Lecture 10: Angular Momentum

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QM I _ Fall 2019
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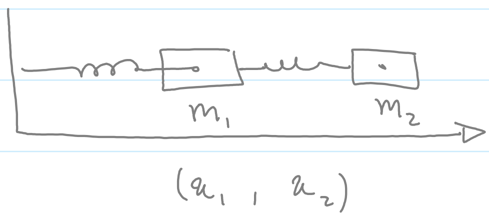
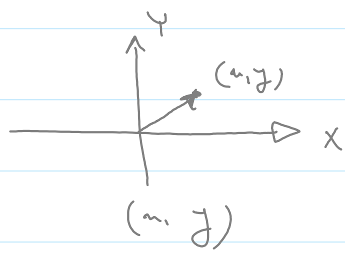
- ☐ Tesor product & Composite Hilbert spaces
- ☐ Angular Momentum operator

Consider a system with 2 degrees of freedom (DOF)





For each DOF, we use a vector in a Hilbert space, how about the full system ?

We use tensor product.

$$\mathcal{H}_{tot} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

Also, for $|v\rangle \in \mathcal{H}_1$ & $|w\rangle \in \mathcal{H}_2$, we

describe the system with

$$|\psi\rangle_{tot} = |v\rangle \otimes |w\rangle \in \mathcal{H}_{tot}$$

Finally, for operators $A \in \mathcal{L}(\mathcal{H}_1)$ & $B \in \mathcal{L}(\mathcal{H}_2)$

we have $A \otimes B \in \mathcal{L}(\mathcal{H}_{tot})$

Example: Take two electrons with states

$|\psi_1\rangle \in \mathcal{H}_1$ & $|\psi_2\rangle \in \mathcal{H}_2$.

The evolution of the electrons are given by $U_1 \in \mathcal{L}(\mathcal{H}_1)$ & $U_2 \in \mathcal{L}(\mathcal{H}_2)$

So the full system after the evolution is given by

$$|\Phi\rangle = (U_1 \otimes U_2)(|\psi_1\rangle \otimes |\psi_2\rangle)$$

$$= (U_1|\psi_1\rangle) \otimes (U_2|\psi_2\rangle).$$

Example: A particle confined in 2D is described by $|\psi\rangle = |\tilde{x}\rangle \otimes |\tilde{y}\rangle$.

A measurement of position is given by

$$\vec{r} = \hat{X} \otimes \hat{Y} = \left(\int dx \; x |x\rangle\langle x|\right)\left(\int dy \; y |y\rangle\langle y|\right)$$

$$Pr(\vec{r} = (x,y)) = \langle\psi| \Big[|x\rangle\langle x| \otimes |y\rangle\langle y| \Big] |\psi\rangle$$

$$= |\langle\tilde{x}|x\rangle|^2 \, |\langle\tilde{y}|y\rangle|^2 .$$

(A) Can we write all the states in $\mathcal{H}_{tot} = \mathcal{H}_1 \otimes \mathcal{H}_2$

as some $|\psi_1\rangle \otimes |\psi_2\rangle$?

Angular Momentum

In 3D, we have 3DOF

$$\vec{R} = (\hat{X}, \hat{Y}, \hat{Z})$$
$$\hat{P} = (\hat{P}_x, \hat{P}_y, \hat{P}_z)$$

with

$$[\hat{X}, \hat{P}_x] = [\hat{Y}, \hat{P}_y] = [\hat{Z}, \hat{P}_z] = i\hbar \mathbb{1}$$
$$[X, Y] = \cdots = [X, P_y] = \cdots = 0$$

In this chapter, we want to investigate systems in 3D, but with rotational symmetry.

$$V(\hat{X}) \xrightarrow{3D} V(\hat{R}) \xrightarrow{sym} V(|\hat{R}|)$$

As we know from classical mechanics, rotations are generated by angular momentum.

Similar to classical mechanics, we can define

$$\vec{L} = \vec{R} \times \vec{P} = \begin{vmatrix} \hat{X} & \hat{Y} & \hat{Z} \\ \hat{P}_x & \hat{P}_y & \hat{P}_z \\ \hat{i} & \hat{j} & \hat{k} \end{vmatrix} = (\hat{X}\hat{P}_y - \hat{Y}\hat{P}_x)\hat{k} + \cdots$$

$$= \hat{L} \quad \hat{\imath} \quad + \hat{L} \quad \hat{\jmath} \quad \hat{L} \quad \hat{k}$$

$$= \hat{L}_x \hat{i} + \hat{L}_y \hat{j} + \hat{L}_z \hat{k}$$

What can we say about $[L_i, L_j]$?

