(x) = \ln(\frac{1}{x})$
- $4x^2 y'' + 4xy' + (4x^2 - 1)y = 0; \quad \phi(x) = \sqrt{\frac{2}{\pi x}} \sin x$
- $(1 - x^2)y'' = -2xy' + 6y; \quad \phi(x) = 3x^2 - 1$
- $(1 - x^2)y'' = -2xy' + 2y + 2; \quad \phi(x) = x \tanh^{-1} x$
- Show that $\phi_1(x) = \frac{1}{9}x^3$ and $\phi_2(x) = \frac{1}{9}(x^{3/2} + 1)^2$ satisfy $(y')^2 - xy = 0$ on the interval $(0, \infty)$. Do their sum satisfy this DE?

Linear differential equations: Exercise

- $y' - 3y^{2/3} = 0$ has the general solution $y = (x+c)^3$. Test if linear combinations of these solutions are solutions. Test the independence of different solutions? Consider the following solutions: a) $\phi(x) = \begin{cases} (x-a)^3 & x \le a \\ 0 & x > a \end{cases}$ b) $\phi(x) = \begin{cases} 0 & x \le b \\ (x-b)^3 & x > b \end{cases}$ c) $\phi(x) = \begin{cases} (x-a)^3 & x \le a \\ 0 & b > x > a \\ (x-b)^3 & x \ge b \end{cases}$
- Show that functions $1, x, x^2, \cdots, x^n$ constitute a linearly independent set.
- Prove that n solutions of the DE $L[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$ are linearly independent iff their Wronskian is nonzero.
- Drive the Abel relation for n=3. To this end show that

$$w' = \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1' & \phi_2' & \phi_3' \\ \phi_1''' & \phi_2''' & \phi_3''' \end{vmatrix}$$

Linear DE with constant coefficients

- $y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = 0$
- $L_n = \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n = D^n + a_1 D^{n-1} + \cdots + a_n$
- $L[y] = (L_1 \cdots L_k)[y]$
- If ϕ is a solution to $L_i[y] = 0$ then
  $L[\phi] = (L_1 \cdots L_{i-1} L_{i+1} \cdots L_k) L_i[\phi] = 0$
- In this way solutions of linear homogeneous DE with constant coefficients of order n can be deduced from solutions of DEs of order one and two.
- E.g.,
  $L_n[y] = y'' + y' - 2y = 0 = (D^2 + D - 2)y = (D-1)(D+2)y = 0$
- $\{e^x, e^{-2x}\}$

Linear DE with constant coefficients: exercise

- Prove that roots of a polynomial with real coefficients appear in complex conjugate pairs.
- Prove that each polynomial of odd degree has at least one real root.
- Prove that each polynomial can be written as a product of first and second order polynomials with real coefficient.
- Write these polynomials as multiplication of first and second dergree polynomials.
- $D^3+1, \quad D^3-1, \quad D^4+1, \quad D^4+2D^2+10, \quad D^3-D^2+D-1.$

L homogeneous second order DE with constant coefficients

- For a second order DE $L[y] = y'' + ay' + by = 0$ try solutions of the form $\phi(x) = e^{sx}$
- $L[e^{sx}] = p(s)e^{sx} \quad p(s) = s^2 + as + b$ is called characteristic polynomial of the DE.
- $p(s) = 0$ is the characteristic equation of the DE.
- $p(s) = 0 \rightarrow s = s_1, s_2$
- $s_1 \neq s_2 \quad \phi(x) = c_1 e^{s_1 x} + c_2 e^{s_2 x}$ including the case of complex conjugate roots.
- If $s_1 = a + bi$ then $s_2 = a - bi$. $\{e^{(a+bi)x}, e^{(a-bi)x}\}$ or $\{e^{ax}\cos bx, e^{ax}\sin bx\}$
- A homogeneous equation in x is said to have a double root, or repeated root, at a if is a factor of the equation. At the double root, the graph of the equation is tangent to the x-axis.
- $s_1 = s_2 \quad \frac{\partial}{\partial s} L[e^{sx}] = L[\frac{\partial}{\partial s} e^{sx}] = L[xe^{sx}]$
- $L[xe^{s_1 x}] = p'(s_1)e^{s_1 x} + p(s_1)xe^{s_1 x} = 0$
- $\phi(x) = (c_1 + c_2 x)e^{s_1 x}$

- E.g., $y'' + 2y' + 10y = 0, \quad y(0) = 1, y'(0) = 0$
- E.g., $y'' + 2y' + y = 0, \quad y(0) = 1, y'(0) = 0$

