Mathematics in Chemistry

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Aim

- Your most valuable asset is your learning ability.
- This course is a practice in learning and specially improves your deduction skills.
- This course provides you with tools applicable in and necessary for modeling many natural phenomena.
- The fundamental laws necessary for the mathematical treatment of a large part of physics and the whole of chemistry are thus completely known, and the difficulty lies only in the fact that application of these laws leads to equations that are too complex to be solved.
- The first part of the course reviews Linear algebra and calculus while introducing some very useful notations. In the second part of the course we study ordinary differential equations.

Course Evaluation

Final exam	29 Khordad 9 AM	60%
Midterm exam	29 Farvardin 10 AM	40%
Tutorials		10%
- Office hours: Mondays 9 AM - 12 PM

To be covered in the course

- In the first part of the course we try leveling the class by reviewing some very useful concepts from (linear) Algebra and calculus.
- Complex numbers, Vector analysis and Linear algebra
- Vector rotation, vector multiplication and vector derivatives
- Series expansion of analytic functions
- Integration and some theorems from calculus
- Dirac delta notation and Fourier transformation
- Curvilinear coordinates.
- Matrices

To be covered in the course

- When we know the relation between change in dependent variable with changes in independent variable we are facing a differential equation.
- The laws of nature are expressed in terms of differential equations. For example, study of chemical kinetics, diffusion and change in a systems state all start with differential equations.
- Analytically solvable ordinary differential equations.
- Due to lack of time a discussion of partial differential equations and a discussion of numerical solutions to differential equations are left to a course in computational chemistry.

References

- "Mathematical methods for physicists", by George Arfken and Hans Weber
- Ordinary differential equations by D. Shadman and B. Mehri (A relatively thin book in Farsi)
- Linear Algebra, Second Edition, Kenneth Hoffman, Ray Kanze
- Applied Mathematics for Physical Chemistry by J. Barrante

Complex numbers

- Real numbers
- Fundamental theorem of algebra: "Every non-constant single-variable polynomial has at least one complex root."
- $X^2 + 1 = 0$ defines $x = i = \sqrt{-1}$. Complex number $x = a + bi = (a, b) = ce^{\theta i}$.
- Complex conjugate, Complex plane, summation, multiplication, division, and logarithm.
- Euler formula, "our jewel", $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$ for real α
- Proof by Taylor expansion

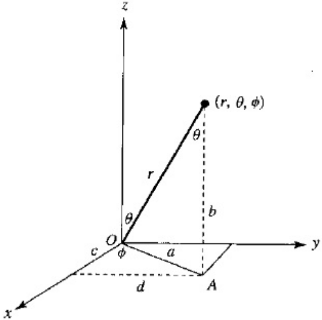
Complex numbers

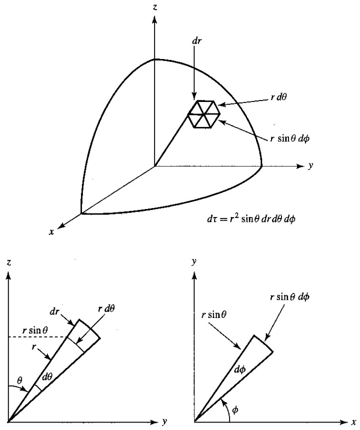
- $\cos x = \frac{e^{ix}+e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix}-e^{-ix}}{2i}.$
- $\cosh(y) = \cos(iy) = \frac{e^{y}+e^{-y}}{2},$
  $i\sinh(y) = \sin(iy) \to \sinh y = \frac{e^{y}-e^{-y}}{2}.$
- $\cos(x)\cdot\cos(y) = \frac{1}{2}[\cos(x+y)+\cos(x-y)],$
- $\cos(x+y) = \cos x\cos y - \sin x\sin y,$
  $\sin(x+y) = \sin x\cos y + \cos x\sin y.$

Coordinate System

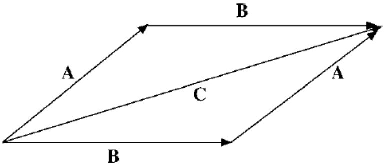
- Rectangular cartesian coordinate system is a one to one correspondence between ordered sets of numbers and points of space.
- Ordinate (vertical) vs. abscissa (horizontal) axes.
- Round or curvilinear coordinate system
- Curvilinear coordinates are a coordinate system for Euclidean space in which the coordinate lines may be curved, e.g., rectangular, spherical, and cylindrical coordinate systems.
- Coordinate surfaces of the curvilinear systems are curved.
- Plane polar coordinate system,
  $x = r\cos\theta, \quad y = r\sin\theta, \qquad dS = rdrd\theta,$
- Spherical polar coordinates
- $x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \quad dV = r^2\sin\theta drd\phi d\theta$
- Rectangular coordinates

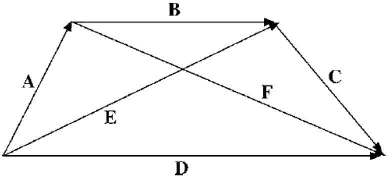
Coordinate System



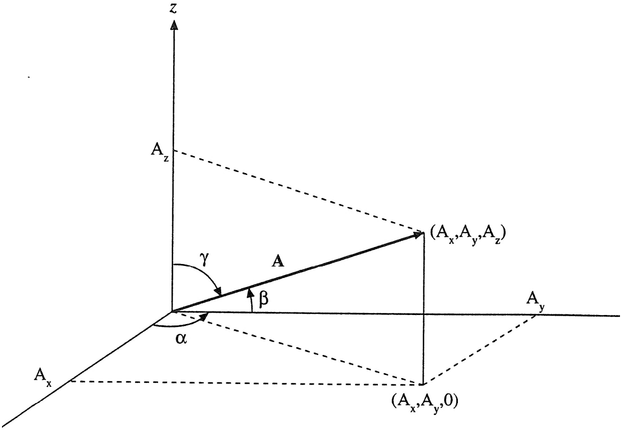


- Scalar quantities have magnitude vs. vector quantities which have magnitude and direction.
- Triangle law of vector addition.
- Parallelogram law of vector addition (Allows for vector subtraction), further it shows commutativity and associativity.





Vector analysis



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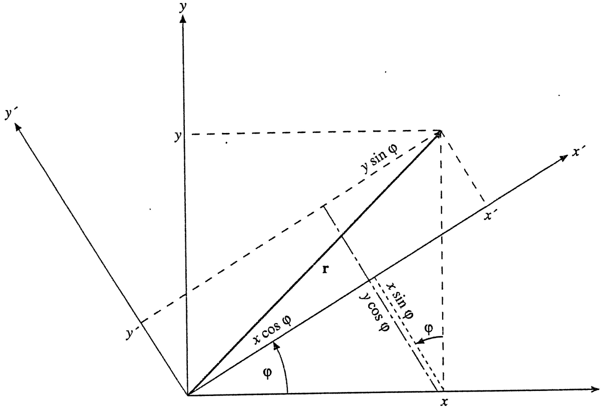
- Direction cosines, projections of $\vec{A}$.
- Geometric or algebraic representation.

- Unit vectors, $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$.
- Expansion of vectors in terms of a set of linearly independent basis allow algebraic definition of vector addition and subtraction, i.e.,
  $\vec{A} \pm \vec{B} = \hat{x}(A_x \pm B_x) + \hat{y}(A_y \pm B_y) + \hat{z}(A_z \pm B_z)$.
- $|A|$, Norm for scalars and vectors.
- $A_x = |A| \cos\alpha, \quad A_y = |A| \cos\beta, \quad A_z = |A| \cos\gamma$
- Pythagorean theorem,
  $|A|^2 = A_x^2 + A_y^2 + A_z^2, \qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$
- Vector field: An space to each point of which a vector is associated.
- Direction of vector r is coordinate system independent.



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- $x' = x \cos\phi + y \sin\phi \qquad y' = -x \sin\phi + y \cos\phi$
- Since each vector can be represented by a point in space a vector field A is defined as an association of vectors to points of space such that

  $A'_x = A_x \cos\phi + A_y \sin\phi \qquad A'_y = -A_x \sin\phi + A_y \cos\phi$

N-dimensional vectors

- $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
- $x \to x_1, \quad y \to x_2, \quad z \to x_3$
- $x'_i = \sum_{j=1}^{N} a_{ij} x_j; \qquad i = 1, 2, \cdots, N; \qquad a_{ij} = \cos(x'_i, x_j)$.
- In Cartesian coordinates, $x'_i = \cos(x'_i, x_1)x_1 + \cos(x'_i, x_2)x_2 + \cdots$ thus $a_{ij} = \frac{\partial x'_i}{\partial x_j}$.
- By considering primed coordinate axis to rotate by $-\phi$, $x_j = \sum_i \cos(x_j, x'_i)x'_i = \sum_i \cos(x'_i, x_j)x'_i = \sum_i a_{ij}x'_i$ resulting in $\frac{\partial x_j}{\partial x'_i} = a_{ij}$.
- A is the matrix whose effect is the same as rotating the coordinate axis, whose elements are a_{ij}.

Matrices

- A two dimensional array of elements is called a matrix.
- A matrix with m rows and n columns is called an m by n matrix.
- If number of rows and columns are equal matrix is called square matrix.
- Matrix A is determined by determining its elements a_{ij}.
- A+B = C iff $a_{ij} + b_{ij} = c_{ij}$
- AB = C iff $c_{ij} = \sum_k a_{ik} b_{kj}$
- The identity matrix $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$.
- If metrix A is composed of elements a_{ij}, transpose of A, A^T, is composed of elements a_{ji}.
- Inverse of the sqare matrix A is defined by $AA^{-1} = A^{-1}A = I$.

Vectors and vector space

- Orthogonality condition for A: $A^T A = I$ or

$$\sum_i a_{ij} a_{ik} = \sum_i \frac{\partial x_i'}{\partial x_j} \frac{\partial x_i'}{\partial x_k} = \sum_i \frac{\partial x_j}{\partial x_i'} \frac{\partial x_i'}{\partial x_k} = \frac{\partial x_j}{\partial x_k} = \delta_{jk}$$

- By depicting a vector as an n-tuple, $B = (B_1, B_2, \cdots, B_n)$, define:
- Vector equality.
- Vector addition
- Scalar multiplication
- Unique Null vector
- Unique Negative of vector
- Addition is commutative and associative. Scalar multiplication is distributive and associative.

Group

- A group is a set equipped with a binary operation which combines any two elements to form a third element in such a way that closure, associativity, identity and invertibility called group axioms are satisfied.
- E.g., the set of integers together with the addition operation, but groups are encountered in numerous areas, and help focusing on essential structural aspects.
- Point groups are used to help understand symmetry phenomena in molecular chemistry.
- A group is a set, G, together with an operation * (called the group law of G) that combines any two elements a and b to form another element, denoted a* b or ab.

Group

- Closure: For all a, b in G, the result of the operation, a*b, is also in G.
- Associativity: For all a, b and c in G, (a*b)*c = a*(b*c).
- Identity element: There exists an element e in G such that, for every element a in G, the equation e*a = a*e = a holds. Such an element is unique, and thus one speaks of the identity element.
- Inverse element: For each a in G, there exists an element b in G, commonly denoted a^{-1} (or -a, if the operation is denoted "+"), such that a*b = b*a = e, where e is the identity element.

Group

- Groups for which the commutativity equation a*b = b*a always holds are called abelian groups
- The symmetry group is an example of a group that is not abelian.
- The identity element of a group G is often written as 1 or 1_G a notation inherited from the multiplicative identity.
- If a group is abelian, then one may choose to denote the group operation by + and the identity element by 0; in that case, the group is called an additive group.
- There can be only one identity element in a group, and each element in a group has exactly one inverse element.
- The existence of inverse elements implies that division is possible

Field

- a field is a set on which addition, subtraction, multiplication, and division are defined, and behave as the corresponding operations on rational and real numbers do.
- There exist an additive inverse -a for all elements a, and a multiplicative inverse b^{-1} for every nonzero element b.
- An operation is a mapping that associates an element of the set to every pair of its elements.
- Associativity of addition and multiplication
- Commutativity of addition and multiplication
- Additive and multiplicative identity
- Additive inverses
- Multiplicative inverses
- Distributivity of multiplication over addition
- The best known fields are the field of rational numbers, the field of real numbers and the field of complex numbers.

Linear vector spaces

- A ring consists of a set equipped with two binary operations that generalize the arithmetic operations of addition and multiplication.
- A vector space over a field F is a set V together with two operations that satisfy axioms listed below.
- Vector addition $+ : V \times V \to V$, takes any two vectors v and w and assigns to them a third vector commonly written as v $+$ w.
- Scalar multiplication $\cdot : F \times V \to V$, takes any scalar a and any vector v and gives another vector av. (The vector av is an element of the set V). Elements of V are commonly called vectors. Elements of F are commonly called scalars.

