Localization of Elastic Waves in Heterogeneous Media with Off-Diagonal Disorder and Long-Range Correlations

F. Shahbazi,[1] Alireza Bahraminasab,[2] S. Mehdi Vaez Allaei,[3] Muhammad Sahimi,[4,*] and M. Reza Rahimi Tabar[2,5]

[1]*Department of Physics, Isfahan University of Technology, Isfahan 84156, Iran*
[2]*Department of Physics, Sharif University of Technology, Tehran 11365-9161, Iran*
[3]*Institute for Advanced Studies in Basic Sciences, Gava Zang, Zanjan 45195-159, Iran*
[4]*Department of Chemical Engineering, University of Southern California, Los Angeles, California 90089-1211, USA*
[5]*CNRS UMR 6529, Observatoire de la Côte d'Azur, BP 4229, 06304 Nice Cedex 4, France*

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Using the Martin-Siggia-Rose method, we study propagation of acoustic waves in strongly heterogeneous media which are characterized by a broad distribution of the elastic constants. Gaussian-white distributed elastic constants, as well as those with long-range correlations with nondecaying power-law correlation functions, are considered. The study is motivated in part by a recent discovery that the elastic moduli of rock at large length scales may be characterized by long-range power-law correlation functions. Depending on the disorder, the renormalization group (RG) flows exhibit a transition to localized regime in *any* dimension. We have numerically checked the RG results using the transfer-matrix method and direct numerical simulations for one- and two-dimensional systems, respectively.

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Understanding how waves propagate in heterogeneous media is fundamental to such important problems as earthquakes, underground nuclear explosions, the morphology of oil and gas reservoirs, oceanography, and medical and materials sciences [1]. For example, seismic wave propagation and reflection are used not only to estimate the hydrocarbon content of a potential oil or gas field, but also to image structures located over a wide area, ranging from the Earth's near surface to the deeper crust and upper mantle. The same essential concepts and techniques are used in such diverse fields as materials science and medicine.

In condensed matter physics, a related problem, namely, the nature of electronic states in disordered materials, has been studied for several decades [2]. It was rigorously shown that, for one-dimensional (1D) systems, even infinitesimally small disorder is sufficient for localizing the wave function, irrespective of the energy [3], and that the envelope of the wave function $\psi(r)$ decays exponentially at large distances r from the domain's center, $\psi(r) \sim \exp(-r/\xi)$, with ξ being the localization length. The scaling theory of localization [4] predicts that, for spatial dimensions $d \leq 2$, all the electronic states are localized for any degree of disorder, while a transition to extended states—the metal-insulator transition—occurs for $d > 2$ if disorder is sufficiently strong. The transition between the two states is characterized by divergence of ξ, $\xi \propto |W - W_c|^{-\nu}$, where W_c is the critical value of the disorder intensity. Wegner [5] derived a field-theoretic formulation for the localization problem which, in agreement with the scaling theory [4], predict a lower critical dimension, $d_c = 2$, for the localization problem. These predictions were confirmed by numerical simulations [6].

Wave characteristics of electrons suggest that the localization phenomenon may occur in other wave propagation processes. For example, consider the propagation of seismic waves in heterogeneous rock. In this case, the interference of the waves that have undergone multiple scattering, caused by the heterogeneities of the medium, may cause their localization. Unlike electrons, however, the classical waves do not interact with one another and, therefore, propagation of such waves in heterogeneous materials provides [7] an ideal model for studying the classical Anderson localization [8–12] in strongly disordered media. This is the focus of this Letter. We study localization of acoustic waves in strongly heterogeneous media, and formulate a field-theoretic method to investigate the problem in the media that are characterized by a broad distribution of the elastic constants. Localization of acoustic waves was previously studied by several groups [13]. In particular, Baluni and Willemsen [14] studied the propagation of acoustic waves in a 1D layered system. However, these studies [13,14] did not consider the type of systems that we consider in the present Letter, which represent the continuum limit of an acoustic system with an *off-diagonal* disorder. Our approach is based on the method first introduced by Martin, Siggia, and Rose [15] for analyzing dynamical critical phenomena. We calculate to one-loop order the beta functions [8,15] for both spatially delta-correlated and power-law correlated disorder in the elastic constants, and show that in any case there is a disorder-induced transition from delocalized to localized states for *any* d. In addition to its general importance, our study is motivated in part by the recent discovery [16] that the distribution of the elastic moduli of heterogeneous rock contains long-range correlations characterized by a *nondecaying* power-law correlation function. However, our results are completely general and apply to any material in which the local elastic constants are distributed by the type of distributions we consider.

