Convergent vs Asymptotic Series

Example of convergent series

$$\frac{\pi}{4} = 1 - 1/3 + 1/5 + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \qquad \boxed{x \leq 1}$$

$$f(x) = \sum_{n}^{\infty} a_n g_n(x) \qquad n \to \infty \quad \text{better \& better approx.}$$

More term ≡ better approx. ←

$$\text{or } f(x) = \sum_{n=0}^{M} a_n g_n(x) + R$$

$$\lim_{\boxed{M \to \infty}} \frac{R}{\sum_{n=0}^{M} a_n g_n(x)} = 0$$

Often, there's a radius of convergence.

Within the window of " " ", the accuracy improves by keeping more terms, and this is for any value of x.

Asymptotic Expansion

In contrast to convergent series, for an asymptotic expansion, we truncate the series at some point.

The accuracy is improved by reducing the approx. parameter.

$$f(x) = \sum_{n} \alpha_n g_n(x) \qquad g_{n+1}(x) < o(g_n(x)) \qquad x \to \text{point we are perturbing.}$$

Grow slower than $g_n(x)$

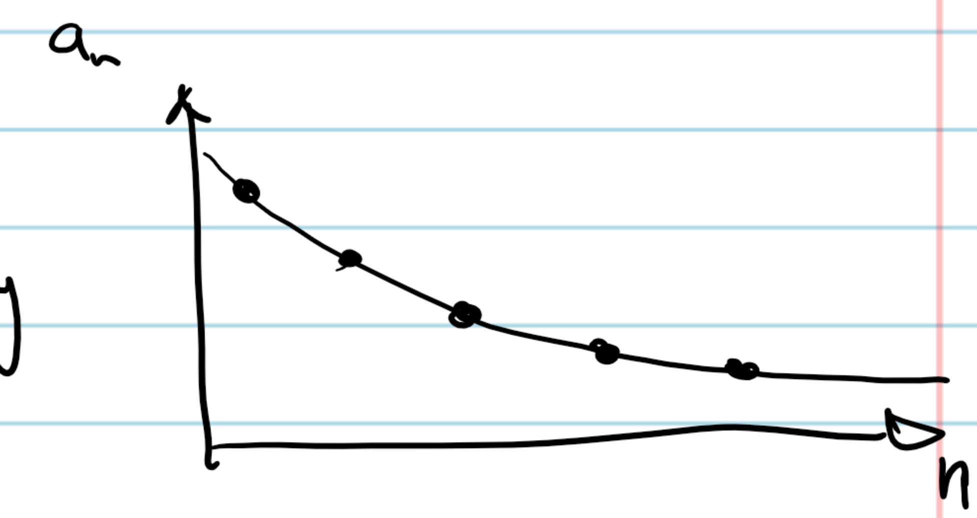
This approximation has fixed number of terms, but it improves if $x \to 0$.

The example of $\tan^{-1}$ is both. $\tan^{-1}(x) \simeq x - \frac{x^3}{3} + \frac{x^5}{5}$

Distinction

For a convergent series, there's a radius of convergence, and within that radius, it converges for any value of the parameter:

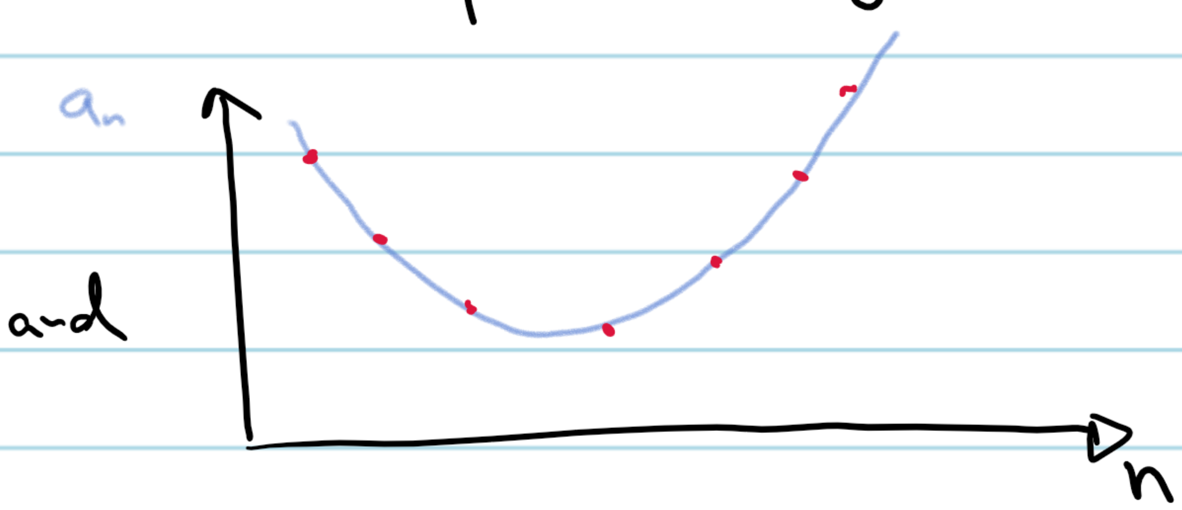
$$\lim_{M \to \infty} \sum_{n=0}^{M} a_n g_n(x) = f(x) \quad : \quad \text{intuitively}$$