Linear vector spaces

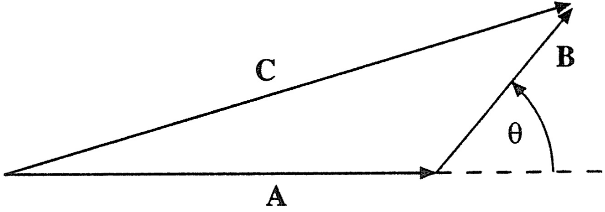
Axiom	Meaning
Associativity of addition	$u + (v + w) = (u + v) + w$
Commutativity of addition	$u + v = v + u$
Identity element of addition	$\exists 0 \in V$, called the zero vector, such that $v + 0 = v \ \forall v \in V$.
Inverse elements of addition for every $v \in V$,	$\exists -v \in V$, called the additive inverse of v, such that $v + (-v) = 0$
Compatibility of scalar multiplication with field multiplication	a(bv) = (ab)v
Identity element of scalar multiplication 1v = v,	1 denotes the multiplicative identity in F
Distributivity of scalar multiplication with respect to vector addition	a(u + v) = au + av
Distributivity of scalar multiplication with respect to field addition	$(a + b)v = av + bv$

Scalar or dot product

- Real n-tuples labeled $\mathbb{R}^n$, complex n-tuples are labeled $\mathbb{C}^n$.
- Inner product should be distributive and associative.
  $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \qquad \vec{A} \cdot (y\vec{B}) = (y\vec{A}) \cdot \vec{B} = y\vec{A} \cdot \vec{B}$
- Algebraic definition: $\vec{A}, \vec{B} \in \mathbb{R}^n \quad \vec{A} \cdot \vec{B} \equiv \sum_i A_i B_i$
- $\vec{A}, \vec{B} \in \mathbb{C}^n \quad \vec{A} \cdot \vec{B} \equiv \sum_i A_i^* B_i$
- Dot product of A by a unit vector is the length of A's projection into unit vectors direction.
- $A_x = |A| \cos\alpha \equiv \vec{A} \cdot \hat{x}, \quad A_y = |A| \cos\beta \equiv \vec{A} \cdot \hat{y}, \quad A_z = |A| \cos\gamma \equiv \vec{A} \cdot \hat{z}$.
- Geometric definition: $\vec{A} \cdot \vec{B} = A_B B = A B_A = AB\cos\theta$
- $\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$
- $\hat{x} \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{z} \cdot \hat{y} = 0$
- Perpendicular or orthogonal vectors.
- $\hat{x} = e_1, \hat{y} = e_2, \hat{z} = e_3; \qquad e_m \cdot e_n = \delta_{mn}$

Invariance of Scalar or dot product under rotation

- $\vec{B}' \cdot \vec{C}' = \sum_l B'_l C'_l = \sum_l \sum_i \sum_j a_{li} B_i a_{lj} C_j = \sum_{ij} (\sum_l a_{li} a_{lj}) B_i C_j = \sum_{ij} \delta_{ij} B_i C_j = \sum_i B_i C_i = \vec{B} \cdot \vec{C}$; thus dot product is scalar.
- $\vec{C} = \vec{A} + \vec{B}, \qquad \vec{C} \cdot \vec{C} = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) = \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} + 2\vec{A} \cdot \vec{B} \rightarrow \overrightarrow{A} \cdot \overrightarrow{B} = \frac{1}{2}(C^2 - A^2 - B^2)$. Therefore, $\overrightarrow{A} \cdot \overrightarrow{B}$ is a scalar.
- Another derivation for cosine law, $C^2 = A^2 + B^2 + 2AB \cos\theta$



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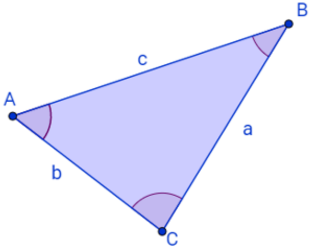
The sine law

- This reminds us of the sine law: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = d$



Law of sines

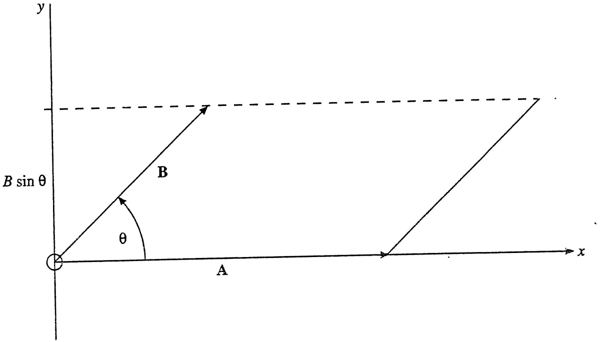
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

- Triangle area,
  $S = \frac{1}{2}ah_a = \frac{1}{2}a(b\sin C) = \frac{1}{2}a(c\sin B) = \frac{1}{2}ch_c = \frac{1}{2}c(b\sin A)$.
- $\frac{1}{2}a(b\sin C) = \frac{1}{2}a(c\sin B) = \frac{1}{2}c(b\sin A)$

Vector or cross product

- Geometric definition: $\overrightarrow{C} = \overrightarrow{A} \times \overrightarrow{B}$ $C = AB\sin\theta$, $\overrightarrow{C}$ is a vector perpendicular to the plane of $\overrightarrow{A}$ and $\overrightarrow{B}$ such that $\overrightarrow{A}$ and $\overrightarrow{B}$ and $\overrightarrow{C}$ form a right-handed system.
- Cross product is non-commutative. $\overrightarrow{A} \times \overrightarrow{B} = -\overrightarrow{B} \times \overrightarrow{A}$
- $\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0$
- $\hat{x} \times \hat{y} = \hat{z}, \quad \hat{x} \times \hat{z} = -\hat{y}, \quad \hat{z} \times \hat{y} = -\hat{x}$
- Angular momentum, $\overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p}$; torque, $\overrightarrow{\tau} = \overrightarrow{r} \times \overrightarrow{F}$ and magnetic force, $\overrightarrow{F}_M = q\overrightarrow{v} \times \overrightarrow{B}$.
- Treating area as a vector quantity.

Vector or cross product



- $\overrightarrow{A} \times \overrightarrow{B} \equiv \overrightarrow{C} = (C_x, C_y, C_z) =$
  $(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) =$
  $(A_xB_y - A_yB_x)\hat{x}\times\hat{y} + (A_xB_z - A_zB_x)\hat{x}\times\hat{z} + (A_yB_z - A_zB_y)\hat{y}\times\hat{z}$
- $C_x = A_yB_z - A_zB_y, \quad C_y = A_zB_x - A_xB_z, \quad C_z = A_xB_y - A_yB_x.$
- $C_i = A_jB_k - A_kB_j$, i,j and k are different.

Vector or cross product

- $\overrightarrow{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$
- $\overrightarrow{A} \cdot \overrightarrow{C} = \overrightarrow{A} \cdot (\overrightarrow{A} \times \overrightarrow{B}) = A_x(A_y B_z - A_z B_y) + A_y(A_z B_x - A_x B_z) + A_z(A_x B_y - A_y B_x) = 0.$
- $\overrightarrow{B} \cdot \overrightarrow{C} = \overrightarrow{B} \cdot (\overrightarrow{A} \times \overrightarrow{B}) = 0.$
- $(\overrightarrow{A} \times \overrightarrow{B}) \cdot (\overrightarrow{A} \times \overrightarrow{B}) = A^2 B^2 \sin^2 \theta.$

Levi-Civita symbol

- Levi-Civita symbol, permutation symbol, antisymmetric symbol, or alternating symbol. $\epsilon_{\cdots i_p \cdots i_q \cdots} = -\epsilon_{\cdots i_q \cdots i_p \cdots}$
- $\epsilon_{i_1 i_2 \cdots i_n} = (-1)^p \epsilon_{12 \cdots n}$.

$$\epsilon_{i_1 i_2 \cdots i_n} = \begin{cases} +1 & \text{if } (i_1, i_2, \cdots, i_n) \text{ is an even permutation of } (1, 2, \cdots, n) \\ -1 & \text{if } (i_1, i_2, \cdots, i_n) \text{ is an odd permutation of } (1, 2, \cdots, n) \\ 0 & \text{otherwise (no permutation, repeated index)} \end{cases}$$

- $\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl}$
- $\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{imn} = \sum_{i=1}^{3}(\delta_{ii}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{ki} + \delta_{in}\delta_{ji}\delta_{km} - \delta_{im}\delta_{ji}\delta_{kn} - \delta_{ii}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{ki}) = \delta_{kn}\delta_{jm} - \delta_{jn}\delta_{km}$

Levi-Civita symbol—applications

- Determinant: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \epsilon_{ijk} a_{1i} a_{2j} a_{3k}$
- $C_i = \sum_{jk} \epsilon_{ijk} A_j B_k, \qquad \vec{C} = \sum_{ijk} \epsilon_{ijk} A_j B_k \hat{e}_i = \epsilon_{ijk} A_j B_k \hat{e}_i$
- $(\overrightarrow{A} \times \overrightarrow{B}) \cdot (\overrightarrow{A} \times \overrightarrow{B}) = (\sum_{ijk} \epsilon_{ijk} A_j B_k \hat{e}_i) \cdot (\sum_{lmn} \epsilon_{lmn} A_m B_n \hat{e}_l) = \sum_{ijklmn} \epsilon_{ijk} \epsilon_{lmn} A_j B_k A_m B_n \delta_{il} = \sum_{ijkmn} \epsilon_{ijk} \epsilon_{imn} A_j B_k A_m B_n = \sum_{jkmn} (\delta_{kn}\delta_{jm} - \delta_{jn}\delta_{km}) A_j B_k A_m B_n = \sum_{jk} A_j B_k (A_j B_k - A_k B_j) = (\sum_j A_j^2)(\sum_k B_K^2) - (\sum_j A_j B_j)(\sum_k A_k B_k) = |A|^2 |B|^2 (1 - \cos^2\theta)$
- $(\overrightarrow{A} \times \overrightarrow{B})^2 = (\overrightarrow{A})^2 (\overrightarrow{B})^2 - (\overrightarrow{A} \cdot \overrightarrow{B})^2$

Triple scalar product

- $\overrightarrow{A} \cdot \overrightarrow{B} \times \overrightarrow{C} = \overrightarrow{A} \cdot (\sum_{ijk} \epsilon_{ijk} B_j C_k \hat{e}_i) = \sum_{ijk} \epsilon_{ijk} A_i B_j C_k = \sum_{jki} \epsilon_{ijk} B_i C_j A_k = \overrightarrow{B} \cdot \overrightarrow{C} \times \overrightarrow{A} = \overrightarrow{C} \cdot \overrightarrow{A} \times \overrightarrow{B} = -\overrightarrow{A} \cdot \overrightarrow{C} \times \overrightarrow{B} = -\vec{C} \cdot \vec{B} \times \vec{A}.$
- $\overrightarrow{A} \cdot \overrightarrow{B} \times \overrightarrow{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$. Volume of the parallelepiped defined by $\overrightarrow{A}$, $\overrightarrow{B}$ and $\overrightarrow{C}$.

Triple vector product

- $\vec{A} \times (\vec{B} \times \vec{C}) = x\vec{B} + y\vec{C}$
- $0 = x\vec{A} \cdot \vec{B} + y\vec{A} \cdot \vec{C} \rightarrow x = z\vec{A} \cdot \vec{C} \qquad y = -z\vec{A} \cdot \vec{B}$
- $\vec{A} \times (\vec{B} \times \vec{C}) = z(\vec{B}\vec{A} \cdot \vec{C} - \vec{C}\vec{A} \cdot \vec{B})$
- z is magnitude independent.

$$
\begin{aligned}
[\hat{A} \times (\hat{B} \times \hat{C})]^2 &= \hat{A}^2(\hat{B} \times \hat{C})^2 - [\hat{A} \cdot (\hat{B} \times \hat{C})]^2 \\
&= 1 - \cos^2\alpha - [\hat{A} \cdot (\hat{B} \times \hat{C})]^2 \\
&= z^2[(\hat{A} \cdot \hat{C})^2 + (\hat{A} \cdot \hat{B})^2 - 2\hat{A} \cdot \hat{B}\hat{A} \cdot \hat{C}\hat{B} \cdot \hat{C}] \\
&= z^2(\cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma)
\end{aligned}
$$

- $[\hat{A} \cdot (\hat{B} \times \hat{C})]^2 =$
  $1 - \cos^2\alpha - z^2(\cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma)$

BAC-CAB

- The volume spanned by three vectors is independent of their order, thus $z^2 = 1$.
- $\hat{x} \times (\hat{x} \times \hat{y}) = z((\hat{x} \cdot \hat{y})\hat{x} - (\hat{x} \cdot \hat{x})\hat{y}) = -z\hat{y}$, also, $\hat{x} \times (\hat{x} \times \hat{y}) = \hat{x} \times \hat{z} = -\hat{y}$ thus $z = 1$.
- Lemma: $\vec{A} \times e_i = \sum_{mno} \epsilon_{mno} e_m A_n \delta_{io} = \sum_{mn} \epsilon_{mni} e_m A_n$
- $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{A} \times (\sum_{ijk} \epsilon_{ijk} e_i B_j C_k) =$
  $\sum_{ijkmn} \epsilon_{ijk} \epsilon_{imn} B_j C_k A_n e_m =$
  $\sum_{jkmn} (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) B_j C_k A_n e_m =$
  $\sum_{jk} B_j C_k A_k e_j - \sum_{jk} B_j C_k A_j e_k = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

Taylor series

- Taylor series of a real or complex valued function f(x) that is infinitely differentiable at a number a: $f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$. When $a=0$, the series is also called a Maclaurin series.
- The Taylor series for any polynomial is the polynomial itself.
- The Maclaurin series for 1/(1- x) is the geometric series $1+x+x^2+x^3+\cdots$ so the Taylor series for 1/x at a = 1 is $1-(x-1)+(x-1)^2-(x-1)^3+\cdots$.
- Integrate the above Maclaurin series, to find $\ln(1-x)=-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4-\cdots$ and the corresponding Taylor series for ln x at a = 1 is $(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\cdots$.

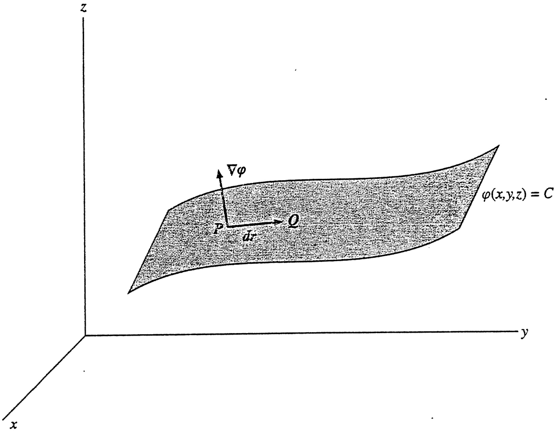
Taylor series

- Taylor series for log x at some $a = x_0$ is: $\log(x_0) + \frac{1}{x_0}(x - x_0) - \frac{1}{x_0^2}\frac{(x-x_0)^2}{2} + \cdots$.
- The Taylor series for the exponential function e^x at $a = 0$ is $\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots =$ $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
- If f(x) is given by a convergent power series in an open disc centered at b in the complex plane, it is analytic in this disc. For x in this disc, f is given by a convergent power series $f(x) = \sum_{n=0}^{\infty} a_n(x - b)^n$.
- Differentiating by x the above formula n times, then setting x = b gives: $\frac{f^{(n)}(b)}{n!} = a_n$ and so the power series expansion agrees with the Taylor series.
- Thus a function is analytic in an open disc centered at b if and only if its Taylor series converges to the value of the function at each point of the disc.

Gradient, ∇

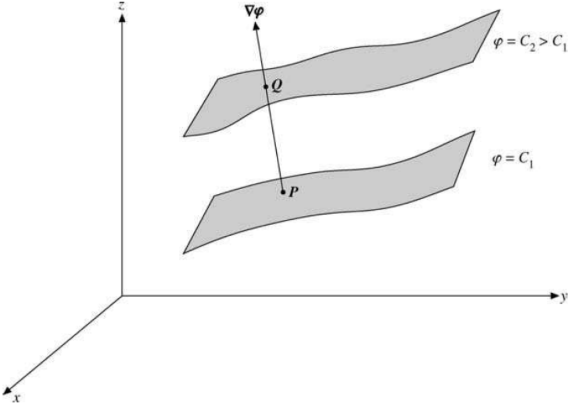
- $\phi'(x'_1, x'_2, x'_3) = \phi(x_1, x_2, x_3)$
- $\frac{\partial \phi'(x'_1, x'_2, x'_3)}{\partial x'_i} = \frac{\partial \phi(x_1, x_2, x_3)}{\partial x'_i} = \sum_j \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial x'_i} = \sum_j a_{ij} \frac{\partial \phi}{\partial x_j}$
- $\frac{\partial \phi}{\partial x_j}$ is behaving as a vector component.
- Del $= \nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$
- Calculate $\nabla f(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$, result is $\hat{r}\frac{df}{dr}$
- $\nabla \phi \cdot d\vec{r} = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi$
- Over a constant ϕ surface $d\phi = \nabla \phi \cdot d\vec{r} = 0$.