Higher order LHDE with constant coefficients

- $L[y] = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$
- $L[e^{sx}] = p(s)e^{sx}$ where $p(s) = s^n + a_1 s^{n-1} + \cdots + a_n$ is the characteristic equation of our DE.
- If $s_1, s_2, \cdots, s_j$ are roots of characteristic equation with multiplicities of $n_1, n_2, \cdots, n_j$ the fundamental set is as follows:
- $$\{e^{s_1 x}, xe^{s_1 x}, \cdots, x^{n_1 - 1} e^{s_1 x}, e^{s_2 x}, xe^{s_2 x}, \cdots, x^{n_2 - 1} e^{s_2 x}, \\ \cdots, e^{s_j x}, xe^{s_j x}, \cdots, x^{n_j - 1} e^{s_j x}\}$$
- E.g., $y^{(6)} + 2y''' + y = 0 \to (D^3 + 1)^2 y = 0$
- $D^3(D-1)^2(D+1)^2 y = 0$

- Write a fundamental set for each of the following equations.
- $D^5 y = 0$
- $(D + 2)^4 y = 0$
- $(D^2 + 4)(D - 3)^2 y = 0$
- $(D^2 + 16)[(D - 1)^2 + 6]^2 y = 0$
- $(D^2 - 1)^2 (D^2 + 2D + 2)^4 y = 0$

Finding private solutions: Variation of parameters

- $L[y] = y'' + p(x)y' + q(x)y = f(x)$ with $\{\phi_1, \phi_2\}$ as a fundamental set.
- Assume $\phi_p = u_1\phi_1 + u_2\phi_2$
- $\phi_p' = u_1'\phi_1 + u_2'\phi_2 + u_1\phi_1' + u_2\phi_2'$
- Assume $u_1'\phi_1 + u_2'\phi_2 = 0$. Thus $\phi_p' = u_1\phi_1' + u_2\phi_2'$.
- $\phi_p'' = u_1\phi_1'' + u_2\phi_2'' + u_1'\phi_1' + u_2'\phi_2'$.
- $L[\phi_p] = \phi_p'' + p(x)\phi_p' + q(x)\phi_p = u_1\phi_1'' + u_2\phi_2'' + u_1'\phi_1' + u_2'\phi_2' + p(x)(u_1\phi_1' + u_2\phi_2') + q(x)(u_1\phi_1 + u_2\phi_2) = u_1(-p\phi_1' - q\phi_1) + u_2(-p\phi_2' - q\phi_2) + u_1'\phi_1' + u_2'\phi_2' + p(x)(u_1\phi_1' + u_2\phi_2') + q(x)(u_1\phi_1 + u_2\phi_2) = f(x)$
- $u_1'\phi_1' + u_2'\phi_2' = f$
- $\begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$
- By Cramer's rule: $u_1' = \frac{-f(x)\phi_2(x)}{W(\phi_1,\phi_2)}$ $\quad u_2' = \frac{f(x)\phi_1(x)}{W(\phi_1,\phi_2)}$
- $u_1(x) = -\int_{x_0}^{x} \frac{f(s)\phi_2(s)}{W(\phi_1,\phi_2)(s)}ds, \quad u_2(x) = \int_{x_0}^{x} \frac{f(s)\phi_1(s)}{W(\phi_1,\phi_2)(s)}ds$
- Finaly, $\phi_p(x) = \int_{x_0}^{x} \frac{\phi_2(x)\phi_1(s) - \phi_1(x)\phi_2(s)}{W(\phi_1,\phi_2)(s)} f(s)ds$

Finding private solutions: Variation of parameters

- Suppose $\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}}{dt^{n-1}}y + \cdots + p_n(t)y = g(t)$
- Solve the corresponding homogeneous differential equation to get: $y_h(t) = C_1 y_1(t) + C_2 y_2(t) + ... + C_n y_n(t)$.
- Assume a particular solution to the nonhomogeneous differential equation is of the form: $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + ... + u_n(t)y_n(t)$.
- Solve the following system of equations for $u_1'(t), u_2'(t), , u_n'(t)$.

$$\begin{aligned}
u_1'(t)y_1(t) + u_2'(t)y_2(t) + ... + u_n'(t)y_n(t) &= 0\\
u_1'(t)y_1'(t) + u_2'(t)y_2'(t) + ... + u_n'(t)y_n'(t) &= 0\\
&\vdots\\
u_1'(t)y_1^{(n-1)}(t) + u_2'(t)y_2^{(n-1)}(t) + ... + u_n'(t)y_n^{(n-1)}(t) &= g(t)
\end{aligned}$$