Terms get smaller & smaller.

But for an asymptotic series, the radius of convergence may be zero, and $a g_n(x)$ may start to explode beyond some n

(it is often the case that the terms reduce to some point and then start to grow again.



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Perturbation Series are asymptotic series.

So, we use them even when the radius of conv. is zero.

Also, it is important that they are only valid for small enough perturbations.

Example: Anharmonic Oscillator

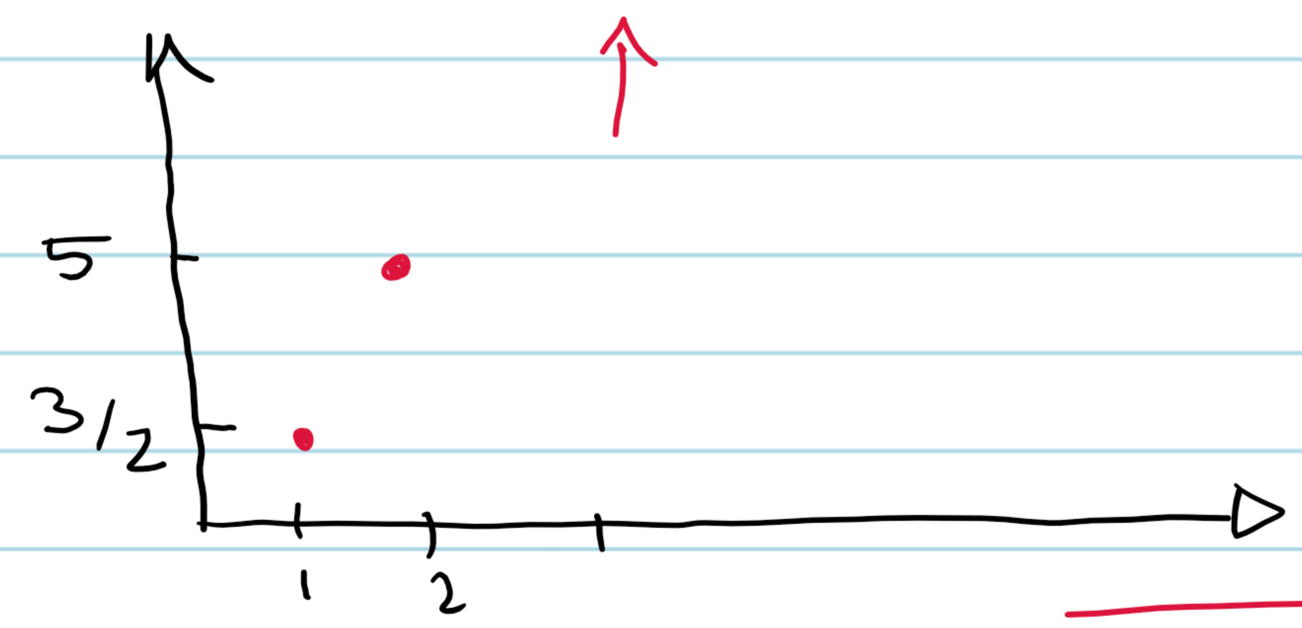
For the Anharmonic Oscillator, the first few terms in the expasion of E_0 are as follows:

Bender & Wu

$$E_0 = \hbar\omega/2 \left(1 + 3/2\,\lambda - \frac{21}{4}\lambda^2 + \frac{333}{8}\lambda^3 - \frac{30855}{64}\lambda^4 + \right.$$

Citation

The plot of the coefficients would show that:



→ For M>> this would go like $(-1)^n 3^n \Gamma(n+1/2) \sim n!\,3^n$

But the expasion is useful b/c we truncate it for a finite M & for that, we can make λ small enough, i.e.