Wave propagation in a medium with a distribution of elastic constants is described by the following equation (for simplicity we consider the scalar wave equation):

$$\frac{\partial^2}{\partial t^2}\psi(\mathbf{x},t) - \boldsymbol{\nabla}\cdot[\lambda(\mathbf{x})\boldsymbol{\nabla}\psi(\mathbf{x},t)] = 0, \tag{1}$$

where $\psi(\mathbf{x},t)$ is the wave amplitude, and $\lambda(\mathbf{x}) = C(\mathbf{x})/m$ is the ratio of the elastic stiffness $C(\mathbf{x})$ and the mean density m of the medium. We then write λ as

$$\lambda(\mathbf{x}) = \lambda_0 + \eta(\mathbf{x}), \tag{2}$$

where $\lambda_0 = \langle\lambda(\mathbf{x})\rangle$. We assume $\eta(\mathbf{x})$ to be a Gaussian random process with a zero mean and the covariance

$$\begin{aligned}\langle\eta(\mathbf{x})\eta(\mathbf{x}')\rangle &= 2K(|\mathbf{x}-\mathbf{x}'|)\\ &= 2D_0\delta^d(\mathbf{x}-\mathbf{x}') + 2D_\rho|\mathbf{x}-\mathbf{x}'|^{2\rho-d},\end{aligned} \tag{3}$$

in which D_0 and D_ρ represent the strength of the disorder due to the delta-correlated and power-law correlated parts of the disorder. Previously, Souillard and co-workers [17] studied wave propagation in disordered fractal media, which is characterized by a *decaying* power-law correlation function. Their study is not, however, related to our work.

Consider a wave component $\psi(\mathbf{x},\omega)$ with angular frequency ω, which is obtained by taking the temporal Fourier transform of Eq. (1) which yields the following equation for propagation of a wave component in a disordered medium:

$$\nabla^2\psi(\mathbf{x},\omega) + \frac{\omega^2}{\lambda_0}\psi(\mathbf{x},\omega) + \boldsymbol{\nabla}\cdot\left[\frac{\eta(\mathbf{x})}{\lambda_0}\boldsymbol{\nabla}\psi(\mathbf{x},\omega)\right] = 0. \tag{4}$$

Since $\eta(\mathbf{x})$ is a Gaussian variable, we obtain a Martin-Siggia-Rose effective action S_e for the probability density functional of the wave function $\psi(\mathbf{x},\omega)$, given by

$$\begin{aligned}S_e(\psi_I,\psi_R,\tilde{\psi},\chi,\chi^*) = \int d\mathbf{x}d\mathbf{x}'\Big[& i\tilde{\psi}_I(\mathbf{x}')\Big(\nabla^2+\frac{\omega^2}{\lambda_0}\Big)\psi_I(\mathbf{x}) + i\tilde{\psi}_R(\mathbf{x}')\Big(\nabla^2+\frac{\omega^2}{\lambda_0}\Big)\psi_R(\mathbf{x}) + \chi^*(\mathbf{x}')\Big(\nabla^2+\frac{\omega^2}{\lambda_0}\Big)\chi(\mathbf{x})\Big]\\ &\times\delta(\mathbf{x}-\mathbf{x}') + (i\boldsymbol{\nabla}\tilde{\psi}_I\boldsymbol{\nabla}\psi_I + i\boldsymbol{\nabla}\tilde{\psi}_R\boldsymbol{\nabla}\psi_R + \boldsymbol{\nabla}\chi\boldsymbol{\nabla}\chi)\frac{K(\mathbf{x}-\mathbf{x}')}{\lambda_0^2}(i\boldsymbol{\nabla}\tilde{\psi}_I\boldsymbol{\nabla}\psi_I + i\boldsymbol{\nabla}\tilde{\psi}_R\boldsymbol{\nabla}\psi_R + \boldsymbol{\nabla}\chi\boldsymbol{\nabla}\chi).\end{aligned} \tag{5}$$

Here, $\tilde{\psi}_I(\mathbf{x})$, $\tilde{\psi}_R(\mathbf{x})$, χ, and χ^* are the auxiliary and Grassmanian fields of the field-theoretic formulation, respectively, and $g_0 = D_0/\lambda_0^2$ and, $g_\rho = D_\rho/\lambda_\rho^2$ are two coupling constants. Thus, we carry out a renormalization group (RG) analysis in the critical limit, $\omega^2/\lambda_0 \to 0$, to derive, to one-loop order, the beta functions [7,15] that govern g_0 and g_ρ under the transformation. The results are given by

$$\beta(\tilde{g}_0) = \frac{\partial\tilde{g}_0}{\partial\ln l} = -d\tilde{g}_0 + 8\tilde{g}_0^2 + 10\tilde{g}_\rho^2 + 20\tilde{g}_0\tilde{g}_\rho, \tag{6}$$