Finding private solutions: Variation of parameters

- $y'' - 2y' + y = \frac{e^x}{1+x^2}$ where the fundamental set is $\{e^x, xe^x\}$
- $y''' + y' = \tan x$
- $y''' - y' + 2y = e^{-x} \sin x$
- $y'' + y = \frac{1}{\cos x}$
- $(D^2 + 10D - 12)y = \frac{(e^{2x}+1)^2}{e^{2x}}$
- $(4D^2 - 8D + 5)y = e^x \tan^2(x/2)$
- $y^{(4)} + y = g(t)$

Undetermined multipliers method for finding PS

- One can guess the general form of the private solution and substitute in the DE to find the undetermined multipliers in the general form.
- $y'' + y = 3x^2 + 4 \to (D^2 + 1)y = 3x^2 + 4$
- Note that $D^3(3x^2 + 4) = 0 \to D^3(D^2 + 1)y = 0$
- $y = c_1 + c_2 x + c_3 x^2 + c_4 \cos x + c_5 \sin x$
- Substituting y into original DE determines multiples except for $\cos x$ and $\sin x$ multiples as they are solutions of the corresponding homogeneous equation and cancel out.
- E.g., $y'' + 2y = e^x$

Undetermined multipliers method for finding PS

- $y''' + y' = \sin x$
- Since $(D^2+1)\sin x = 0, (D^2+1)(D^3+D)y = 0$
- $(D-2)^3 y = 3e^{2x}$
- Since $(D-2)(3e^{2x}) = 0, (D-2)^4 y = 0$. Thus $\phi_p(x) = cx^3e^{2x}$
- The method of undetermined multiples has the following limitations.
- In $L[y] = f(x)$, L must contain only constant coefficients.
- f(x) must contain functions which satisfy a homogeneous linear DE with constant coefficient.

Undetermined multipliers method for finding PS

- If $f(x) = p_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \to \phi_p(x) = x^r(A_0x^n + A_1x^{n-1} + \cdots + A_n)$
- If $f(x) = p_n(x)e^{\alpha x} \to \phi_p(x) = x^r(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{\alpha x}$
- If $f(x) = p_n(x)e^{\alpha x}\sin\beta x$ or $f(x) = p_n(x)e^{\alpha x}\cos\beta x$ then $\phi_p(x) = x^r(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{\alpha x}\cos\beta x + x^r(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{\alpha x}\sin\beta x$
- $L[y] = y''' + y'' = 3x^3 - 1$
- $y'' + 4y = xe^x$
- $y'' - y = x^2e^x\sin x$
- If $L[y] = f_1(x) + f_2(x) + \cdots + f_k(x)$ and $L[\phi_{p1}] = f_1(x), L[\phi_{p2}] = f_2(x), \cdots, L[\phi_{pk}] = f_k(x)$ then by linearity of L, $L[\phi_{p_1} + \phi_{p_2} + \cdots + \phi_{p_k}] = f_1(x) + f_2(x) + \cdots + f_k(x)$

Undetermined multipliers method for finding PS

- $y'' + 4y = xe^x + x\sin 2x$
- $y''' + 3y'' = 2 + x^2$
- $y'' + 4y' + 4y = xe^{-x}$
- $y'' + 9y = 2x\sin 3x$
- $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 8y = e^{2t}(1 + \sin 2t)$

Euler differential equation

- nth order homogeneous Euler equation:
  $(x-x_0)^n y^{(n)} + a_1(x-x_0)^{n-1}y^{(n-1)} + \cdots + a_n y = 0$
- x_0 is the singularity of the Euler equation.
- Consider $L[y] = x^2y'' + axy' + by = 0, \quad x > 0$
- Impose the change of variable $t = \ln x$. $y' = \frac{1}{x}\frac{dy}{dt}$
  $y'' = \frac{d^2y}{dt^2}(\frac{dt}{dx})^2 + \frac{d^2t}{dx^2}\frac{dy}{dt} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$
- $\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0$
- Characteristic equation: $s^2 + (a-1)s + b = 0$
- Depending on Δ for the characteristic equation fundamental set is $\{e^{s_1 t} = x^{s_1}, e^{s_2 t} = x^{s_2}\}, \quad \{e^{s_1 t} = x^{s_1}, te^{s_1 t} = x^{s_1}\ln x\}, \quad \{x^\alpha \cos(\beta \ln x), x^\alpha \sin(\beta \ln x)\}$
- If we substitute x^s for y, $L[x^s] = [s^2 + (a-1)s + b]x^s = 0$