Gradient, ∇

- 

- $d\phi = C_1 - C_2 = \Delta C = (\nabla\phi) \cdot dr$

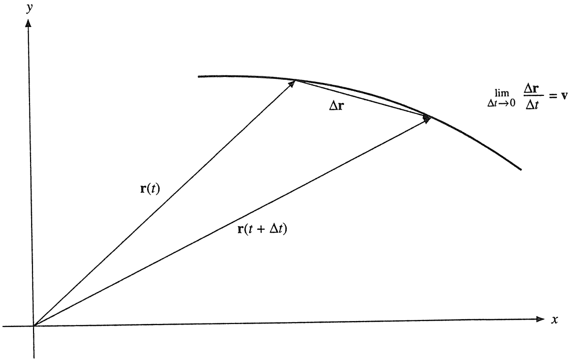
Gradient, ∇

- 

- Consider $\phi(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$, find $\nabla\phi$ and direction cosines of $\nabla\phi$ at (3,2,1).
- $\int \vec{A}(r) \cdot \nabla f(r) d^3r = -\int f(r) \nabla \cdot \vec{A}(r) d^3r$ where A or f vanish at infinity.
- $\vec{F} = -\nabla U$
- Prove $\nabla(uv) = v\nabla u + u\nabla v$.

Divergence, ∇

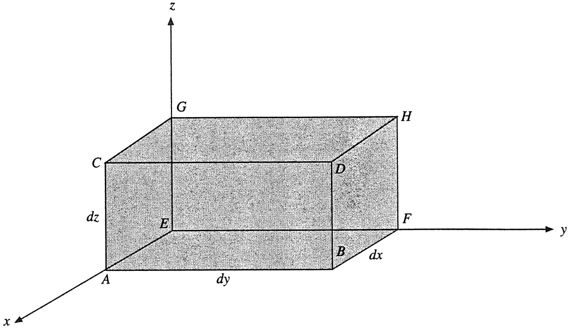
- $\frac{d\vec{r}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \vec{v}$



-
- $\nabla \cdot \vec{r} = (\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}) \cdot (\hat{x}x + \hat{y}y + \hat{z}z) = 3,$
  $\nabla \cdot (\vec{r}f(r)) = ?, \qquad \nabla \cdot (\vec{r}r^{n-1}) = ?.$
- $\int \vec{A}(r) \cdot \nabla f(r) d^3r = -\int f(r) \nabla \cdot \vec{A}(r) d^3r$ where A or f vanish at infinity.

Divergence, ∇

- Divergence of $\vec{V}$, $\quad \nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$
- $\nabla \cdot (\rho \vec{V})$ for a compressible fluid.
- The flow going through a differential volume per unit time is:



- (rate of flow in)$_{EFGH} = (\rho v_x)|_{x=0} dydz$
- (rate of flow out)$_{ABCD} = (\rho v_x)|_{x=dx} dydz = [\rho v_x + \frac{\partial}{\partial x}(\rho v_x) dx]_{x=0} dydz$

Divergence, ∇

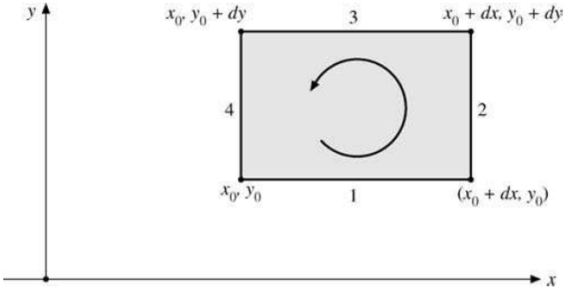
- Net rate of flow out$|_x = \frac{\partial}{\partial x}(\rho v_x)|_{(0,0,0)} dxdydz$
- $\lim_{\Delta x \to 0} \frac{\rho v_x(\Delta x,0,0) - \rho v_x(0,0,0)}{\Delta x} \equiv \frac{\partial[\rho v_x(x,y,z)]}{\partial x}|_{(0,0,0)}$
- Net flow out (per unit time) $= \nabla \cdot (\rho \vec{v})\, dxdydz$.
- Continuity equation: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$.
- $\nabla \cdot (f\vec{V}) = \nabla f \cdot \vec{V} + f \nabla \cdot \vec{V}$
- $\vec{B}$ is solenoidal if and only if $\nabla \cdot B = 0$
- A circular orbit can be represented by $\vec{r} = \hat{x} r \cos \omega t + \hat{y} r \sin \omega t$. Evaluate $r \times \dot{\vec{r}}$ and $\ddot{\vec{r}} + \omega^2 \vec{r} =$
- Divergence of electrostatic field due to a point charge, $\nabla \cdot \vec{E} = \nabla \cdot \frac{q\hat{r}}{4\pi\epsilon_0 r^2}$.

Curl, $\nabla\times$

- $\nabla \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}.$
- $\nabla \times (f\vec{V}) = f\nabla \times \vec{V} + (\nabla f) \times \vec{V}$
- $\nabla \times (\vec{r}F(r)) = 0$
- Show that electrostatic and gravitational forces are irrotational.
- Show that the curl of a vector field is a vector field.
- Curl can be measured by inserting a paddle wheel inside the vector field.

Circulation

- Circulation of a fluid around a differential loop in the xy-plane.



- $\int \vec{V} \cdot d\lambda = \int_1 V_x(x,y)d\lambda_x + \int_2 V_y(x,y)d\lambda_y + \int_3 V_x(x,y)d\lambda_x + \int_4 V_y(x,y)d\lambda_y = V_x(x_0,y_0)dx + [V_y(x_0,y_0) + \frac{\partial V_y}{\partial x}dx]dy + [V_x(x_0,y_0) + \frac{\partial V_x}{\partial y}dy](-dx) + V_y(x_0,y_0)(-dy) = (\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y})dxdy = \nabla \times \vec{V}|_z$

Successive applications of ∇

- Show that $\vec{u} \times \vec{v}$ is solenoidal if u and v are each irrotational.
- If $\vec{A}$ is irrotational show that $\vec{A} \times \vec{r}$ is solenoidal
- $\nabla \cdot \nabla\phi = \nabla^2\phi = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.
- $\nabla \times \nabla\phi = 0.$
- $\nabla \cdot \nabla \times \vec{V} = 0$
- $\nabla \cdot \nabla\vec{V} = \hat{i}\nabla \cdot \nabla V_x + \hat{j}\nabla \cdot \nabla V_y + \hat{k}\nabla \cdot \nabla V_z$
- $\nabla \times (\nabla \times \vec{V}) = \nabla\nabla \cdot \vec{V} - \nabla \cdot \nabla\vec{V}$

Electromagnetic wave equation

- The set of Maxwell equations:
- $\nabla \cdot \vec{B} = 0$
- $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
- $\nabla \times \vec{B} = \mu_0(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t})$
- $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Electromagnetic wave equation

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- $\nabla \cdot \vec{B} = 0$
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- $\nabla \times \vec{B} = \mu_0(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t})$
- $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- Eliminating B between the last two equations, by noting that $\frac{\partial}{\partial t} \nabla \times \vec{B} = \nabla \times \frac{\partial \vec{B}}{\partial t}$ and assuming no charge flux.
- $\nabla \times (\nabla \times \vec{E}) = -\epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$

Review: Integrals

- $\int x(x+a)^n dx =$
- $\int \frac{1}{a^2+x^2} dx =$
- $\int \frac{x}{a^2+x^2} dx$
- $\int \frac{x^2}{a^2+x^2} dx =$
- $\int \frac{x^3}{a^2+x^2} dx =$
- $\int \tan(ax+b)dx =$
- $\int cotan(ax+b)dx =$

Review: Integrals

- $\int x(x+a)^n dx =$
- $\int \frac{1}{a^2+x^2} dx =$
- $\int \frac{x}{a^2+x^2} dx$
- $= \frac{1}{2} \ln |a^2 + x^2|$
- $\int \frac{x^2}{a^2+x^2} dx =$
- $\int \frac{x^3}{a^2+x^2} dx =$
- $\int \tan(ax+b) dx =$
- $-\frac{1}{a} \ln|\cos(ax+b)|$
- $\int cotan(ax+b) dx =$
- $\frac{1}{a} \ln|\sin(ax+b)|$

Review: Integrals

- $\int \sec(ax + b)dx =$
- $\int cosec(ax + b)dx =$
- $\int \sec^2(x)dx =$
- $\int cosec^2(x)dx =$
- $\int \tan(x)\sec(x)dx =$
- $\int cotan(x)cosec(x)dx =$

Review: Integrals

- $\int \sec(ax+b)dx =$
- $\frac{1}{a}\ln|\sec(ax+b)+\tan(ax+b)|$
- $\int cosec(ax+b)dx =$
- $-\frac{1}{a}\ln|cosec(ax+b)+cotan(ax+b)|$
- $\int \sec^2(x)dx =$
- $\tan(x)$
- $\int cosec^2(x)dx =$
- $cotan(x)$
- $\int \tan(x)\sec(x)dx =$
- $\sec(x)$
- $\int cotan(x)cosec(x)dx =$
- $cosec(x)$

Review: Integrals

- $\int \frac{1}{ax^2+bx+c}dx = \int \frac{dx}{a(x+\frac{b}{2a})^2+c-\frac{b^2}{4a}} = \frac{1}{a}\int \frac{dx}{(x+\frac{b}{2a})^2+c/a-\frac{b^2}{4a^2}} =$
  $\frac{1}{a}\int \frac{du}{u^2+(c/a-\frac{b^2}{4a^2})} = \frac{1}{a}\tan^{-1}(\frac{u}{\sqrt{c/a-\frac{b^2}{4a^2}}}) = \frac{1}{a}\tan^{-1}(\frac{x+\frac{b}{2a}}{\sqrt{c/a-\frac{b^2}{4a^2}}})$
- $\int \frac{1}{(x+a)(x+b)}dx =$
- $\int \frac{x}{ax^2+bx+c}dx =$
- $\int \frac{1}{\sqrt{x\pm a}}dx =$
- $\int x\sqrt{x-a}dx$
- $\int \sqrt{ax+b}dx =$
- $\int \frac{x}{\sqrt{x\pm a}}dx =$
- $\int \sqrt{\frac{x}{a-x}}dx$

Vector integration over a contour

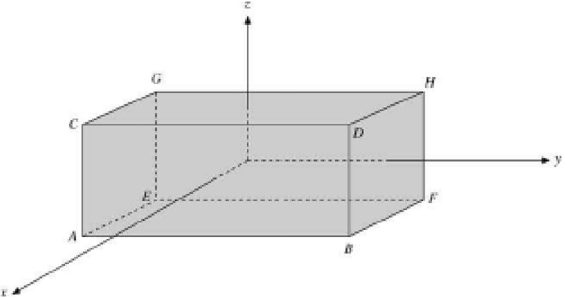
- $\int_C \phi d\vec{r} =$
  $\hat{x}\int_C \phi(x,y,z)dx + \hat{y}\int_C \phi(x,y,z)dy + \hat{z}\int_C \phi(x,y,z)dz$
- $\int_C \vec{V}\cdot d\vec{r}$, e.g., $w = \int F\cdot d\vec{r} =$
  $\int_C \vec{F}_x(x,y,z)dx + \int_C F_y(x,y,z)dy + \int_C F_z(x,y,z)dz$
- $\int_C \vec{V}\times d\vec{r} =$
  $\hat{x}\int_C(V_y dz - V_z dy) - \hat{y}\int_C(V_x dz - V_z dx) + \hat{z}\int_C(V_x dy - V_y dx)$
- Reduce each vector integral to scalar integrals.
- E.g., $\int_{0,0}^{1,1} r^2 dr = \int_{0,0}^{1,1}(x^2+y^2)dr = \int_{0,0}^{1,1}(x^2+y^2)(\hat{x}dx + \hat{y}dy)$
- E.g., Calculate W for $F = -\hat{x}y + \hat{y}x$

Surface and volume integration

- $\int \phi d\vec{\sigma}$
- $\int \vec{V} \cdot d\vec{\sigma}$ (flow or flux through a given surface),
- $\int \vec{V} \times d\vec{\sigma}$
- Convention for the direction of surface normal: Outward from a closed surface. In the direction of thumb when contiguous right hand fingers are traversing the perimeter of the surface.
- Volume integrals:
  $\int_v \vec{V} d\tau = \hat{x} \int_v V_x d\tau + \hat{y} \int_v V_y d\tau + \hat{z} \int_v V_z d\tau$

Integral definition of gradient

- $\nabla\phi = \lim_{d\tau\to 0}\frac{\int_{S_{d\tau}}\phi d\vec{\sigma}}{d\tau}$
- $d\tau = dxdydz$. Place origin at the center of the differential

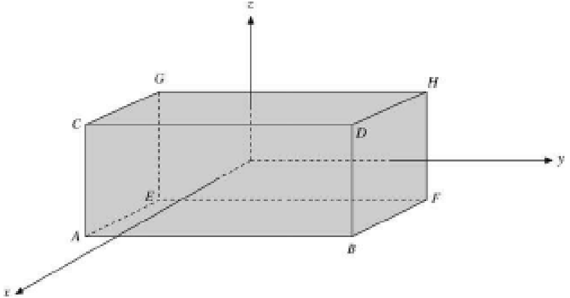


volume.