$$[L_x, L_y] = [\hat{Y}\hat{P}_z - \hat{P}_y Z, \hat{P}_x \hat{Z} - \hat{X}\hat{P}_z] =$$

$$= \hat{P}_x \hat{Y} (-i\hbar) + \hat{X}\hat{P}_y (i\hbar) = i\hbar \hat{L}_z$$

Ⓐ Show that $[L_i, L_j] = i\hbar\, \varepsilon_{ijk} L_k \quad i,j,k \in \{x,y,z\}$

Rotations

How do you guess the rotations are specified ?

$$R_x(\theta) = e^{i\frac{\theta L_x}{\hbar}}$$

$$R_y(\theta) = e^{i\frac{\theta L_y}{\hbar}} \qquad R_{\vec{n}}(\theta) = e^{i\theta\frac{\vec{n}\cdot\vec{L}}{\hbar}}$$

$$R_z(\theta) = e^{i\frac{\theta L_z}{\hbar}} \qquad \vec{n} = (n_x, n_y, n_z)$$

→ We'll come back to this later in QM 2.

What does it mean to be symmetric under

rotations?

$$[H, L_z] = [H, L_x] = [H, L_y] = 0$$

$i \neq j$ $\quad [L_i, L_j] \neq 0 \Rightarrow$ So, whats a CSCO for this system?

$\{H, \hat{L}^2, L_z\}$

! It still may be incomplete, but that's for later.

Generalization of angular momentum

Here we consider a general setting in which

$$[J_i, J_j] = i \epsilon_{ijk} J_k .$$

The point is that $\vec{J}$ does not have to be $\vec{R} \times \vec{P}$.

One example is the Pauli algebra.

Check that $J_i = \hbar \sigma_i$ would satisfy the commutation relation above.

(A) Show that the commutation relation above is equivalent to

$\vec{J} \times \vec{J} \quad = i \vec{J}$

above is equivalent to

$$\vec{J} \times \vec{J} = i\hbar \vec{J}.$$

Ⓐ Check that $[\vec{J}^2, J_i] = 0$ for $i = x, y, z$

& $\vec{J}^2 = \vec{J} \cdot \vec{J}$.

Eigen states of $\vec{J}^2$ & $\vec{J}_z$

Next we'll find the eigenstates of $\vec{J}^2$ & J_z:

$$\vec{J}^2 |\alpha, \beta\rangle = \hbar^2 \alpha |\alpha, \beta\rangle$$

$$J_z |\alpha, \beta\rangle = \hbar\beta |\alpha, \beta\rangle .$$

Part 1

To this end, we define

$$J_\pm = J_x \pm i J_y \Rightarrow \begin{cases} J_x = \frac{1}{2}(J_+ + J_-) \\ J_y = \frac{1}{2i}(J_+ - J_-) \end{cases}$$

Ⓐ Show that:

$$[\vec{J}^2, J_\pm] = 0 \qquad (1)$$

$$[J_z, J_\pm] = \pm\hbar J_\pm \qquad (2)$$

$$[J_+, J_-] = 2\hbar J_z$$

Now we apply $J_\pm |\alpha,\beta\rangle$.

First note that

$$\vec{J}^2 (J_\pm |\alpha,\beta\rangle) = (J_\pm) \vec{J}^2 |\alpha,\beta\rangle = \alpha\hbar^2 (J_\pm |\alpha,\beta\rangle)$$

So α (or the total angular momentum) does not change.
How about $\vec{J}_z$?

$$\vec{J}_z (J_\pm |\alpha,\beta\rangle) \overset{(2)}{=} (J_\pm J_z \pm \hbar J_\pm) |\alpha,\beta\rangle$$

$$= (\hbar\beta J_\pm \pm \hbar J_\pm) |\alpha,\beta\rangle = (\hbar\beta \pm \hbar)(J_\pm |\alpha,\beta\rangle)$$

$$\boxed{\Rightarrow J_\pm |\alpha,\beta\rangle = C^\pm_{\alpha,\beta} |\alpha, \beta \pm 1\rangle}$$

$$|C^\pm_{\alpha,\beta}|^2 = \langle \alpha,\beta | J_\mp J_\pm | \alpha,\beta\rangle$$

Ⓐ Check that $J_+ J_- = J_x^2 + J_y^2 + \hbar J_z$

$$J_- J_+ = J_x^2 + J_y^2 - \hbar J_z .$$

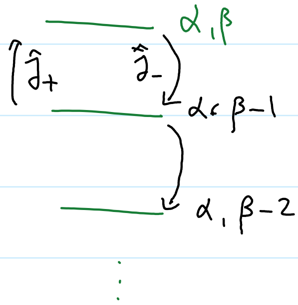
This gives

$$|C^\pm_{\alpha,\beta}| = \langle \alpha,\beta | \vec{J}^2 - J_z^2 \pm \hbar J_z | \alpha,\beta\rangle$$

$$= \left(\alpha^2 - \beta^2 \pm \beta\right)\hbar^2$$

Part 2 : limits of α, β

How far can this ladder go?