$$\lambda \ll \frac{1}{n!\,3^n}$$

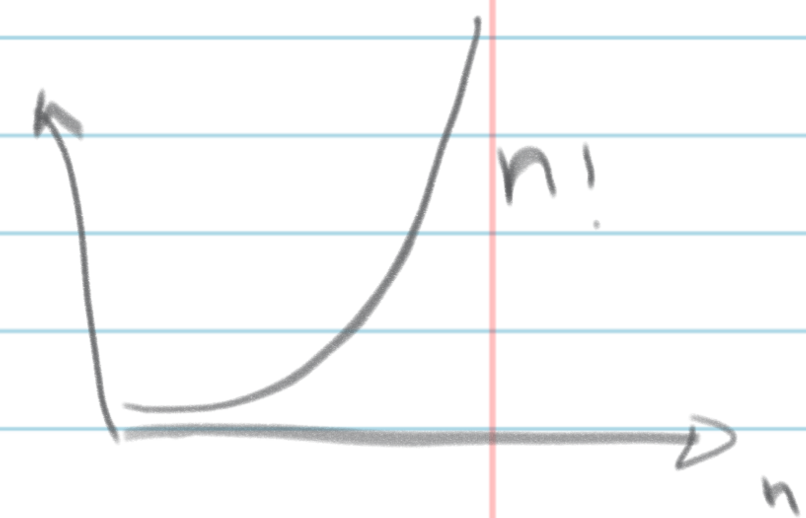
such that terms beyond M+1 have a small contribution and the sum gives a close approx of the function.

Example:

$$\int_0^\infty \frac{1}{(1+t)} e^{-t/\varepsilon} \, dt$$

$$\sim \int_0^\infty \left(1 - t + t^2 - t^3 + t^4 \dots\right) e^{-t/\varepsilon} \, dt \quad \leftarrow \text{only } t \leq 1$$

$$\sim \varepsilon - \varepsilon^2 + \boxed{2}\,\varepsilon^3 - \boxed{3!}\,\varepsilon^4 + \boxed{4!}\,\varepsilon^5 - \dots$$



The sum diverges.

$\varepsilon \to 0$

$$g_1 = \varepsilon, \quad g_2 = -\varepsilon^2, \quad g_3 = 2!\,\varepsilon^3 \dots \quad g_n(\varepsilon) = (n-1)!\,\varepsilon^n$$

$$\lim_{\varepsilon \to 0} \frac{g_{n+1}}{g_n} = 0 \quad \Rightarrow \quad g_{n+1} \in O(g_n(\varepsilon)) \quad \text{for } \varepsilon \to 0$$

So, if we truncate the series at a finite point, we can get close to f by $\varepsilon \to 0$.