- $\int_{S d\tau}\phi d\vec{\sigma} = -i\int_{EFHG}(\phi - \frac{\partial\phi}{\partial x}\frac{dx}{2})dydz + i\int_{ABDC}(\phi + \frac{\partial\phi}{\partial x}\frac{dx}{2})dydz - j\int_{AEGC}(\phi - \frac{\partial\phi}{\partial y}\frac{dy}{2})dxdz + j\int_{BFHD}(\phi + \frac{\partial\phi}{\partial y}\frac{dy}{2})dxdz - k\int_{ABFE}(\phi - \frac{\partial\phi}{\partial z}\frac{dz}{2})dydx + k\int_{CDHG}(\phi + \frac{\partial\phi}{\partial z}\frac{dz}{2})dydx$
- $\int\phi d\vec{\sigma} = (i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z})dxdydz$

Integral definitions of divergence and curl

- $\nabla \cdot \vec{V} = \lim_{d\tau \to 0} \frac{\int_{S_{d\tau}} \vec{V} \cdot d\vec{\sigma}}{d\tau}$



- $\int_{S_{d\tau}} \vec{V} \cdot d\vec{\sigma} = \int_{EFHG} \vec{V} \cdot d\vec{\sigma} + \int_{ABDC} \vec{V} \cdot d\vec{\sigma} + \int_{AEGC} \vec{V} \cdot d\vec{\sigma} + \int_{BFHD} \vec{V} \cdot d\vec{\sigma} + \int_{ABFE} \vec{V} \cdot d\vec{\sigma} + \int_{CDHG} \vec{V} \cdot d\vec{\sigma} = -\int_{EFHG} (V_x - \frac{\partial V_x}{\partial x}\frac{dx}{2}) dydz + \int_{ABDC} (V_x + \frac{\partial V_x}{\partial x}\frac{dx}{2}) dydz - \int_{AEGC} (V_y - \frac{\partial V_y}{\partial y}\frac{dy}{2}) dxdz + \int_{BFHD} (V_y + \frac{\partial V_y}{\partial y}\frac{dy}{2}) dxdz - \int_{ABFE} (V_z - \frac{\partial V_z}{\partial z}\frac{dz}{2}) dydx + \int_{CDHG} (V_z + \frac{\partial V_z}{\partial z}\frac{dz}{2}) dydx = (\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}) dxdydz$

Integral definitions of divergence and curl

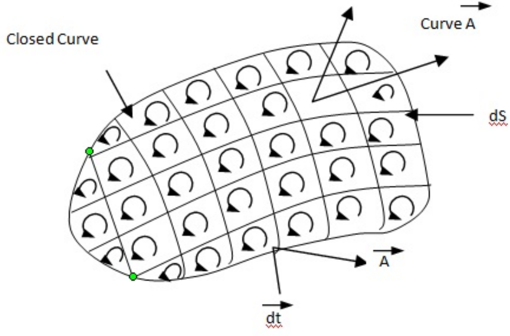
- $\nabla \times \vec{V} = \lim_{d\tau \to 0} \frac{\int_{S_{d\tau}} d\vec{\sigma} \times \vec{V}}{d\tau}$
- $\int_{S_{d\tau}} \vec{V} \times d\vec{\sigma} = \int_{EFHG} \vec{V} \times d\vec{\sigma} + \int_{ABDC} \vec{V} \times d\vec{\sigma} + \int_{AEGC} \vec{V} \times d\vec{\sigma} + \int_{BFHD} \vec{V} \times d\vec{\sigma} + \int_{ABFE} \vec{V} \times d\vec{\sigma} + \int_{CDHG} \vec{V} \times d\vec{\sigma} = -dydz\vec{V}(-dx/2, 0, 0) \times \hat{x} + dydz\vec{V}(dx/2, 0, 0) \times \hat{x} - dxdz\vec{V}(0, -dy/2, 0) \times \hat{y} + dxdz\vec{V}(0, dy/2, 0) \times \hat{y} - dxdy\vec{V}(0, 0, -dz/2) \times \hat{z} + dxdy\vec{V}(0, 0, dz/2) \times \hat{z} = -dydz(V_z(-dx/2, 0, 0)\hat{y} - V_y(-dx/2, 0, 0)\hat{z}) + dydz(-V_z(dx/2, 0, 0)\hat{y} - V_y(dx/2, 0, 0)\hat{z}) - dxdz(-V_z(0, -dy/2, 0)\hat{x} + V_x(0, -dy/2, 0)\hat{z}) + dxdz(-V_z(0, dy/2, 0)\hat{x} + V_x(0, dy/2, 0)\hat{z}) - dxdy(V_y(0, 0, -dz/2)\hat{x} - V_x(0, 0, -dz/2)\hat{y}) + dxdy(V_y(0, 0, dz/2)\hat{x} - V_x(0, 0, dz/2)\hat{y})$

Theorems

- Gauss's theorem, $\int_S \vec{V} \cdot d\vec{\sigma} = \int_V \nabla \cdot \vec{V} d\tau$, equates the flow out of a surface S with the sources inside the volume enclosed by it.
- Alternate form: $\int_S \phi d\vec{\sigma} = \int_V \nabla\phi d\tau$ using $\vec{V} = \phi(x, y, z)\vec{a}$
- Alternate form: $\int_S d\vec{\sigma} \times \vec{P} = \int_V \nabla \times \vec{P} d\tau$ using $\vec{V} = \vec{a} \times \vec{P}$
- Prove Green's theorem, $\int_V (u\nabla^2 v - v\nabla^2 u) d\tau = \int_S (u\nabla v - v\nabla u) \cdot d\vec{\sigma}$, by applying Gauss's theorem to the difference of $\nabla \cdot (u\nabla v) = u\nabla^2 v + \nabla u \cdot \nabla v$ and $\nabla \cdot (v\nabla u)$.
- Alternative form, $\int_S u\nabla v \cdot d\vec{\sigma} = \int_V (u\nabla^2 v + \nabla u \cdot \nabla v) d\tau$

Theorems

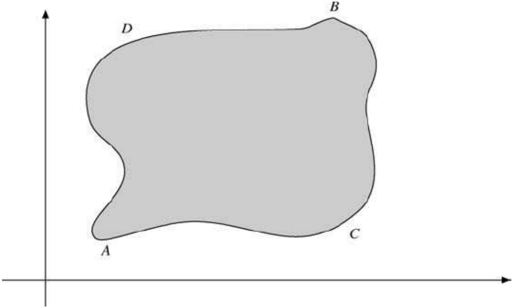
- Stokes theorem: $\oint \vec{V} \cdot d\vec{\lambda} = \int_S \nabla \times \vec{V} \cdot d\vec{\sigma}$



- Alternate form: $\int_S d\sigma \times \nabla\phi = \oint_{\partial S} \phi d\lambda$ using $\vec{V} = \vec{a}\phi$

Potential theory

- Scalar potential
- Conservative force
  $\iff F = -\nabla\phi \iff \nabla \times F = 0 \iff \oint F \cdot dr = 0$
- $\nabla \times F = -\nabla \times \nabla\phi = 0$
- $\oint F \cdot dr = -\oint \nabla\phi \cdot dr = -\oint d\phi = 0$
- $\oint_{ACBDA} F \cdot dr = 0 \iff \int_{ACB} F \cdot dr = -\int_{BDA} F \cdot dr = \int_{ADB} F \cdot dr \iff$ the work is path independent.

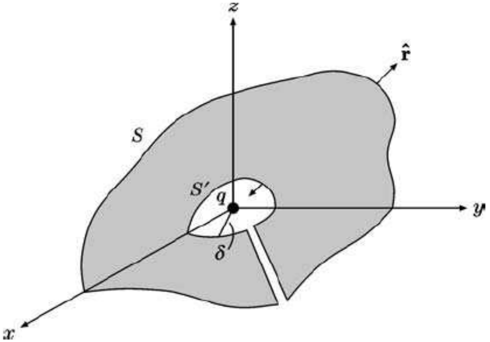


Potential theory

- Thus $\int_A^B F \cdot dr = \phi(A) - \phi(B) \rightarrow F \cdot dr = -d\phi = -\nabla\phi \cdot dr$. Therefore $(F + \nabla\phi) \cdot dr = 0$
- $\oint F \cdot dr = \int \nabla \times F \cdot d\sigma$ by integrating over the perimeter of an arbitrary differentil volume $d\sigma$ we see that $\oint F \cdot dr = 0$ result in $\nabla \times F = 0$.
- Scalar potential for the gravitational force on a unit mass m_1, $F_G = -\frac{Gm_1 m_2 \hat{r}}{r^2} = -\frac{k\hat{r}}{r^2}$?
- Scalar potential for the centrifugal force and simple harmonic oscillator on a unit mass m_1, $\vec{F}_c = \omega^2 \vec{r}$ and $\vec{F}_{SHO} = -k\vec{r}$.
- Exact differentials. How to know if integral of $df = P(x, y)dx + Q(x, y)dy$ is path dependent or independent.
- Vector potential $\vec{B} = \nabla \times \vec{A}$

Gauss's law, Poisson's equation

- Only a point charge at the origin $\vec{E} = \frac{q\hat{r}}{4\pi\epsilon_0 r^2}$
- Gauss's law: $\int_S \vec{E}\cdot d\vec{\sigma} = \begin{cases} 0 & \text{S does not contain the origin,} \\ \frac{q}{\epsilon_0} & \text{S contains the origin.} \end{cases}$
- Closed surface S not including the origin
  $\int_S \frac{\hat{r}\cdot d\vec{\sigma}}{r^2} = \int_V \nabla\cdot(\frac{\hat{r}}{r^2})d\tau$



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- $\int_S \frac{\hat{r}\cdot d\vec{\sigma}}{r^2} + \int_{S'} \frac{\hat{r}\cdot d\vec{\sigma}'}{\delta^2} = 0$

Gauss's law, Poisson's equation

- $d\sigma' = -\hat{r}\delta^2 d\Omega$
- $\int_S \vec{E} \cdot d\vec{\sigma} = \frac{q}{\epsilon_0} = \int_V \frac{\rho}{\epsilon_0} d\tau$. Further, $\int_S \vec{E} \cdot d\vec{\sigma} = \int_V \nabla \cdot \vec{E} d\tau$
- Maxwell equation: $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$
- Poisson's equation: $\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$.
- Laplace's equation $\nabla^2 \phi = 0$
- Substitute ϕ for E into the Gauss's law.

Dirac delta function

- $\int_v \nabla^2(\frac{1}{r})d\tau = \begin{cases} -4\pi & 0 \in v, \\ 0 & 0 \notin v. \end{cases}$ Thus
  $\nabla^2(\frac{1}{r}) = -4\pi\delta(\vec{r}) = -4\pi\delta(x)\delta(y)\delta(z).$
- Dirac Delta properties $\begin{cases} \delta(x) = 0 & x \neq 0, \\ f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx. \end{cases}$
- See functions approximating δ in a Mathematica notebook.
  $$\delta_n(x) = \begin{cases} 0 & x < -\frac{1}{2n}, \\ n, & -\frac{1}{2n} < x < \frac{1}{2n}, \\ 0 & x > \frac{1}{2n}. \end{cases}$$
- $\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}.$
- $\delta_n(x) = \frac{n}{\pi} \frac{1}{1+n^2x^2}.$
- $\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi} \int_{-n}^{n} e^{ixt} dt.$
- $\int_{-\infty}^{\infty} f(x)\delta(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\delta_n(x)dx$

Dirac delta function

- $\delta(x)$ is a distribution defined by the sequences $\delta_n(x)$
- Evenness: $\delta(x) = \delta(-x)$.
- $\int_{-\infty}^{\infty} f(x)\delta(ax)dx = \frac{1}{a}\int_{-\infty}^{\infty} f(\frac{y}{a})\delta(y)dy = \frac{1}{a}f(0)$. Thus $\delta(ax) = \frac{1}{|a|}\delta(x)$.
- $\int_{-\infty}^{\infty} f(x)\delta(g(x))dx = \sum_a \int_{a-\epsilon}^{a+\epsilon} f(x)\delta((x-a)g'(a))dx$. Thus $\delta(g(x)) = \sum_{a,g(a)=0,g'(a)\neq 0} \frac{\delta(x-a)}{|g'(a)|}$.
- Derivative: $\int f(x)\delta'(x-x_0)dx = -\int f'(x)\delta(x-x_0)dx = -f'(x_0)$.
- Delta Operator: $\mathcal{L}(x_0) = \int dx\delta(x-x_0)$.
- $\iiint_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \delta(\vec{r})r^2 dr \sin\theta d\theta d\phi$

Representation of Dirac delta by orthogonal functions

- Consider an infinite dimensional vector space where elements of the underlying set are functions.
  $(f+g)(x) = f(x) + g(x) \quad (cf)(x) = cf(x)$.
- Inner product maybe defined as $f(x) \cdot g(x) = \int_a^b f(x)g(x)dx$ where either a, b or both can be ∞.
- No good and natural example but Real orthogonal functions $\{\phi_n(x), n = 0, 1, 2, \cdots\}$ form a basis for this vector space.
- Their orthonormality relation is
  $\phi_m \cdot \phi_n = \int_a^b \phi_m(x)\phi_n(x)dx = \delta_{mn}$
- Around any point x_0 an example is the set $\{(x-x_0)^0, (x-x_0), (x-x_0)^2, \cdots\}$ which is not orthonormal.
- Use Gram-Schmidt orthonormalization.
- For square integrable functions use $\{\sin(n\pi x), \cos(n\pi x)\}$
- Expanding delta function in this bases:
  $\delta(x-t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x)$: closure.
- Take the inner product of both sides by $\phi_m(x)$ to derive coefficients.

Fourier transform

- $\delta(x-t) = \sum_{n=0}^{\infty} \phi_n(t)\phi_n(x) = \delta(t-x)$
- $\int F(t)\delta(x-t)dt = \int \sum_{p=0}^{\infty} a_p\phi_p(t) \sum_{n=0}^{\infty} \phi_n(t)\phi_n(x)dt = \sum_{n,p=0}^{\infty} a_p\phi_n(x)\delta_{np} = \sum_{p=0}^{\infty} a_p\phi_p(x) = F(x)$
- Fourier integral translates a function from one domain into another, $\mathcal{F}(f(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt = F(\omega)$, $\mathcal{F}(\psi(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)e^{ixp}dx = \psi(p)$
- Inverse Fourier transform is $\mathcal{F}^{-1}(F(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\exp(-i\omega t)d\omega = f(t)$, $\mathcal{F}^{-1}(\psi(p)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(p)\exp(-ixp)dp = \psi(x)$
- $\mathcal{F}(\delta(x)) = \frac{1}{\sqrt{2\pi}}$, $\delta(x) = \mathcal{F}^{-1}(\mathcal{F}(\delta(x))) = \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ixp)dp$

Probability distributions

- Average of a discrete random variable, $\bar{u} = \frac{\sum_{j=1}^{M} u_j p(u_j)}{\sum_{j=1}^{M} p(u_j)}$
- Average of any function of u: $\overline{f(u)} = \sum_{j=1}^{M} f(u_j) p(u_j)$
- m'th moment of distribution $\overline{u^m}$
- m'th central moment of distribution $\overline{(u - \bar{u})^m}$ including variance.
- Poisson distribution: $P(m) = \frac{a^m e^{-a}}{m!}$
- $\overline{f(u)} = \int f(u) p(u) du$
- Gauss distribution: $p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$

Important equations in physics

- Laplace's equation: $\nabla^2 \phi = 0$ or $\Delta\phi = 0$. Its solutions describe the behaviour of electric, gravitational and fluid potentials. Laplace's equation is also the steady-state heat equation.
- Helmholtz equation represents a time-independent form of the wave equation: $\nabla^2 A + k^2 A = 0$, where k is the wavenumber and A is amplitude. HE commonly results from separation of variable in a PDE involving both time and space varibles. E.g., the wave equation $(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})u(r,t) = 0$
- Diffusion equation: $\frac{\partial \phi(r,t)}{\partial t} = \nabla \cdot [D(\phi, r)\nabla(\phi(r,t))]$, where $\phi(r,t)$ is the density of the diffusing material at location r and time t, $D(\phi, r)$ is the collective diffusion coefficient for density at location r. If D is constant, $\frac{\partial \phi(r,t)}{\partial t} = D\Delta\phi(r,t)$ also called heat equation.
- Schrodinger wave equation: $i\hbar\frac{\partial}{\partial t}|\psi(r,t)\rangle = \hat{H}|\psi(r,t)\rangle$.

