Harmonic Oscillator

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QM I _ Fall 2019

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$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}^2$$

There are at least two ways to solve this:

① Solving the Sch eq.

- Guess $\psi(x) = f(x)\, e^{-x^2/2\sigma^2}$
- Take $f(x) = \sum_{n=0} c_n x_n$
- Terminate $f(x)$ at some n and this would give different solutions.

② Algebraic method: Finding the eigensystem of $\hat{H}$

To begin with, we introduce a & $a^\dagger$ such that:

$$\hat{X} = X_0 \left(\frac{a+a^\dagger}{\sqrt{2}}\right) \quad , \quad \hat{P} = P_0 \left(\frac{a-a^\dagger}{\sqrt{2}\, i}\right)$$

$$X_0 = \sqrt{\frac{m\omega}{\hbar}} \qquad\qquad P_0 = \sqrt{m\hbar\omega}$$

$$\Longrightarrow \hat{H} = \frac{\hbar\omega}{2}\left[\frac{-1}{2}(a-a^\dagger)^2 + \frac{1}{2}(a+a^\dagger)^2\right]$$

$$= \frac{\hbar\omega}{2}\left[aa^\dagger + a^\dagger a\right]$$

Now, what is $aa^\dagger + a^\dagger a$? We need to calculate $[a, a^\dagger]$:

$$\rightarrow a = \frac{1}{\sqrt{2}}\left(\frac{\hat{X}}{X_\circ} + i\frac{\hat{P}}{P_\circ}\right) \quad , \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\hat{X}}{X_\circ} - i\frac{\hat{P}}{P_\circ}\right)$$

$$[a, a^\dagger] = \frac{i}{2X_\circ P_\circ}\left(-\overbrace{[\hat{X}, \hat{P}]}^{i\hbar \mathbb{1}} + \overbrace{[\hat{P}, \hat{X}]}^{-i\hbar \mathbb{1}}\right)$$

$$= \frac{\hbar}{X_\circ P_\circ} = 1$$

$$\Rightarrow \hat{a}\hat{a}^\dagger = \mathbb{1} + \hat{a}^\dagger \hat{a}$$

$$\Rightarrow \hat{H} = \frac{\hbar\omega}{2}\left[2\hat{a}^\dagger\hat{a} + \mathbb{1}\right] = \hbar\omega\left[\hat{N} + \mathbb{1}/2\right]$$

$$\hat{N} = \hat{a}^\dagger \hat{a} \longrightarrow \text{Number operator}$$

Eigen system of $\hat{H}$

$\hat{N} = a^\dagger a$ is Hermitian.

So there is a basis $\{|n\rangle\}$. $\hat{N}|n\rangle = |n\rangle$

Here we want to find $|n\rangle$ & n.

We start with $|\phi\rangle = a|n\rangle$

$\hat{N}|\phi\rangle = \;?$

$$[\hat{N}, a] = [a^\dagger a, a] = [a^\dagger, a]\, a = -\hat{a}$$

$$\hat{N}\hat{a} = \hat{a}\hat{N} - \hat{a}$$

$$\hat{N}|\phi\rangle = \hat{N}\hat{a}|n\rangle = \hat{a}(\hat{N}-1)|n\rangle =$$

$$(n-1)\,\hat{a}|n\rangle = (n-1)|\phi\rangle$$

$|\phi\rangle$ is also an eigen state.

a is a lowering operator $\quad N(a|n\rangle) = (n-1)(a|n\rangle)$

Similarly we can show that $a^\dagger$ is a rising operator.

$$N(a^\dagger|n\rangle) = (n+1)(a^\dagger|n\rangle)$$

So

$$a|n\rangle = c_n|n-1\rangle$$ What if N is degenerate?

$$a^\dagger|n\rangle = d_n|n+1\rangle$$

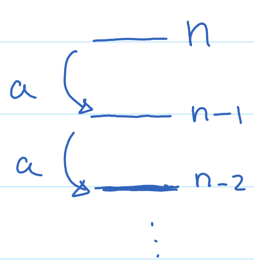
We will

* Find C_n
* Show that n has to be an integer
* It starts from $n=0$

Take $|\phi\rangle$ and calculate $\langle\phi|\phi\rangle =$

$$\langle n|\underbrace{a^\dagger a}_{\hat{N}}|n\rangle = |C_n|^2 = n$$

$$\begin{cases} (*) \rightarrow |C_n| = \sqrt{n} \rightarrow \text{Similarly find } d_n. \\ * \quad n \geq 0 \end{cases}$$



→ Eventually n becomes negative.

$\boxed{?}$

But note that $a|n=0\rangle = 0$, So if n is an integer, somewhere in the chain $(a^{n+1}|n\rangle = 0)$ and $\langle\hat{N}\rangle$ stays non-negative

But if n is not an integer, for $|\phi\rangle = a^{n+1}|n\rangle \rightarrow \langle N\rangle$ becomes negative which is not possible.