First, note that $\vec{J}^2 - J_z^2$ is a positive operator.

$$\vec{J}^2 - J_z^2 = J_x^2 + J_y^2 \longrightarrow \text{This indicates that}$$

$$\alpha \geq \beta^2$$

$\longrightarrow$ There should be a β_{max} & β_{min} such that.

$$J_+ |\alpha, \beta_{max}\rangle = 0 \quad ,$$

$$J_- |\alpha, \beta_{min}\rangle = 0$$

$$J_- \left(J_+ |\alpha, \beta_{max}\rangle = 0 \right) \Rightarrow$$

From the last assignment, we have

$$J_- J_+ = J_x^2 + J_y^2 - \hbar J_z = \vec{J}^2 - J_z^2 - \hbar J_z$$

Which gives

$$\hbar^2 \left(\alpha - \beta_{max}^2 - \beta_{max} \right) = 0$$

$$\Rightarrow \quad \alpha = \beta_{max}(\beta_{max}+1) \qquad (*1)$$

Similarly:

$$J_+\left(J_- |\alpha, \beta_{min}\rangle = 0\right) \quad \& \quad J_+J_- = \vec{J}^2 - J_z^2 + \hbar J_z$$

$$\Rightarrow \quad \hbar^2(\alpha - \beta_{min}^2 + \beta_{min}) = 0 \Rightarrow \beta_{min}(\beta_{min}-1) = \alpha \qquad (*2)$$

Also note that $\beta_{max} = \beta_{min} + n$ where n is some integer. This comes from $J_-^n |\alpha, \beta_{max}\rangle \propto |\alpha, \beta_{min}\rangle$.

Solving these eqs we get.

$$\boxed{\beta_{max} = -\beta_{min} \quad , \quad \beta_{max} = \frac{n}{2}}$$

We define $j = \beta_{max}$ and have:

$$\alpha = j(j+1)$$

We also change the notation for the state.

$$|\alpha, \beta\rangle \longrightarrow |\underset{\substack{\uparrow \\ \beta_{max}}}{j}, \underset{\substack{\uparrow \\ \beta}}{m}\rangle \ .$$

$\Rightarrow$ We also get

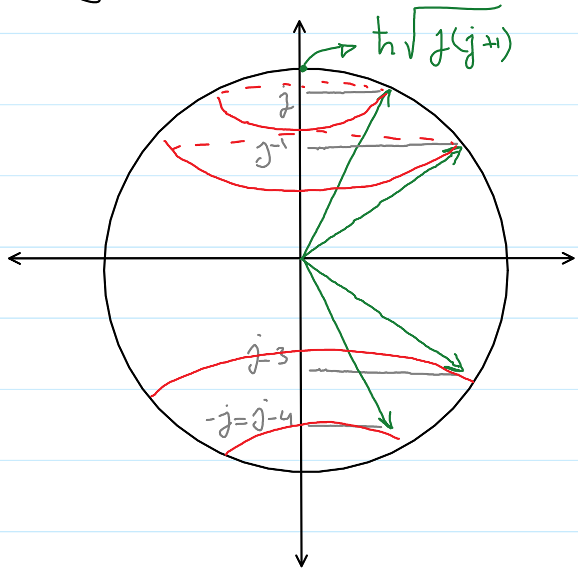
$$C^{\pm}_{\alpha\beta} \to C^{\pm}_{jm} = \hbar\sqrt{j(j+1)-m(m\pm1)}$$

or

$$J_{\pm}|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\;|j,m\pm1\rangle$$

(A) Calculate the variances of J_x, J_y & J_z for $|jm\rangle$.

Geometrically, this is



$Y_{lm}(\theta,\varphi)$

We know (from math. phys) that

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

$$\hat{L}_\pm = L_x \pm i L_y = \pm \hbar e^{\pm i\varphi}\left[\frac{\partial}{\partial\theta} \pm i\cot\theta \frac{\partial}{\partial\varphi}\right]$$

(See appendix B of Zettel.)

In this section we want to find the spatial distribution of $|l,m\rangle$, sort of

$$\psi(\vec{r}) = \langle \vec{r} | l,m\rangle$$

but there's no radial part. So it's mostly the distribution in θ, & φ.

$$\psi(\theta,\varphi) = \langle \theta,\varphi | l,m\rangle$$

We refer to these functions a $Y_{lm}(\theta,\varphi)$ (spherical Harmonics).