Important equations in physics

- For the nonrelativistic relative motion of two particles in the coordinate basis, $i\hbar\frac{\partial}{\partial t}\psi(r,t) = [-\frac{\hbar^2}{2\mu}\nabla^2 + V(r,t)]\psi(r,t)$.
- When Hamiltonian is not explicitly dependent on time, we have the time independent Schrodinger equation: $\hat{H}\psi = E\psi$.
- For the nonrelativistic relative motion of two particle in the coordinate basis, $[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)]\psi(r) = E\psi(r)$.
- All have the form $\nabla^2\psi + k^2\psi = 0$.
- Any coordinate system in which this equation is separable is of great interest.
- Thus finding expressions for gradient, divergence, curl and laplacian in a general coordinate system is of great interest.

Curvilinear coordinates

- A point can be specified as the intersection of the 3 planes x = constant, y = constant and z = constant.
- A point can be desdcribed by the intersection of three curvilinear coordinate surfaces q_1 = constant, q_2 = constant, q_3 = constant.
- Associate a unit vector $\hat{q}_i$ normal to the surface q_i = constant and in the direction of increasing q_i.
- General vector $\vec{V} = \hat{q}_1 V_1 + \hat{q}_2 V_2 + \hat{q}_3 V_3$.
- While coordinate or position vectors can be simpler, e.g., $\vec{r} = r\hat{r}$ in spherical polar coordinates and $\vec{r} = \rho\hat{\rho} + z\hat{z}$ for cylindrical coordinates.
- $\hat{q}_i^2 = 1$, for a right handed coordinate system $\hat{q}_1 \cdot (\hat{q}_2 \times \hat{q}_3) > 0$.
- $ds^2 = dx^2 + dy^2 + dz^2 = \sum_{ij} h_{ij}^2 dq_i dq_j$
- h_{ij} are referred to as the metric.
- $dx = (\frac{\partial x}{\partial q_1})dq_1 + (\frac{\partial x}{\partial q_2})dq_2 + (\frac{\partial x}{\partial q_3})dq_3$
- $dy = (\frac{\partial y}{\partial q_1})dq_1 + (\frac{\partial y}{\partial q_2})dq_2 + (\frac{\partial y}{\partial q_3})dq_3$

Curvilinear coordinates

- $dz = (\frac{\partial z}{\partial q_1})dq_1 + (\frac{\partial z}{\partial q_2})dq_2 + (\frac{\partial z}{\partial q_3})dq_3$
- $ds^2 = d\vec{r} \cdot d\vec{r} = d\vec{r}^2 = \sum_{ij} \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j} dq_i dq_j$. Thus: $h_{ij}^2 = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j}$, valid in metric or Riemannian spaces.
- For orthogonal coordinate systems:
  $h_{ij} = 0, \quad i \neq j$ or $\hat{q}_i \cdot \hat{q}_j = \delta_{ij}$. Thus, setting
  $h_{ii} = h_i > 0 \quad ds^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$.
- ds_i is the differential length in the direction of increasing q_i.
- Scale factors may be identified as $ds_i = h_i dq_i$ with length dimension. $\frac{\partial \vec{r}}{\partial q_i} = h_i \hat{q}_i$
- The differential distance vector
  $d\vec{r} = h_1 dq_1 \hat{q}_1 + h_2 dq_2 \hat{q}_2 + h_3 dq_3 \hat{q}_3$
- $\int \vec{V} \cdot d\vec{r} = \sum_i \int V_i h_i dq_i$
- For orthogonal coordinates: $d\sigma_{ij} = ds_i ds_j = h_i h_j dq_i dq_j$ and
  $d\tau = ds_1 ds_2 ds_3 = h_1 h_2 h_3 dq_1 dq_2 dq_3$
- $d\vec{\sigma} = ds_2 ds_3 \hat{q}_1 + ds_1 ds_3 \hat{q}_2 + ds_2 ds_1 \hat{q}_3 =$
  $h_2 h_3 dq_2 dq_3 \hat{q}_1 + h_1 h_3 dq_1 dq_3 \hat{q}_2 + h_2 h_1 dq_2 dq_1 \hat{q}_3$

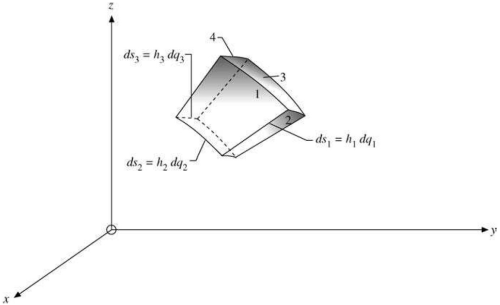
Curvilinear coordinates

- $\int_S \vec{V} \cdot d\vec{\sigma} =$
  $\int V_1 h_2 h_3 dq_2 dq_3 + \int V_2 h_1 h_3 dq_1 dq_3 + \int V_3 h_2 h_1 dq_2 dq_1$
- vector algebra is the same in orthogonal curvilinear coordinates as in Cartesian coordinates.
  $\vec{A} \cdot \vec{B} = \sum_{ik} A_i \hat{q}_i \cdot \hat{q}_k B_k = \sum_{ik} A_i B_k \delta_{ik} = \sum_i A_i B_i$
- $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{q}_1 & \hat{q}_2 & \hat{q}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$
- To perform a double integral in a curvilinear coordinate one needs to express a cartesian surface element in terms of the curvilinear coordinates.
- $d\vec{r}_1 = \vec{r}(q_1 + dq_1, q_2) - \vec{r}(q_1, q_2) = \frac{\partial \vec{r}}{\partial q_1} dq_1 \quad d\vec{r}_2 = \vec{r}(q_1, q_2 + dq_2) - \vec{r}(q_1, q_2) = \frac{\partial \vec{r}}{\partial q_2} dq_2$
- $dxdy = d\vec{r}_1 \times d\vec{r}_2|_z = [\frac{\partial x}{\partial q_1}\frac{\partial y}{\partial q_2} - \frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_1}] dq_1 dq_2 = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{vmatrix} dq_1 dq_2$

Curvilinear coordinates

- The transformation coefficient in determinant form is called the Jacobian
- Similarly, $dxdydz = dr_1 \cdot (dr_2 \times dr_3)$
- $dxdydz = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{vmatrix} dq_1 dq_2 dq_3$
- Volume Jacobian is $h_1 h_2 h_3 (\hat{q}_1 \times \hat{q}_2) \cdot \hat{q}_3$
- In polar coordinates: $x = \rho \cos\phi \quad y = \rho \sin\phi \quad J = ?$
- In spherical coordinates:
  $x = r \sin\theta \cos\phi \quad y = r \sin\theta \sin\phi \quad z = r \cos\theta \quad J = ?$

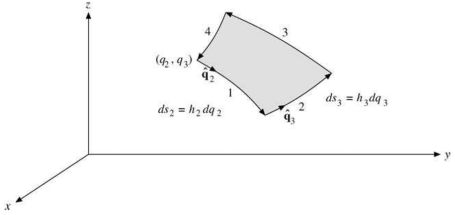
Differential vector operations

- Gradient is the vector of maximum space rate of change
- Since ds_i is the differential length in the direction of increasing q_i, this direction is depicted by the unit vector $\hat{q}_i$.
  $\nabla\psi \cdot \hat{q}_i = \nabla\psi|_i = \frac{\partial\psi}{\partial s_i} = \frac{\partial\psi}{h_i \partial q_i}$.
- $\nabla\psi(q_1, q_2, q_3) = \hat{q}_1 \frac{\partial\psi}{\partial s_1} + \hat{q}_2 \frac{\partial\psi}{\partial s_2} + \hat{q}_3 \frac{\partial\psi}{\partial s_3} = \hat{q}_1 \frac{\partial\psi}{h_1 \partial q_1} + \hat{q}_2 \frac{\partial\psi}{h_2 \partial q_2} + \hat{q}_3 \frac{\partial\psi}{h_3 \partial q_3}$
- $d\psi =$
- $\nabla \cdot \vec{V}(q_1, q_2, q_3) = \lim_{d\tau \to 0} \frac{\int_{S_{d\tau}} \vec{V} \cdot d\vec{\sigma}}{d\tau}$
- 

Differential vector operations: Divergence

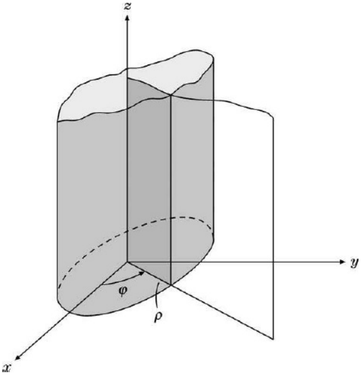
- Area integrals for the two q_1 = constant surfaces are $V_1(q_1 + dq_1, q_2, q_3)ds_2 ds_3 - V_1(q_1, q_2, q_3)ds_2 ds_3 = [V_1 h_2 h_3 + \frac{\partial}{\partial q_1}(V_1 h_2 h_3)dq_1]dq_2 dq_3 - V_1 h_2 h_3 dq_2 dq_3 = \frac{\partial}{\partial q_1}(V_1 h_2 h_3)dq_1 dq_2 dq_3$
- $\int V \cdot d\sigma = [\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_1 h_3) + \frac{\partial}{\partial q_3}(V_3 h_2 h_1)]dq_1 dq_2 dq_3$ where $V_i = \hat{q}_i \cdot \vec{V}$
- $\nabla \cdot \vec{V}(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_1 h_3) + \frac{\partial}{\partial q_3}(V_3 h_2 h_1)]$
- Using $V = \nabla\psi(q_1, q_2, q_3), \quad \nabla \cdot V = \nabla^2\psi = \frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial q_1}(\frac{h_2 h_3}{h_1}\frac{\partial\psi}{\partial q_1}) + \frac{\partial}{\partial q_2}(\frac{h_1 h_3}{h_2}\frac{\partial\psi}{\partial q_2}) + \frac{\partial}{\partial q_3}(\frac{h_2 h_1}{h_3}\frac{\partial\psi}{\partial q_3})]$

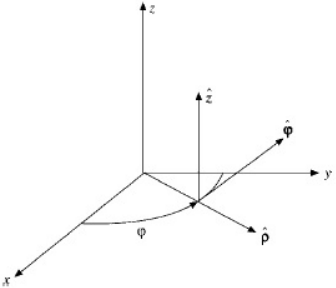
Differential vector operations: Curl

- Assuming the surface s to lay on $q_1 = $ constant surface.
- $\lim_{s\to 0} \int_s \nabla \times \vec{V} \cdot d\vec{\sigma} = \hat{q}_1 \cdot (\nabla \times \vec{V}) h_2 h_3 dq_2 dq_3 = \oint_{\partial_s} \hat{V} \cdot d\vec{r}$
- 

- $\oint_{\partial_s} \vec{V} \cdot d\vec{r} = V_2 h_2 dq_2 + [V_3 h_3 + \frac{\partial}{\partial q_2}(V_3 h_3) dq_2] dq_3 - [V_2 h_2 + \frac{\partial}{\partial q_3}(V_2 h_2) dq_3] dq_2 - V_3 h_3 dq_3 = [\frac{\partial}{\partial q_2}(V_3 h_3) - \frac{\partial}{\partial q_3}(V_2 h_2)] dq_2 dq_3$
- $\nabla \times \vec{V}|_1 = \frac{1}{h_2 h_3}[\frac{\partial}{\partial q_2}(V_3 h_3) - \frac{\partial}{\partial q_3}(V_2 h_2)]$
- Permuting the indices $\nabla \times \vec{V}|_2 = \frac{1}{h_3 h_1}[\frac{\partial}{\partial q_3}(V_1 h_1) - \frac{\partial}{\partial q_1}(V_3 h_3)]$
- Thus $\nabla \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{q}_1 & h_2 \hat{q}_2 & h_3 \hat{q}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$

Circular cylinder coordinates

- $(\rho, \phi, z), \quad 0 \leq \rho < \infty, \quad 0 \leq \phi \leq 2\pi$, and $-\infty < z < \infty$
- 



- $x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$
- Using: $h_{ij}^2 = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j}$
- $h_1 = h_\rho = 1, \quad h_2 = h_\phi = \rho, \quad h_3 = h_z = 1.$
- $\vec{r} = \hat{\rho}\rho + \hat{z}z, \quad \vec{V} = \hat{\rho}V_\rho + \hat{\phi}V_\phi + \hat{z}V_z$

Spherical polar coordinates

- $(r, \theta, \phi), \quad 0 \leq r < \infty, \quad 0 \leq \theta \leq \pi$, and $0 < \phi < 2\pi$
- $x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta$
- $h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_\phi = r\sin\theta.$
- $r_0 = \hat{i} \sin\theta \cos\phi + \hat{j} \sin\theta \cos\phi + \hat{k} \cos\theta, \quad \theta_0 = \hat{i} \cos\theta \cos\phi + \hat{j} \cos\theta \sin\phi - \hat{k} \sin\theta, \quad \phi_0 = -\hat{i} \sin\phi + \hat{j} \cos\phi$
- $\nabla\psi = \hat{r}_0 \frac{\partial\psi}{\partial r} + \hat{\theta}_0 \frac{1}{r} \frac{\partial\psi}{\partial\theta} + \hat{\phi}_0 \frac{1}{r\sin\theta} \frac{\partial\psi}{\partial\phi}$
- $\nabla \cdot \vec{V} = \frac{1}{r^2 \sin\theta}[\sin\theta \frac{\partial}{\partial r}(r^2 V_r) + r\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + r\frac{\partial V_\phi}{\partial\phi}]$
- $\nabla \cdot \nabla\psi =$
- $\nabla \times \vec{V} =$

Matrices

- Det(A) $= \epsilon_{i_1 \cdots i_n} a_{1i_1} \cdots a_{ni_n}$
- Theorem: $Det(AB) = Det(A)Det(B)$. Thus $Det(A^{-1}A) = Det(I) \rightarrow Det(A^{-1})Det(A) = 1 \rightarrow Det(A^{-1}) = \frac{1}{Det(A)}$
- Matrix A is invertible iff $Det(A) \neq 0$
- Consider a system of n first order linear equations in n unknowns,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots &= \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

Matrices

- Such a system can be written in matrix form
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$
- $AX = B$
- If $Det(A) \neq 0, \quad X = A^{-1}B$ and is uniquely determined.
- If $B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ the above system of linear equations is called homogeneous.
- In order for this system to have any solution other than the trivial $X = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, Det(A) must equal zero.