So n has to be $\in \mathbb{Z} = \{0, 1, \ldots\}$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$$

$$\Longrightarrow \quad E_n = \hbar\omega \left(n + 1/2\right) \quad . \quad n = 0, 1, \ldots$$

$\ast$ $E_0 > 0$. Ground-state energy is non-zero.

$\ast$ $E_{n+1} - E_n = \hbar\omega$

Next we calculate the $\langle x|n\rangle = \psi_n(x)$

$\psi_n(x)$

Now, we need to find the $\psi_n(x)$.
Note that, so far we have the eigen states,
but have no way to calculate the probability
distributions like $P_r(x)$.

$$\psi_n(x) = \langle x|n\rangle \quad ?$$

Let's take $\hat{a}|0\rangle = 0$ and project to position space.

$$a\psi_0(x) = \langle x|\hat{a}|0\rangle = 0$$

Also, $\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{X}}{X_0} + i\,\frac{\hat{P}}{P_0}\right)$

$$\Rightarrow a = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} + i\left(-i\boxed{\frac{\hbar}{p_0}}\frac{\partial}{\partial x}\right)\right) = \frac{1}{\sqrt{2}\,x_0}\left(x + x_0^2\frac{\partial}{\partial x}\right)$$

x_0

$$\Rightarrow \left(x + x_0^2\frac{\partial}{\partial x}\right)\psi_0(x) = 0 \Rightarrow \frac{\partial}{\partial x}\psi_0(x) = -\frac{x}{x_0^2}\psi_0(x)$$

$$\Rightarrow \psi_0(x) = A\, e^{-\frac{x^2}{2x_0^2}}$$

Ⓐ Calculate the normalization factor of $\psi_0(x)$ and show that

$$A = \frac{1}{\sqrt{\sqrt{\pi}\, x_0}}$$

Now, to calculate $\psi_n(x)$ we use the fact that

$$|n\rangle = \frac{\hat{a}^{\dagger n}}{\sqrt{n!}}|0\rangle \qquad n > 0$$

$$\Rightarrow \psi_n(x) = \langle x|n\rangle = \frac{1}{\sqrt{n!}}\langle x|\hat{a}^{\dagger n}|0\rangle =$$

$$\frac{1}{\sqrt{n!}}\, a^{\dagger n}\psi_0(x)$$

Ⓐ Show that $a^\dagger \psi(x) = \frac{1}{\sqrt{2}\,x_0}\left(x - x_0^2\frac{\partial}{\partial x}\right)$

$$\text{So} \quad \psi_n(x) = \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}\,x_0}\right)^n\left(x - x_0^2\frac{\partial}{\partial x}\right)^n\psi_0(x)$$

Example:

$$\psi_1(x) = \frac{A}{\sqrt{1!}} \frac{1}{\sqrt{2}\, x_0} \left(x - x_0^2 \frac{\partial}{\partial x} \right) e^{-\frac{x^2}{2x_0^2}}$$

$$= \frac{A}{\sqrt{2}\, x_0} (x + x)\, e^{-\frac{x^2}{2x_0^2}} = \sqrt{\frac{2}{\sqrt{\pi}\, x_0^3}}\; x\, e^{-\frac{x^2}{2x_0^2}}$$

We can use the Hermite functions $H_n(y)$

$$H_n(y) = (-1)^n\, e^{y^2} \frac{d^n}{dx^n} e^{-y^2}$$

to express $\psi_n(x)$ as

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}\, 2^n\, n!\, x_0}}\, e^{-x^2/2x_0^2}\, H_n\left(\frac{x}{x_0}\right)$$

Remarks

* $\psi_{2n}(x)$ are even & $\psi_{2n+1}(x)$ are odd functions.

* $\langle x \rangle$ & $\langle p \rangle$ on the eigenstates are zero

(A) Calculate $\langle n | \hat{x} | n \rangle$ & $\langle n | \hat{p} | n \rangle$

Ⓐ " $\langle n | \hat{X}^2 | n \rangle$ & $\langle n | \hat{P}^2 | n \rangle$.

Ⓐ " ΔX & ΔP for $|n\rangle$.

Ⓐ Find the eigensystem of $H = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}^2 + \underline{\xi \hat{X}}$.

Ⓐ For a HO with $f = 10$ GHz, find the X_0 & P_0 & $\langle \Delta X \rangle$ & $\langle \Delta P \rangle$. Take the mass to be 10 ng.

Coherent States

From classical HO, our expectation is that

$$\langle \hat{X}(t) \rangle = A \cos(\omega t)$$
$$\langle \hat{P}(t) \rangle = B \sin(\omega t)$$

Ⓐ Find $\hat{X}(t)$ & $\hat{P}(t)$ is the Heisenberg picture.