$\boxed{\text{Step 2}}$

$$\hat{L}_z |l,m\rangle = \hbar m |l,m\rangle$$

$$\Rightarrow L_z Y_{lm}(\theta,\varphi) = -i\hbar \frac{\partial}{\partial\varphi} Y_{lm}(\theta,\varphi) = \hbar m Y_{lm}(\theta,\varphi)$$

$$\Rightarrow Y_{lm}(\theta,\varphi) = A(\theta)\, e^{im\varphi}$$

Remark: $Y_{lm}(\theta, \varphi + 2\pi) = Y_{lm}(\theta,\varphi)$

$$\Rightarrow 2\pi m \overset{2\pi}{\equiv} 0 \Rightarrow m \in \mathbb{Z}$$

This is only for spatial AM.

Step 2. There are two ways. One is to use

$$\hat{L}^2 |l, m\rangle = \hbar^2 l(l+1) |l\, m\rangle$$

The other is to use

$$\hat{L}_+ |l\, l\rangle = 0 \quad .$$

Let's do the first one :

$$-\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \underbrace{\frac{\partial^2}{\partial\varphi^2}}_{-m^2} \right] Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1)\, Y_{lm}(\theta,\varphi)$$

$e^{-im\varphi} \times$

$$\left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) - \frac{m^2}{\sin\theta} \right] A(\theta) = l(l+1)\, A(\theta)$$

$$\Rightarrow \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial A(\theta)}{\partial\theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right) A(\theta) = 0 \qquad (*)$$

$\rightarrow$ Legendre Diff. eq.

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + l(l+1)\, y = 0 \qquad (m=0)$$

Ⓐ Turn the diff eq to Legendre form.

The solutions of the diff. eq $(*)$ are known as

Associated Legendre functions $P_\ell^m(\cos(\theta))$

$$\Rightarrow \quad Y_{\ell m}(\theta,\varphi) = \underset{\substack{\downarrow \\ \text{Normalization factors}}}{C_{\ell m}}\, P_\ell^m(\cos\theta)\, e^{im\varphi}$$

$$C_{\ell m} = (-1)^m \sqrt{\frac{(2\ell+1)}{2}\;\frac{(\ell-1)!}{(\ell+1)!}} \qquad m \geq 0$$

This gives a way to look up the $Y_{\ell m}(\theta,\varphi)$ from tables of $P_m^\ell(x)$.

The alternative solution is use $L_\pm$.

$$L_+ Y_{\ell,\ell}(\theta,\varphi) = 0 \qquad m = \ell$$

$$\Rightarrow \quad +\hbar e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\,\frac{\partial}{\partial\varphi}\right] A(\theta)\, e^{i\ell\varphi} = 0$$

$$\Rightarrow \quad \frac{\partial A}{\partial\theta} - \cot(\theta)\,\ell\, A(\theta) = 0$$

$$\Rightarrow \quad \frac{1}{A(\theta)}\frac{\partial A}{\partial\theta} = \ell\cot(\theta) \Rightarrow A_{\ell,\ell}(\theta) = c_\ell \sin^\ell(\theta)$$

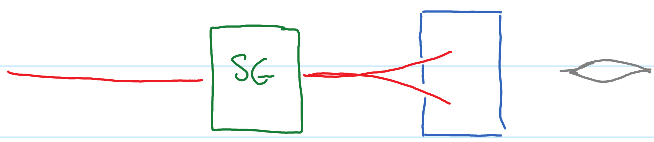
$A_{\ell,\ell-1}(\theta)$ $\qquad L_- Y_{\ell\ell}(\theta,\varphi) = \hbar\sqrt{2\ell}\; Y_{\ell,\ell-1}(\theta,\varphi)$

we also have

$$L_- = \hbar e^{-i\varphi}\left[\frac{\partial}{\partial\theta} - i\cot\theta \frac{\partial}{\partial\varphi}\right]$$

Spin

Lets go back to the SG experiment.



Initially, the splitting was attributed to the orbital AM of the silver atoms. But this cannot be true.

Ⓐ Why?

Beside the issue that you'll answer in the assignment, further analysis of of the atom shows that in the ground state, orbital AM is $\ell = 0$ (we'll see similar analysis for the Hydrogen atom). This indicates that there should be no splitting if

$\vec{J}^2$ & J would have formal a

there should be no splitting if L^2 & L_z would have formed a CSCO. This means that there should be some other DOF. This is why Goudsmit & Uhlenbeck postulated that there should be an intrinsic AM of spin. Initially, there were some attempts to justify this by the prssesion of the electron charge on the surface of a sphere (particle), but it became clear that it would not work.

It was later understood/predicted when realativistic QM was developped by Dirac.

$$|\psi\rangle = |l, m\rangle \otimes \underbrace{|j_s, m_s\rangle}_{\downarrow}$$

Intrinsic spin of the electron

Two spots indicates $j = 1/2$

- $|1/2, 1/2\rangle$
- $|1/2, -1/2\rangle$