Leibniz integral rule

- $\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) =$
  $f\left(x, b(x)\right) \cdot \frac{d}{dx} b(x) - f\left(x, a(x)\right) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt,$
- A generalisation of the fundamental theorem of calculus; if $F(x) = \int_a^x f(t)\,dt$ then F'(x)=f(x),
- $F(x_1 + \Delta x) - F(x_1) = \int_a^{x_1+\Delta x} f(t)\,dt - \int_a^{x_1} f(t)\,dt = \int_{x_1}^{x_1+\Delta x} f(t)\,dt.$
- $F(x_1 + \Delta x) - F(x_1) = f(c) \cdot \Delta x.$
- $\lim_{\Delta x \to 0} \frac{F(x_1+\Delta x) - F(x_1)}{\Delta x} = \lim_{\Delta x \to 0} f(c).$

Differential equations

- Ordinary differential equations only contain functions of a single variable.
- Differential equations with partial derivatives include functions of more than one variable.
- The highest order derivative in the differential equation determines the order of the differential equation.
- $(y'')^3 + 2yy' + 5xy = \sin x$ is an ordinary differential equation of order 2.
- $(\frac{dy}{dx})^2 - [\sin(xy) - 4x]^2 = 0$ is an ordinary differential equation of the first order.
- $\frac{\partial^3 u}{\partial x^3} + x\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x \partial t} = 0$ is a differential equation with partial derivatives of the third order.

Ordinary differential equations

- $F(x, y, y', \cdots, y^{(n)}) = 0$ on an interval I.
- F is rewritten as, $y^{(n)} = f(x, y, \cdots, y^{(n-1)})$
- A function ϕ such that $\phi^{(n)} = f(x, \phi, \cdots, \phi^{(n-1)})$ is a solution to this differential equation on I.
- Initial conditions are restrictions on the solution at a single point, while boundary conditions are restrictions on the solution at different points.
- E.g., $y' = 2y - 4x \rightarrow y = ce^{2x} + 2x + 1$
- E.g., $y'' + y = x \rightarrow y = c_1 \cos x + c_2 \sin x + x$
- $a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x)$ is a linear ordinary differential equation which constitutes our focus in this section of the course.

- $y^{(4)} + 4y''' + 3y = x; \qquad y_1 = \frac{x}{3}, \quad y_2 = e^{-x} + \frac{x}{3}$
- $x^2y'' + 5xy' + 4y = 0,\ x > 0; \quad y_1 = x^{-2}, y_2 = x^{-2}\ln x$
- $y' - 2xy = 1; \quad y = e^{x^2}\int_0^x e^{-t^2}dt + e^{x^2}$
- $u_{xx} + u_{yy} = 0; \quad u_1 = x^2 + y^2, u_2 = xy$
- $u_{tt} - c^2u_{xx} = 0; \qquad u_1 = \sin(x + ct), u_2 = \sin(x - ct), u_3 = f(x + ct) + g(x - ct)$
- $u_{xx} + u_{yy} + u_{zz} = 0; u = (x^2 + y^2 + z^2)^{-1/2}$
- $x^2y'' + xy' + y = 0, y(1) = 1, y'(1) = -1; \quad y = \cos(\ln x) - \sin(\ln x)$

First order differential equations

- $y' = f(x,y) \quad y(x_0) = y_0$ there exists a unique solution if f and $\frac{\partial f}{\partial x}$ are continuous around (x_0, y_0).
- First order linear differential equations: $\frac{dy}{dx} + a(x)y = f(x)$
- Assuming $A(x) = \int^x a(t)dt$, $\frac{d}{dx}(e^{A(x)}y) = e^{A(x)}(y' + a(x)y) = e^{A(x)}f(x)$
- General solution is: $y = e^{-A(x)} \int^x e^{A(t)} f(t)dt + ce^{-A(x)}$
- Imposing the initial condition, $y(x_0) = y_0$, $y = e^{-A(x)} \int_{x_0}^x e^{A(t)} f(t)dt + y_0 e^{-(A(x) - A(x_0))}$
- e.g., $y' = y + \sin x$, $e^{-x}(y' - y) = (e^{-x}y)' = e^{-x} \sin x$
- $e^{-x}y = \int^x e^{-t} \sin t dt + c = \frac{-1}{2} e^{-x}(\sin x + \cos x) + c$

- Solve $y' = y + \sin x, \quad y(0) = 1$
- $(x \ln x)y' + y = 6x^3, \quad x > 1$, thus $(y \ln x)' = 6x^2$,
  $y = \frac{2x^3+c}{\ln x} \quad x > 1$.
- Assuming $a(x)$ and $f(x)$ to be continuous on the interval (α, β) for every $x_0 \in (\alpha, \beta)$, the initial value problem $y' + a(x)y = f(x) \quad y(x_0) = y_0$, for every value of y_0 has one and only one solution on the interval (α, β).
- $xy' + 2y = 4x^2, \quad x > 0, \qquad y(1) = 2$, result in $y = x^2 + \frac{c}{x^2}$.
- Solve it for y(1)=1.

First order differential equations

- $y' + \frac{y}{x} = 3\cos 2x, x > 0$
- $y' + 3y = x + e^{-2x}$
- $(x^2+1)y' + y + 1 = 0$
- $y' \sin 2x = y \cos 2x$
- $xy' + y + 4 = 0, x > 0$
- $x^2y' - xy = x^2 + 4, x > 0$
- $y' + 2y = xe^{-2x}; y(1) = 0$
- $y' + \frac{2}{x}y = \frac{\cos x}{x^2}; y(\pi) = 0$
- $y' + y\cot x = 2x - x^2 \cot x, y(\frac{\pi}{2}) = \frac{\pi^2}{4} + 1$
- $y' - x^3 y = -4x^3; y(0) = 6$
- $y' + y\tan x = \sin 2x; y(0) = 1$
- $\sin x y' + \cos x y = \cos 2x, x \in (0, \pi); y(\frac{\pi}{2}) = 1/2$
- $y' + \frac{y}{x} = e^{x^2},\ x > 0;\ y(1) = 0$
- $y' + y = xe^{-x}; y(0) = 1$

- For nonlinear equations there is no general method for solving the DE.
- Separable differential equations:
  $y' = f(x, y) \to p(x) + q(y)y' = 0$
- $p(x)dx + q(y)dy = 0 \to d[P(x) + Q(y)] = 0 \to$
  $P(x) + Q(y) = c \to y = \phi(x, c)$
- E.g., $y' = \frac{2+\sin x}{3(y-1)^2} \to (2 + \sin x)dx - 3(y - 1)^2 dy = 0 \to$
  $2x - \cos x - (y - 1)^3 = c \to y = 1 + (2x - \cos x - c)^{1/3}$
- E.g., $y' = \frac{x^3 y - y}{y^4 - y^2 + 1}, y(0) = 1 \to (y^3 - y + 1/y)dy =$
  $(x^3 - 1)dx \to y^4/4 - y^2/2 + \ln |y| = x^4/4 - x + c$

Complete first order DE

- $y' = -\frac{p(x,y)}{q(x,y)} \to p(x,y)dx + q(x,y)dy = 0$ this equation is complete in a region D if and only if there is a g such that $dg(x,y) = p(x,y)dx + q(x,y)dy$
- $\frac{\partial g}{\partial x} = p(x,y), \quad \frac{\partial g}{\partial y} = q(x,y)$
- E.g., For $(4x - y)dx + (2y - x)dy = 0, \quad g(x,y) = 2x^2 - xy + y^2$, g is an integral of the differential equation and the curves $g(x,y) = c$ are its integral curves.
- Theorem: The necessary and sufficient condition for completeness of $p(x,y)dx + q(x,y)dy = 0$ in a region D of the xy plane is to have $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}, \quad (x,y) \in D$

Complete first order DE

- The condition is necessary since $g_{xy} = g_{yx}$, to prove sufficiency consider g such that $g_x(x,y) = p(x,y), \quad g_y(x,y) = q(x,y)$ thus we have $g(x,y) = \int^x p(t,y)dt + h(y) \to g_y(x,y) = \int^x \frac{\partial p(t,y)}{\partial y} dt + h'(y) = q(x,y)$ thus $h'(y) = q(x,y) - \int^x \frac{\partial p(t,y)}{\partial y} dt$
- If we show that the right hand side is only a function of y, we have an algorithm for evaluating g.
- $\frac{\partial}{\partial x}[q(x,y) - \int^x \frac{\partial p(t,y)}{\partial y} dt] = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 0$
- E.g., $(4x - y)dx + (2y - x)dy = 0$ for which $\frac{\partial p}{\partial y} = -1, \quad \frac{\partial q}{\partial x} = -1$. Thus $dg(x,y) = (4x - y)dx + (2y - x)dy$

Completing a first order DE

- $g(x, y) = 2x^2 - xy + h(y)$ so
  $-x + h'(y) = 2y - x \quad h(y) = y^2 + c,$
- $g(x, y) = 2x^2 - xy + y^2 + c$
- Integration factor
- $\mu(x, y)p(x, y)dx + \mu(x, y)q(x, y)dy = 0$
- $\frac{\partial}{\partial y}(\mu p) = \frac{\partial}{\partial x}(\mu q)$
- $p(x, y)\frac{\partial \mu}{\partial y} - q(x, y)\frac{\partial \mu}{\partial x} + (\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x})\mu = 0.$ This PDE must be solved to find the integrating factor.
- E.g., $x^2 - y^2 + 2xyy' = 0$, Assuming
  $\mu = \mu(x), \quad \mu(x)(x^2 - y^2)dx + \mu(x)(2xy)dy = 0$
- $\frac{\partial}{\partial y}[\mu(x^2 - y^2)] = \frac{\partial}{\partial x}[\mu(2xy)] \rightarrow x\mu' + 2\mu = 0 \rightarrow \mu(x) = x^{-2}$
- $(1 - \frac{y^2}{x^2})dx + (\frac{2y}{x})dy = 0 \rightarrow x + y^2/x = c \rightarrow y^2 + (x - a)^2 = a^2$

Completing a first order DE: excersize

- $y' = x^3 y^{-2}$
- $(1+x^2)^{1/2} y' = 1 + y^2$
- $y' = xy^2 + y^2 + xy + y$; $Y(1) = 1$
- $(x+1)y' + y^2 = 0; y(0) = 1$
- $(2x - y)dx - xdy = 0$
- $(x - 2y)dx + (4y - 2x)dy = 0$
- $\frac{ydx - xdy}{y^2} + xdx = 0$
- $3(x-1)^2 dx - 2ydy = 0$
- $e^{x^2 y}(1 + 2x^2 y)dx + x^3 e^{x^2 y} dy = 0$
- $(x^2 + y^2)^2 (xdx + ydy) + 2dx + 3dy = 0$
- $(x^2 + y^2)dx + 2xydy = 0, y(1) = 1$
- $\frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y^2} = 0, y(2) = 2$
- $(x - y)dx + (2y - x)dy = 0, y(0) = 1$
- If $\mu = \mu(x)$, $\frac{\partial \mu}{\partial y} = 0, \quad \frac{d\mu}{\mu} = \frac{p_y - q_x}{q} dx$

Completing a first order DE

- If $\mu = \mu(y)$, $\quad \frac{d\mu}{\mu} = \frac{q_x - p_y}{p} dy$
- $(x^2 - y^2) - 2xyy' = 0$
- $y + (y^2 - x)y' = 0$
- $(3xy + y^2) + (x^2 + xy)y' = 0$
- $(3xy + y^2)dx + (3xy + x^2)dy = 0$

- Bernoulli equation: $y' + a(X)y = b(x)y^\alpha$ use $z = y^{1-\alpha}$ get $z' + (1-\alpha)a(x)z - (1-\alpha)b(x) = 0$
- $xy' - y = e^x y^3$
- Riccati equation: $y' = a(x)y + b(x)y^2 + c(x)$ assume $y = \phi(x)$ to be a private solution and use $y = \phi(x) + 1/z$ to derive $z' + [a(x) + 2\phi(x)b(x)]z = -b(x)$.
- $y' = 1 + x^2 - 2xy + y^2, \quad \phi(x) = x$
- $y' - xy^2 + (2x-1)y = x - 1, \quad \phi(x) = 1$
- $y' + xy^2 - 2x^2y + x^3 = x + 1, \quad \phi(x) = x - 1$
- $y' + y^2 - (1 + 2e^x)y + e^{2x} = 0, \quad \phi(x) = e^x$
- $y' + y^2 - 2y + 1 = 0$

Completing a first order DE

- If $\mu = \mu(y)$, $\quad \frac{d\mu}{\mu} = \frac{q_x - p_y}{p} dy$
- $(x^2 - y^2) - 2xyy' = 0$
- $y + (y^2 - x)y' = 0$
- $(3xy + y^2) + (x^2 + xy)y' = 0$
- $(3xy + y^2)dx + (3xy + x^2)dy = 0$
- $\mu = x + y$
- Bernoulli equation: $y' + a(X)y = b(x)y^\alpha$ use $z = y^{1-\alpha}$ get $z' + (1-\alpha)a(x)z - (1-\alpha)b(x) = 0$
- $xy' - y = e^x y^3$
- Riccati equation: $y' = a(x)y + b(x)y^2 + c(x)$ assume $y = \phi(x)$ to be a private solution and use $y = \phi(x) + 1/z$ to derive $z' + [a(x) + 2\phi(x)b(x)]z = -b(x)$.
- $y' = 1 + x^2 - 2xy + y^2, \quad \phi(x) = x$
- $y' - xy^2 + (2x-1)y = x - 1, \quad \phi(x) = 1$
- $y' + xy^2 - 2x^2y + x^3 = x + 1, \quad \phi(x) = x - 1$
- $y' + y^2 - (1 + 2e^x)y + e^{2x} = 0, \quad \phi(x) = e^x$
- $y' + y^2 - 2y + 1 = 0$

Linear differential equations

- $a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x)$
- $y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x)$
- $L_n \equiv \frac{d^n}{dx^n} + p_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x)$
- $L_n[y] = f(x)$
- Existence and uniqueness theorem: If $p_1, p_2, \cdots, p_n$ and f are continuous on the interval I, $\forall x_0 \in I$ the above equation has one and only one solution $y = \phi(x)$ satisfying $\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \phi''(x_0) = \alpha_3, \cdots, \phi^{(n-1)}(x_0) = \alpha_n$.
- $y'' + p(x)y' + q(x)y = 0; \quad y(x_0) = 0, y'(x_0) = 0$ only has the trivial solution.