Euler differential equation

- The characteristic equation $p(s) = s^2 + (a-1)s + b = 0$. If $\Delta > 0 \to \phi(x) = c_1 x^{s_1} + c_2 x^{s_2}, x > 0$ where $x^{s_1} = e^{s_1 \ln x}$
- If $\Delta = 0$ we note that $\frac{\partial}{\partial s} L[x^s] = L[x^s \ln x] = p'(s)x^s + p(s)x^s \ln x$
- At $s = s_1, L[x^{s_1} \ln x] = p'(s_1)x^{s_1} + p(s_1)x^{s_1} \ln x = 0$. Thus $\phi(x) = c_1 x^{s_1} + c_2 x^{s_1} \ln x, x > 0$
- If $\Delta < 0 \to \phi(x) = e^{\alpha x}(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)), x > 0$
- For $x < 0$ we make the change of variable $\zeta = -x$. Euler equation become $\zeta^2 \frac{d^2y}{d\zeta^2} + a\zeta \frac{dy}{d\zeta} + by = 0$
- $$\phi(\zeta) = \begin{cases} c_1 \zeta^{s_1} + c_2 \zeta^{s_2} & s_1 \neq s_2 \in \Re \\ c_1 \zeta^{s_1} + c_2 \zeta^{s_1} \ln \zeta & s_1 = s_2 \in \Re \\ c_1 \zeta^{\alpha} \cos(\beta \ln \zeta) + c_2 \zeta^{\alpha} \sin(\beta \ln \zeta) & s = \alpha \pm i\beta \end{cases}$$

Euler differential equation

- Combining solutions for $x > 0$ and $x < 0$.
  $$\phi(|x|) = \begin{cases} c_1|x|^{s_1} + c_2|x|^{s_2} \\ c_1|x|^{s_1} + c_2|x|^{s_1}\ln|x| \\ c_1|x|^{\alpha}\cos(\beta\ln|x|) + c_2|x|^{\alpha}\sin(\beta\ln|x|) \end{cases}$$
- $x^2y'' + 2xy' + 2y = 0;\ y(1) = 0, y'(1) = 0$
- $x^2y - 5xy' + 13y = 0$
- $x^2y'' + 5xy' + 4y = 0$
- $x^2y'' - 3xy' + 4y = \ln x$
- $x^2y'' + 4xy' - 6y = 0$
- Order reduction technique:
  $L[y] = x^2y'' + x^3y' - 2(1 + x^2)y = x$

Series

- Every power series defines a continuous differentiable function over its radius of convergence. $\sum_{k=0}^{\infty} a_k x^k = f(x)$
- $(\sum_{k=0}^{\infty} a_k x^k)(\sum_{k=0}^{\infty} b_k x^k) = \sum_{k=0}^{\infty} c_k x^k$ where $c_k = \sum_{m=0}^{k} a_{k-m} b_m = \sum_{m=0}^{k} b_{k-m} a_m$
- Uniqueness of the taylor series.
- Find the convergence interval for $\sum_{n=0}^{\infty} \frac{2^n}{n+1} x^n$ and $\sum_{n=1}^{\infty} \frac{(x+1)^n}{2^n n}$
- $\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{(1-x)} = \sum_{n=1}^{\infty} n x^{n-1}$
- Linear indepence of power series starting from different powers of x.
- If p(x) and q(x) are analytic around x_0 then $y'' + p(x)y' + q(x)y = 0$ has analytic solution around the point x_0.
- E.g., Determine a series solution for the following differential equation about $x_0 = 0$, $y'' + xy' + y = 0$.
- $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$

Series

- $\sum_{k=0}^{\infty}(k+2)(k+1)a_k x^k + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0$
- $\phi(x) = a_0[1 + \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k}}{(2k)(2k-2)\cdots(4)(2)}] + a_1[x + \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)(2k-1)\cdots(5)(3)}]$
- Legendre differential equation, $(1-x^2)y'' - 2xy' + \lambda(\lambda+1)y = 0$
- Solution would converge on the interval (-1,1).
- $\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} + (\lambda-k)(\lambda+k+1)a_k]x^k = 0$
- For natural values of λ one of the solutions would be a polynomial. These are Legendre polynomials.
- If p(x) and q(x) are analytic around x_0 then $y'' + p(x)y' + q(x)y = f(x)$ has solution $\phi(x)$ such that $\phi(x_0) = a$ and $\phi'(x_0) = b$, Taylor series of the solution would have a convergence radius greater than the smallest of the convergence radius of p, q and f at x_0.

Numeric solution to a differential equation

- Start by substituting Taylor series of p and q in the corresponding homogenous equation. To derive $\phi_h(x) = a_0 + a_1 x + \sum_{k=2}^{\infty}(\alpha_k a_0 + \beta_k a_1)x^k$
- Lemma: If $\sum c_k x^k$ has convergence radius $R^* > 0 \quad \forall r < R^* \quad \exists M : |c_k| r^k \leq M$
- Numerically Solve the equation $\frac{dy(t)}{dt} = -\lambda y(t)$ and compare the resulting solution to exact solution.