Ⓐ Calculate $\langle \hat{X}(t) \rangle$ & $\langle \hat{P}(t) \rangle$ for the number states

Next we'll find a set of states that

resemble the classical behavior.

Let's find the eigen states of $\hat{a}$:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Take $|\alpha\rangle = \sum_n c_n |n\rangle$

$$\hat{a}|\alpha\rangle = \sum_n c_n \hat{a}|n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}\,|n-1\rangle$$

$$= \alpha \sum_{n=0}^{\infty} c_n |n\rangle$$

$$\Longrightarrow \quad c_{n+1}\sqrt{n+1} = \alpha\, c_n$$

$$c_{n+1} = \frac{\alpha c_n}{\sqrt{n+1}} = \frac{\alpha^2}{\sqrt{(n+1)(n)}}\, c_{n-1} = \cdots$$

$$= \frac{\alpha^{n+1}}{\sqrt{(n+1)!}}\, c_0 \longrightarrow \boxed{c_0}$$

(A) Calculate the normalization factor of coherent state, c_0.

$$\Longrightarrow |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Ⓐ Calculate $\langle \alpha_1 | \alpha_2 \rangle$ for α_1 & α_2 two coherent states.

For this state we have.

$$\langle X \rangle = \frac{X_0}{\sqrt{2}} \langle a + a^\dagger \rangle = \frac{X_0}{\sqrt{2}} \langle \alpha | a | \alpha \rangle + \frac{X_0}{\sqrt{2}} \langle \alpha | a^\dagger | \alpha \rangle$$

$$= \sqrt{2} X_0 \, Re(\alpha)$$

$$\langle P \rangle = \sqrt{2} \, P_0 \, Im(\alpha)$$

This suggests that we can rewrite α as

$$\alpha = \frac{1}{\sqrt{2}} \left(\langle X \rangle + i \langle P \rangle \right)$$

Next we want to calculate the time evolution of $\langle \hat{X} \rangle$ & $\langle \hat{P} \rangle$. We can use the $\hat{X}(t)$ & $\hat{P}(t)$ in the Heisenberg picture or directly calculate the

$$|\alpha(t)\rangle = e^{-i\frac{Ht}{\hbar}} |\alpha\rangle =$$

$$e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\frac{Ht}{\hbar}} |n\rangle$$

$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i(n+1/2)\omega t} |n\rangle$$

$$(\alpha e^{-i\omega t})^n e^{-i\frac{\omega t}{2}}$$

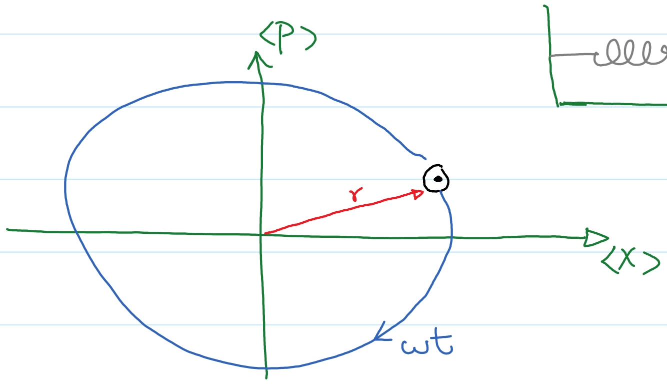
$$= e^{-\frac{i\omega t}{2}} e^{-|\alpha|^2/2} \sum^{\infty} \frac{(\alpha e^{-i\omega t})^n}{} |n\rangle$$

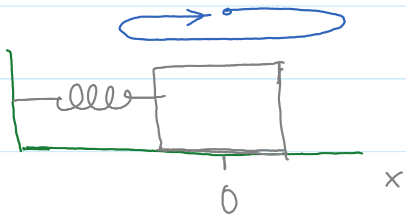
$$= e^{\frac{-i\omega t}{2}} e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle$$

$$= |\alpha e^{i\omega t}\rangle$$

$$\Rightarrow \langle X \rangle = \sqrt{2}\, X_0\, \mathrm{Re}(\alpha e^{-i\omega t})$$

$$\langle P \rangle = \sqrt{2}\, P_0\, \mathrm{Im}(\alpha e^{-i\omega t})$$





For real α we'll have:

$$\langle X \rangle = \sqrt{2}\, X_0 \cos(\omega t)$$

$$\langle P \rangle = \sqrt{2}\, P_0 \sin(\omega t)$$

(A) Calculate $\langle \hat{N} \rangle$ for $|\alpha\rangle$.

(A) What does r, the distance from the centre depends on?

How does it depend on energy?

(A) Calculate $\langle \Delta X(t) \rangle$ & $\langle \Delta P(t) \rangle$ for $|\alpha\rangle$.

(A) Find the $\psi_\alpha(x) = \langle x | \alpha \rangle$.