Linear differential equations

- $xy'' + (\cos x)y' + \frac{x}{1+x}y = 2x$ solutions can be determined for each of the intervals $(-\infty, -1), (-1, 0)$ and $(0, \infty)$.
- Homogeneous differential equations have f(x)=0. E.g., $y'' + p(x)y' + q(x)y = 0$.
- Operator L is called linear iff for arbitrary constants $c_1, c_2, c_3, \cdots, c_k$ and functions $\phi_1, \phi_2, \cdots, \phi_k; \quad L[c_1\phi_1 + c_2\phi_2 + \cdots + c_k\phi_k] = c_1 L[\phi_1] + c_2 L[\phi_2] + \cdots + c_k L[\phi_k]$.
- $c_1\phi_1 + c_2\phi_2 + \cdots + c_k\phi_k = \sum_i c_i\phi_i$ is a linear combination of the k functions ϕ_i.
- If $\phi_1, \phi_2, \cdots, \phi_k$ are solutions of $L_n[y] = 0$ each linear combination of them is a solution as $L_n[\sum_{i=1}^k c_i\phi_i] = \sum_{i=1}^k c_i L_n[\phi_i] = 0$.

Homogeneous Linear differential equations

- $L_2[y] = y'' - y = 0$
- $y''' + y' = 0$
- m functions $g_1, g_2, \cdots, g_m$ are linearly independent on the interval I iff $c_1 g_1(x) + c_2 g_2(x) + \cdots + c_m g_m(x) = 0$ implies that $c_1 = c_2 = \cdots = c_m = 0$.
- The set of functions $g_1, g_2, \cdots, g_m$ are linearly dependent on the interval I if there is a set of constants $c_1, c_2, \cdots, c_m$ including at least one non zero c_i such that for $\forall x \in I \quad c_1 g_1(x) + c_2 g_2(x) + \cdots + c_m g_m(x) = 0$.
- E.g., $\{e^{r_1 x}, e^{r_2 x}\}$.
- E.g., $\{e^x, e^{-x}, \cosh x\}$.

Wronskian

- Introduced by Polish mathematician Jozef Wronski.
- If $f_1, f_2, \cdots, f_n$ are (n-1) times differentiable functions on I,

$$W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

- E.g., $W(x^2, x^3) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4$
- E.g., $W(1, e^x, e^{-x}) = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 2$

Wronskian

- Theorem: Given $p(x)$ and $q(x)$ continuous on I, two solutions of $L_2[y] = y'' + p(x)y' + q(x)y = 0$ are linearly independent on I iff their Wronskian is non-zero on I.
- If ϕ_1 and ϕ_2 are dependent $\exists b_1, b_2 \neq 0| \quad b_1\phi_1 + b_2\phi_2 = 0 \quad b_1\phi_1' + b_2\phi_2' = 0$
- $\begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 0$
- Nonzero Wronskian implies $b_1 = b_2 = 0$ and that ϕ_1 is linearly independent from ϕ_2.
- Assume $\{\phi_1, \phi_2\}$ are linearly independent and $\exists x_0 \quad W(\phi_1, \phi_2)(x_0) = 0$
- $\begin{bmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 0$ has nontrivial solutions b_{10}, b_{20}

Wronskian

- Define $\psi(x) = b_{10}\phi_1(x) + b_{20}\phi_2(x)$
- $\psi(x_0) = b_{10}\phi_1(x_0) + b_{20}\phi_2(x_0) = 0$
- $\psi'(x_0) = b_{10}\phi_1'(x_0) + b_{20}\phi_2'(x_0) = 0$
- $\psi(x)$ is the solution to $L_n[y] = 0, \quad \psi(x_0) = 0, \quad \psi'(x_0) = 0$
  According to the existence and uniqueness theorem $\psi \equiv 0$.
- This implies linear dependence of $\{\phi_1, \phi_2\}$.

Wronskian

- Theorem: Wronskian of the solutions to the $L_2[y] = 0$ on I are either never zero or always zero.
- Proof: $W(\phi_1, \phi_2)(x) = \phi_1\phi_2' - \phi_2\phi_1'$, $\quad \frac{dW}{dx} = \phi_1\phi_2'' - \phi_2\phi_1'' = p(x)(\phi_1'\phi_2 - \phi_2'\phi_1) = -p(x)W$
- Abel relation: $W(\phi_1, \phi_2)(x) = ce^{-\int_{x_0}^{x} p(t)dt}, \quad x \in I$
- $W(\phi_1, \phi_2)(x) = W(\phi_1, \phi_2)(x_0)e^{-\int_{x_0}^{x} p(t)dt}, \quad x \in I$
- If $p_1(x), p_2(x), \cdots, p_n(x)$ are continuous on the interval I, then solutions $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$ of $L_n[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$ are linearly independent iff their Wronskian is nonzero.
- Further, $\frac{dW}{dx} + p_1(x)W = 0$

Wronskian

- $W(\phi_1, \cdots, \phi_n)(x) = W(\phi_1, \cdots, \phi_n)(x_0) e^{-\int_{x_0}^{x} p_1(t)dt}, \quad x \in I$
- $y''' - 4y'' + 5y' - 2y = 0$ has solutions
  $\phi_1 = e^x,\ \phi_2 = xe^x,\ \phi_3 = e^{2x}$, these constitute a fundamental set of solutions.
- Theorem: Linear homogeneous differential equation of order n has n linearly independent solutions.
- Proof: consider

$$\begin{aligned}
L_n[y] = 0; &\quad y(x_0) = 1, y'(x_0) = 0, y''(x_0) = 0, \cdots, y^{(n-1)}(x_0) = 0 \\
L_n[y] = 0; &\quad y(x_0) = 0, y'(x_0) = 1, y''(x_0) = 0, \cdots, y^{(n-1)}(x_0) = 0 \\
\vdots & \qquad\qquad\qquad\qquad \vdots \\
L_n[y] = 0; &\quad y(x_0) = 0, y'(x_0) = 0, y''(x_0) = 0, \cdots, y^{(n-1)}(x_0) = 1
\end{aligned}$$

of solutions of a LHDE

- By existence and uniqueness theorem the above equations have solutions $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$
- $$\begin{aligned} c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) &= 0 \\ c_1\phi_1'(x) + c_2\phi_2'(x) + \cdots + c_n\phi_n'(x) &= 0 \\ c_1\phi_1''(x) + c_2\phi_2''(x) + \cdots + c_n\phi_n''(x) &= 0 \\ \vdots &= \vdots \\ c_1\phi_1^{(n-1)}(x) + c_2\phi_2^{(n-1)}(x) + \cdots + c_n\phi_n^{(n-1)}(x) &= 0 \end{aligned}$$
- Substitute $x = x_0$ to derive $c_1 = c_2 = \cdots = c_n = 0$
- n linearly independent solutions of a linear differential equation of order n are called a fundamental set of that equation.

Linear vector space of solutions

- Theorem: If $p_1(x), p_2(x), \cdots, p_n(x)$ are continuous on the interval I, and if solutions $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$ are a fundamental set of $L_n[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$ on I, for every solution $\phi(x)$ there is a unique set $c_1, \cdots, c_n$ such that $\phi(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)$
- Proof: Assume $\phi(x_0) = \alpha_0,\ \phi'(x_0) = \alpha_1,\ \cdots,\ \phi^{(n-1)}(x_0) = \alpha_{n-1}$

$$
\begin{aligned}
c_1\phi_1(x_0) + c_2\phi_2(x_0) + \cdots + c_n\phi_n(x_0) &= \alpha_0 \\
c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \cdots + c_n\phi_n'(x_0) &= \alpha_1 \\
\vdots &= \vdots \\
c_1\phi_1^{(n-1)}(x_0) + c_2\phi_2^{(n-1)}(x_0) + \cdots + c_n\phi_n^{(n-1)}(x_0) &= \alpha_{n-1}
\end{aligned}
$$

Linear vector space of solutions

- if solutions $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$ are a fundamental set of $L_n[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$ on I, $W(\phi_1, \cdots, \phi_n)(x) \neq 0$. Thus the above system has unique solutions $c_1^0, \cdots, c_n^0$. Define $\psi = c_1^0\phi_1(x) + c_2^0\phi_2(x) + \cdots + c_n^0\phi_n(x)$
- According to existence and uniqueness theorem $\psi = \phi$.

- Consider a private solution $\phi_p(x)$ of $L_n[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x)$ where $p_i(x)$ and $f(x)$ are continuous on I, and $\{\phi_1(x), \phi_2(x), \cdots, \phi_n(x)\}$ is a fundamental set of the corresponding linear homogeneous DE. If $\phi(x)$ is any other solution to the $L_n[y] = f(x)$ then $L_n[\phi - \phi_p] = L_n[\phi] - L_n[\phi_p] = 0$ thus $\phi = c_i\phi_i + \phi_p$

- Theorem: If $\phi_p(x)$ is a private solution of $L_n[y] = f(x)$, every solution can be written as $\phi(x) = c_k\phi_k(x) + \phi_p(x)$ this is called a general solution.
- Find the general solution to $y^{(4)} + 2y'' + y = x$
- $\phi_p = x, \quad \{\cos x, \sin x, x\cos x, x\sin x\}, \quad \phi(x) = ?$
- E.g., $y'' - y = x, \quad y(0) = 0, y'(0) = 1$
- $\phi_p = -x \quad \{e^x, e^{-x}\}$
- E.g.,
  $x^2y'' + 4xy' + 2y = 6x + 1, \quad x > 0, \quad y(1) = 2, \quad y(2) = 1$
- $\phi_p = x + 1/2, \{1/x, 1/x^2\}$

Linear differential equations: Exercise

- If $L[y] = y'' + ay' + by$, find a) $L[\cos x]$, b) $L[x^2]$, c) $L[x^r]$, d) $L[e^{rx}]$
- If $L[y] = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$ determine $L[e^{rx}]$
- $L[y] = x^2 y'' + axy' + by$ determine $L[x^r]$, do the same for $L[y] = x^3 y''' + a_1 x^2 y'' + a_2 xy' + a_3 y$
- Check validity of given solution and determine its validity integral. $xy'' + y' = 0; \quad \phi(x) = \ln(\frac{1}{x})$
- $4x^2 y'' + 4xy' + (4x^2 - 1)y = 0; \quad \phi(x) = \sqrt{\frac{2}{\pi x}} \sin x$
- $(1 - x^2)y'' = -2xy' + 6y; \quad \phi(x) = 3x^2 - 1$
- $(1 - x^2)y'' = -2xy' + 2y + 2; \quad \phi(x) = x \tanh^{-1} x$
- Show that $\phi_1(x) = \frac{1}{9}x^3$ and $\phi_2(x) = \frac{1}{9}(x^{3/2} + 1)^2$ satisfy $(y')^2 - xy = 0$ on the interval $(0, \infty)$. Do their sum satisfy this DE?

- $y' - 3y^{2/3} = 0$ has the general solution $y = (x + c)^3$. Test if linear combinations of these solutions are solutions. Test the independence of different solutions? Consider the following solutions: a) $\phi(x) = \begin{cases} (x-a)^3 & x \le a \\ 0 & x > a \end{cases}$ b) $\phi(x) = \begin{cases} 0 & x \le b \\ (x-b)^3 & x > b \end{cases}$ c) $\phi(x) = \begin{cases} (x-a)^3 & x \le a \\ 0 & b > x > a \\ (x-b)^3 & x \ge b \end{cases}$
- Show that functions $1, x, x^2, \cdots, x^n$ constitute a linearly independent set.
- Prove that n solutions of the DE $L[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0$ are linearly independent iff their Wronskian is nonzero.
- Drive the Abel relation for n=3. To this end show that $w' = \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1' & \phi_2' & \phi_3' \\ \phi_1''' & \phi_2''' & \phi_3''' \end{vmatrix}$

Linear DE with constant coefficients

- $y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = 0$
- $L_n = \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n = D^n + a_1 D^{n-1} + \cdots + a_n$
- $L[y] = (L_1 \cdots L_k)[y]$
- If ϕ is a solution to $L_i[y] = 0$ then $L[\phi] = (L_1 \cdots L_{i-1} L_{i+1} \cdots L_k) L_i[\phi] = 0$
- In this way solutions of linear homogeneous DE with constant coefficients of order n can be deduced from solutions of DEs of order one and two.
- E.g., $L_n[y] = y'' + y' - 2y = 0 = (D^2 + D - 2)y = (D-1)(D+2)y = 0$
- $\{e^x, e^{-2x}\}$

Linear DE with constant coefficients: exercise

- Prove that roots of a polynomial with real coefficients appear in complex conjugate pairs.
- Prove that each polynomial of odd degree has at least one real root.
- Prove that each polynomial can be written as a product of first and second order polynomials with real coefficient.
- Write these polynomials as multiplication of first and second dergree polynomials.
- $D^3+1, \quad D^3-1, \quad D^4+1, \quad D^4+2D^2+10, \quad D^3-D^2+D-1.$

L homogeneous second order DE with constant coefficients

- For a second order DE $L[y] = y'' + ay' + by = 0$ try solutions of the form $\phi(x) = e^{sx}$
- $L[e^{sx}] = p(s)e^{sx} \quad p(s) = s^2 + as + b$ is called characteristic polynomial of the DE.
- $p(s) = 0$ is the characteristic equation of the DE.
- $p(s) = 0 \rightarrow s = s_1, s_2$
- $s_1 \neq s_2 \quad \phi(x) = c_1 e^{s_1 x} + c_2 e^{s_2 x}$ including the case of complex conjugate roots.
- If $s_1 = a + bi$ then $s_2 = a - bi$. $\{e^{(a+bi)x}, e^{(a-bi)x}\}$ or $\{e^{ax}\cos bx, e^{ax}\sin bx\}$
- A homogeneous equation in x is said to have a double root, or repeated root, at a if is a factor of the equation. At the double root, the graph of the equation is tangent to the x-axis.
- $s_1 = s_2 \quad \frac{\partial}{\partial s} L[e^{sx}] = L[\frac{\partial}{\partial s} e^{sx}] = L[xe^{sx}]$
- $L[xe^{s_1 x}] = p'(s_1)e^{s_1 x} + p(s_1)xe^{s_1 x} = 0$
- $\phi(x) = (c_1 + c_2 x)e^{s_1 x}$

L homogeneous second order DE with constant coefficients

- E.g., $y'' + 2y' + 10y = 0, \quad y(0) = 1, y'(0) = 0$
- E.g., $y'' + 2y' + y = 0, \quad y(0) = 1, y'(0) = 0$