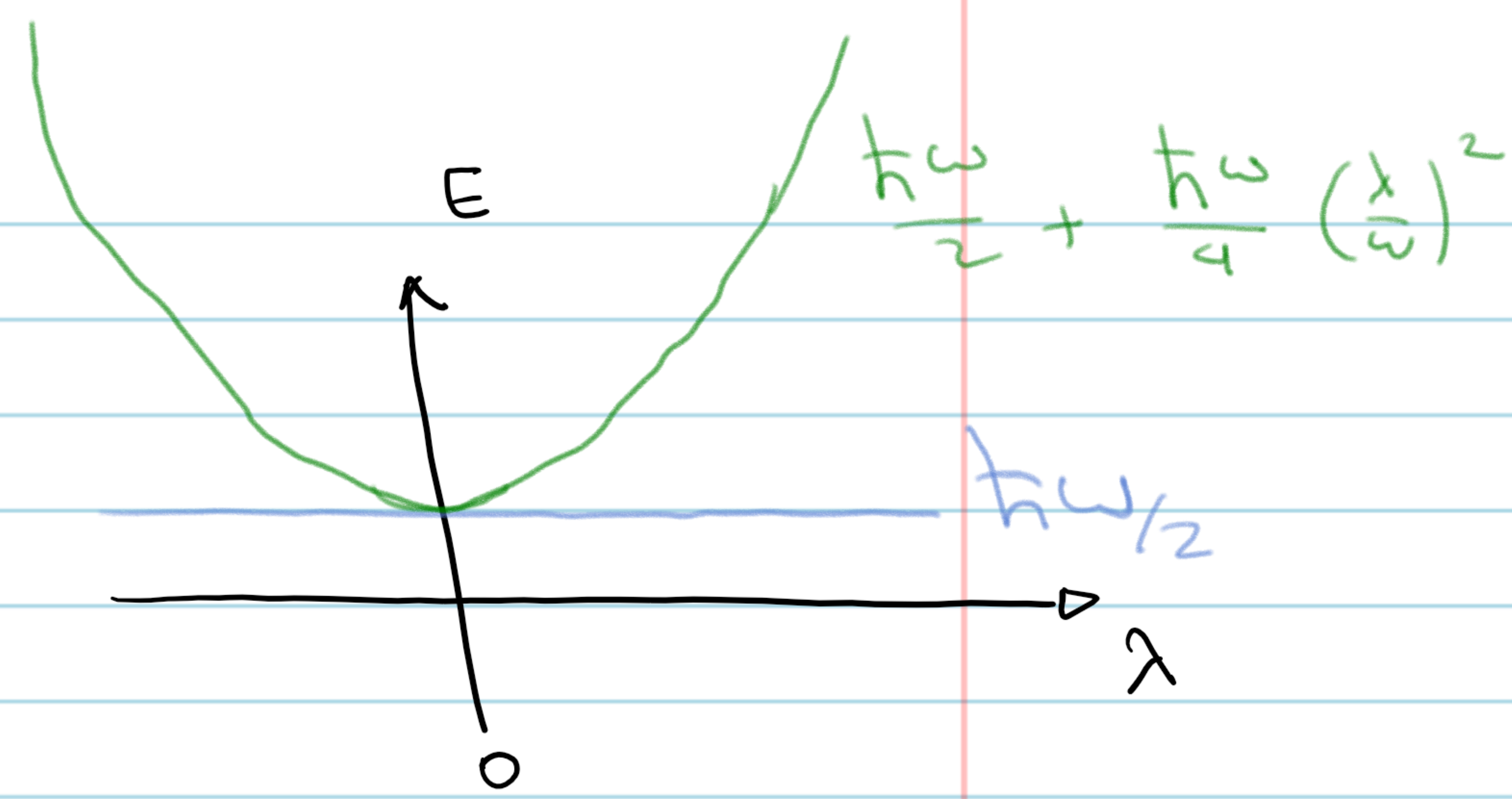
Note: g_n are not always polynomials.

Assignment: Find the solutions of $\left(\frac{d^2}{dx^2} + a(x)\right) y(0) = 0$ with perturbation (take $x \in [a, b]$ i.e. x is finite).
→ Note that this is basically the Schrödinger's eq.)

Assignment: $\left(\varepsilon \frac{d^2}{dx^2} + 2\frac{d}{dx} + 2\right) y(x) = 0$; $x \in [0,1]$, $y(0) = 0$, $y(1) = 1$

ε small

Let's look at a simple example:



$$H = \underbrace{\frac{\hbar\omega}{2}\sigma_z}_{H_0} + \frac{\hbar\lambda}{2}\underbrace{\sigma_x}_{W}$$

In this case the exact eigenenergies are

$$H = \vec{n}\cdot\vec{\sigma} : n = \left(\frac{\hbar\lambda}{2}, 0, \frac{\hbar\omega}{2}\right)$$

$$E_\pm = \pm\sqrt{\left(\frac{\hbar\lambda}{2}\right)^2 + \left(\frac{\hbar\omega}{2}\right)^2} = \pm\frac{\hbar}{2}\sqrt{\lambda^2+\omega^2} = \pm\frac{\hbar}{2}\omega\sqrt{1+\underbrace{\left(\frac{\lambda}{\omega}\right)^2}_{x}}$$

$$E_\pm = \pm\frac{\hbar\omega}{2}\left(1+\frac{1}{2}\left(\frac{\lambda}{\omega}\right)^2 - \frac{1}{8}\left(\frac{\lambda}{\omega}\right)^4 + \frac{1}{16}\left(\frac{\lambda}{\omega}\right)^6 + \cdots\right)$$

⇝ This gives both a convergent series (for $\left(\frac{\lambda}{\omega}\right) < 1$) and an asymptotic expansion.

Note that $\left(\frac{\lambda}{\omega}\right) \ll 1 \Rightarrow \boxed{\lambda \ll \omega}$ This gives a notion of being small.

The perturbation should be small compared to the gap.

<u>Note that</u> if $\lambda \simeq \omega$, then it is not clear what H_0 is & what the perturbation is.

<u>Plot:</u> ⇝ <u>small ε</u>

<u>Plot $\chi(x)$</u>