Higher order LHDE with constant coefficients

- $L[y] = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$
- $L[e^{sx}] = p(s)e^{sx}$ where $p(s) = s^n + a_1 s^{n-1} + \cdots + a_n$ is the characteristic equation of our DE.
- If $s_1, s_2, \cdots, s_j$ are roots of characteristic equation with multiplicities of $n_1, n_2, \cdots, n_j$ the fundamental set is as follows:
- $$\{e^{s_1 x}, xe^{s_1 x}, \cdots, x^{n_1-1}e^{s_1 x}, e^{s_2 x}, xe^{s_2 x}, \cdots, x^{n_2-1}e^{s_2 x}, \\ \cdots, e^{s_j x}, xe^{s_j x}, \cdots, x^{n_j-1}e^{s_j x}\}$$
- E.g., $y^{(6)} + 2y''' + y = 0 \rightarrow (D^3+1)^2 y = 0$
- $D^3(D-1)^2(D+1)^2 y = 0$

- Write a fundamental set for each of the following equations.
- $D^5 y = 0$
- $(D+2)^4 y = 0$
- $(D^2+4)(D-3)^2 y = 0$
- $(D^2+16)[(D-1)^2+6]^2 y = 0$
- $(D^2-1)^2(D^2+2D+2)^4 y = 0$

Finding private solutions: Variation of parameters

- $L[y] = y'' + p(x)y' + q(x)y = f(x)$ with $\{\phi_1, \phi_2\}$ as a fundamental set.
- Assume $\phi_p = u_1\phi_1 + u_2\phi_2$
- $\phi_p' = u_1'\phi_1 + u_2'\phi_2 + u_1\phi_1' + u_2\phi_2'$
- Assume $u_1'\phi_1 + u_2'\phi_2 = 0$. Thus $\phi_p' = u_1\phi_1' + u_2\phi_2'$.
- $\phi_p'' = u_1\phi_1'' + u_2\phi_2'' + u_1'\phi_1' + u_2'\phi_2'$.
- $L[\phi_p] = \phi_p'' + p(x)\phi_p' + q(x)\phi_p = u_1\phi_1'' + u_2\phi_2'' + u_1'\phi_1' + u_2'\phi_2' + p(x)(u_1\phi_1' + u_2\phi_2') + q(x)(u_1\phi_1 + u_2\phi_2) = u_1(-p\phi_1' - q\phi_1) + u_2(-p\phi_2' - q\phi_2) + u_1'\phi_1' + u_2'\phi_2' + p(x)(u_1\phi_1' + u_2\phi_2') + q(x)(u_1\phi_1 + u_2\phi_2) = f(x)$
- $u_1'\phi_1' + u_2'\phi_2' = f$
- $\begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$
- By Cramer's rule: $u_1' = \frac{-f(x)\phi_2(x)}{W(\phi_1,\phi_2)}$ $\quad u_2' = \frac{f(x)\phi_1(x)}{W(\phi_1,\phi_2)}$
- $u_1(x) = -\int_{x_0}^{x} \frac{f(s)\phi_2(s)}{W(\phi_1,\phi_2)(s)} ds, \quad u_2(x) = \int_{x_0}^{x} \frac{f(s)\phi_1(s)}{W(\phi_1,\phi_2)(s)} ds$
- Finaly, $\phi_p(x) = \int_{x_0}^{x} \frac{\phi_2(x)\phi_1(s) - \phi_1(x)\phi_2(s)}{W(\phi_1,\phi_2)(s)} f(s) ds$

Finding private solutions: Variation of parameters

- Suppose $\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}}{dt^{n-1}}y + \cdots + p_n(t)y = g(t)$
- Solve the corresponding homogeneous differential equation to get: $y_h(t) = C_1 y_1(t) + C_2 y_2(t) + ... + C_n y_n(t)$.
- Assume a particular solution to the nonhomogeneous differential equation is of the form: $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + ... + u_n(t)y_n(t)$.
- Solve the following system of equations for $u_1'(t), u_2'(t), \ldots, u_n'(t)$.

$$\begin{aligned}
u_1'(t)y_1(t) + u_2'(t)y_2(t) + ... + u_n'(t)y_n(t) &= 0 \\
u_1'(t)y_1'(t) + u_2'(t)y_2'(t) + ... + u_n'(t)y_n'(t) &= 0 \\
&\vdots \\
u_1'(t)y_1^{(n-1)}(t) + u_2'(t)y_2^{(n-1)}(t) + ... + u_n'(t)y_n^{(n-1)}(t) &= g(t)
\end{aligned}$$

Finding private solutions: Variation of parameters

- $y'' - 2y' + y = \frac{e^x}{1+x^2}$ where the fundamental set is $\{e^x, xe^x\}$
- $y''' + y' = \tan x$
- $y''' - y' + 2y = e^{-x} \sin x$
- $y'' + y = \frac{1}{\cos x}$
- $(D^2 + 10D - 12)y = \frac{(e^{2x}+1)^2}{e^{2x}}$
- $(4D^2 - 8D + 5)y = e^x \tan^2(x/2)$
- $y^{(4)} + y = g(t)$

Undetermined multipliers method for finding PS

- One can guess the general form of the private solution and substitute in the DE to find the undetermined multipliers in the general form.
- $y'' + y = 3x^2 + 4 \rightarrow (D^2 + 1)y = 3x^2 + 4$
- Note that $D^3(3x^2 + 4) = 0 \rightarrow D^3(D^2 + 1)y = 0$
- $y = c_1 + c_2x + c_3x^2 + c_4 \cos x + c_5 \sin x$
- Substituting y into original DE determines multiples except for $\cos x$ and $\sin x$ multiples as they are solutions of the corresponding homogeneous equation and cancel out.
- E.g., $y'' + 2y = e^x$

Undetermined multipliers method for finding PS

- $y''' + y' = \sin x$
- Since $(D^2 + 1)\sin x = 0, (D^2 + 1)(D^3 + D)y = 0$
- $(D - 2)^3 y = 3e^{2x}$
- Since $(D - 2)(3e^{2x}) = 0, (D - 2)^4 y = 0$. Thus $\phi_p(x) = cx^3 e^{2x}$
- The method of undetermined multiples has the following limitations.
- In $L[y] = f(x)$, L must contain only constant coefficients.
- f(x) must contain functions which satisfy a homogeneous linear DE with constant coefficient.

Undetermined multipliers method for finding PS

- If $f(x) = p_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \to \phi_p(x) = x^r(A_0x^n + A_1x^{n-1} + \cdots + A_n)$
- If
  $f(x) = p_n(x)e^{\alpha x} \to \phi_p(x) = x^r(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{\alpha x}$
- If $f(x) = p_n(x)e^{\alpha x}\sin\beta x$ or $f(x) = p_n(x)e^{\alpha x}\cos\beta x$ then $\phi_p(x) = x^r(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{\alpha x}\cos\beta x + x^r(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{\alpha x}\sin\beta x$
- $L[y] = y''' + y'' = 3x^3 - 1$
- $y'' + 4y = xe^x$
- $y'' - y = x^2e^x\sin x$
- If $L[y] = f_1(x) + f_2(x) + \cdots + f_k(x)$ and $L[\phi_{p1}] = f_1(x), L[\phi_{p2}] = f_2(x), \cdots, L[\phi_{pk}] = f_k(x)$ then by linearity of L,
  $L[\phi_{p_1} + \phi_{p_2} + \cdots + \phi_{p_k}] = f_1(x) + f_2(x) + \cdots + f_k(x)$

Undetermined multipliers method for finding PS

- $y'' + 4y = xe^x + x\sin 2x$
- $y''' + 3y'' = 2 + x^2$
- $y'' + 4y' + 4y = xe^{-x}$
- $y'' + 9y = 2x\sin 3x$
- $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 8y = e^{2t}(1 + \sin 2t)$

Euler differential equation

- nth order homogeneous Euler equation:
  $(x-x_0)^n y^{(n)} + a_1(x-x_0)^{n-1}y^{(n-1)} + \cdots + a_n y = 0$
- x_0 is the singularity of the Euler equation.
- Consider $L[y] = x^2y'' + axy' + by = 0, \quad x > 0$
- Impose the change of variable $t = \ln x$. $y' = \frac{1}{x}\frac{dy}{dt}$
  $y'' = \frac{d^2y}{dt^2}(\frac{dt}{dx})^2 + \frac{d^2t}{dx^2}\frac{dy}{dt} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$
- $\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0$
- Characteristic equation: $s^2 + (a-1)s + b = 0$
- Depending on Δ for the characteristic equation fundamental set is $\{e^{s_1t} = x^{s_1}, e^{s_2t} = x^{s_2}\}, \quad \{e^{s_1t} = x^{s_1}, te^{s_1t} = x^{s_1}\ln x\}, \quad \{x^\alpha\cos(\beta\ln x), x^\alpha\sin(\beta\ln x)\}$
- If we substitute x^s for y, $L[x^s] = [s^2 + (a-1)s + b]x^s = 0$

Euler differential equation

- The characteristic equation $p(s) = s^2 + (a-1)s + b = 0$. If $\Delta > 0 \rightarrow \phi(x) = c_1 x^{s_1} + c_2 x^{s_2}, x > 0$ where $x^{s_1} = e^{s_1 \ln x}$
- If $\Delta = 0$ we note that $\frac{\partial}{\partial s} L[x^s] = L[x^s \ln x] = p'(s)x^s + p(s)x^s \ln x$
- At $s = s_1, L[x^{s_1} \ln x] = p'(s_1)x^{s_1} + p(s_1)x^{s_1} \ln x = 0$. Thus $\phi(x) = c_1 x^{s_1} + c_2 x^{s_1} \ln x, x > 0$
- If $\Delta < 0 \rightarrow \phi(x) = e^{\alpha x}(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)), x > 0$
- For $x < 0$ we make the change of variable $\zeta = -x$. Euler equation become $\zeta^2 \frac{d^2y}{d\zeta^2} + a\zeta \frac{dy}{d\zeta} + by = 0$
-

$$\phi(\zeta) = \begin{cases} c_1\zeta^{s_1} + c_2\zeta^{s_2} & s_1 \neq s_2 \in \Re \\ c_1\zeta^{s_1} + c_2\zeta^{s_1} \ln \zeta & s_1 = s_2 \in \Re \\ c_1\zeta^{\alpha} \cos(\beta \ln \zeta) + c_2\zeta^{\alpha} \sin(\beta \ln \zeta) & s = \alpha \pm i\beta \end{cases}$$

Euler differential equation

- Combining solutions for $x > 0$ and $x < 0$.
$$\phi(|x|) = \begin{cases} c_1|x|^{s_1} + c_2|x|^{s_2} \\ c_1|x|^{s_1} + c_2|x|^{s_1}\ln|x| \\ c_1|x|^{\alpha}\cos(\beta\ln|x|) + c_2|x|^{\alpha}\sin(\beta\ln|x|) \end{cases}$$
- $x^2y'' + 2xy' + 2y = 0; y(1) = 0, y'(1) = 0$
- $x^2y - 5xy' + 13y = 0$
- $x^2y'' + 5xy' + 4y = 0$
- $x^2y'' - 3xy' + 4y = \ln x$
- $x^2y'' + 4xy' - 6y = 0$
- Order reduction technique:
$L[y] = x^2y'' + x^3y' - 2(1 + x^2)y = x$

Series

- $\sum_{k=o}^{\infty} a_k(x-a)^k = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$ where $a_k \in \mathbb{R}, k \in \mathbb{N}$
- The sequence $\{s_n(x)\}$ where $s_n(x) = \sum_{k=o}^{n} a_k(x-a)^k$ is a partial sum sequence for the above series.
- The above power series is convergent at point x_0 if the partial sum sequence $\{s_n(x)\}$ is convergent at point x_0. I.e., $\lim_{n\to\infty} s_n(x_0) = s(x_0)$
- $s(x_0)$ is the sum of the above series at point x_0.
- $\lim_{n\to\infty} \sum_{k=o}^{n} a_k(x_0-a)^k = \sum_{k=o}^{\infty} a_k(x_0-a)^k = s(x_0)$
- Set $a=0$, $\quad \sum_{k=o}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$. This is absolutely convergent iff $\sum |a_k x^k|$ is convergent.
- Convergence radius, convergence interval or region of convergence.
- $\lim_{n\to\infty} |\frac{a_{n+1}x^{n+1}}{a_n x^n}|$

Series

- Every power series defines a continuous differentiable function over its radius of convergence. $\sum_{k=0}^{\infty} a_k x^k = f(x)$
- $(\sum_{k=0}^{\infty} a_k x^k)(\sum_{k=0}^{\infty} b_k x^k) = \sum_{k=0}^{\infty} c_k x^k$ where $c_k = \sum_{m=0}^{k} a_{k-m} b_m = \sum_{m=0}^{k} b_{k-m} a_m$
- Uniqueness of the taylor series.
- Find the convergence interval for $\sum_{n=0}^{\infty} \frac{2^n}{n+1} x^n$ and $\sum_{n=1}^{\infty} \frac{(x+1)^n}{2^n n}$
- $\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{(1-x)} = \sum_{n=1}^{\infty} n x^{n-1}$
- Linear indepence of power series starting from different powers of x.
- If p(x) and q(x) are analytic around x_0 then $y'' + p(x)y' + q(x)y = 0$ has analytic solution around the point x_0.
- E.g., Determine a series solution for the following differential equation about $x_0 = 0$, $y'' + xy' + y = 0$.
- $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$

Series

- $\sum_{k=0}^{\infty}(k+2)(k+1)a_k x^k + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0$
- $\phi(x) = a_0[1 + \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k}}{(2k)(2k-2)\cdots(4)(2)}] + a_1[x + \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)(2k-1)\cdots(5)(3)}]$
- Legendre differential equation, $(1-x^2)y'' - 2xy' + \lambda(\lambda+1)y = 0$
- Solution would converge on the interval (-1,1).
- $\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} + (\lambda - k)(\lambda + k + 1)a_k]x^k = 0$
- For natural values of λ one of the solutions would be a polynomial. These are Legendre polynomials.
- If p(x) and q(x) are analytic around x_0 then $y'' + p(x)y' + q(x)y = f(x)$ has solution $\phi(x)$ such that $\phi(x_0) = a$ and $\phi'(x_0) = b$, Taylor series of the solution would have a convergence radius greater than the smallest of the convergence radius of p, q and f at x_0.

Numeric solution to a differential equation

- Start by substituting Taylor series of p and q in the corresponding homogenous equation. To derive $\phi_h(x) = a_0 + a_1 x + \sum_{k=2}^{\infty} (\alpha_k a_0 + \beta_k a_1) x^k$
- Lemma: If $\sum c_k x^k$ has convergence radius $R^* > 0 \quad \forall r < R^* \quad \exists M : |c_k| r^k \leq M$
- Numerically Solve the equation $\frac{dy(t)}{dt} = -\lambda y(t)$ and compare the resulting solution to exact solution.