$$\beta(\tilde{g}_\rho) = \frac{\partial\tilde{g}_\rho}{\partial\ln l} = (2\rho-d)\tilde{g}_\rho + 12\tilde{g}_0\tilde{g}_\rho + 16\tilde{g}_\rho, \tag{7}$$

where $l > 1$ is the rescaling parameter, and $\tilde{g}_0$ and $\tilde{g}_\rho$ are given by

$$\tilde{g}_0 = k_d\left[\frac{d+5}{2d(d+2)}\right]g_0, \tag{8}$$

$$\tilde{g}_\rho = k_d\left[\frac{d+5}{2d(d+2)}\right]g_\rho, \tag{9}$$

with $k_d = S_d/(2\pi^d)$, and S_d being the surface area of the d-dimensional unit sphere. Examining the RG flows, Eqs. (6) and (7), reveals that, depending on ρ, there are two distinct regimes.

(i) For $0 < \rho < d/2$ there are three sets of fixed points: The trivial Gaussian fixed point ($g_0^* = g_\rho^* = 0$), which is stable, and two nontrivial fixed points and eigendirections. One is $\{g_0^* = d/8, g_\rho^* = 0\}$, while the other set is given by

$$\begin{aligned}g_0^* &= -\frac{4}{41}\left[d+\frac{5}{16}(2\rho-d)\right] - \frac{4}{41}\\ &\quad\times\sqrt{\left[d+\frac{5}{16}(2\rho-d)\right] + \frac{205}{256}(2\rho-d)^2},\\ g_\rho^* &= \frac{3}{4}g_0^* + \frac{1}{16}(d-2\rho),\end{aligned}$$

which is stable in one eigendirection but unstable in the other eigendirection. The corresponding RG flow diagram is shown in Fig. 1. Therefore, for $0 < \rho < d/2$ the RG calculation indicates that the system with uncorrelated disorder is unstable against long-range correlated disorder toward a new fixed point in the coupling constants space, for which there is a phase transition from delocalized to localized states with increasing the disorder intensity.

(ii) For $\rho > d/2$ there are two fixed points: the Gaussian fixed point which is stable on the g_0 axis but unstable on the g_ρ axis, and the nontrivial fixed point, $\{g_0^* = d/8, g_\rho^* = 0\}$, which is unstable in all directions; see Fig. 2. Thus, while the power-law correlated disorder is relevant, no new fixed point exists to one-loop order and, therefore, the long-wavelength behavior of the system is determined by the long-range component of the disorder. That is, for $\rho > d/2$ the waves are localized for *any* d. In addition, in both cases (i) and (ii) the system undergoes a disorder-induced transition when only the uncorrelated disorder is present. These results are general so long as $D_\rho > 0$. For $D_\rho < 0$ the above phase space is valid for $\rho > \frac{1}{2}(d+1)$.

To test these predictions, we carried out numerical simulations of the problem in both 1D and 2D systems.

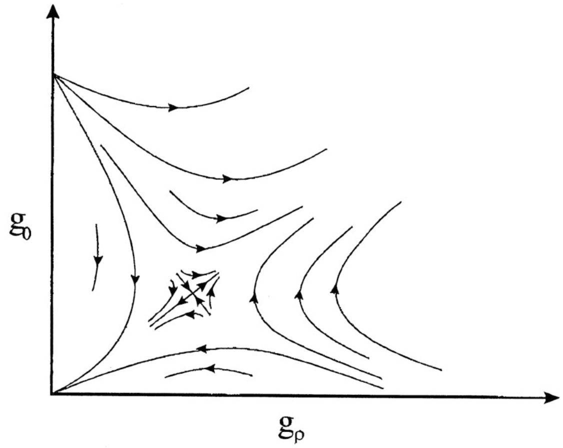

FIG. 1. RG flows in the coupling constants space for $0 < \rho < d/2$.

In 1D disordered systems the waves are localized when the wave functions are of the form $\psi(x) = f(x) \times \exp(-|x - x_0|/\xi)$, where $f(x)$ is a stochastic function that depends on the particular realization of the chain, and ξ is the localization length. The simplest 1D system that exhibits wave localization is [18] a 15 m long steel wire with a 0.178 mm diameter, suspended vertically. The tension in the wire is maintained with a weight attached at its lower end, and $\psi(x, t)$ consists of transverse waves in the wire with an electromechanical actuator at one end of the wire. It was shown [18] that, even for small deviations (less than 1%) from periodicity, the diagonal disorder (e.g., variations in the resonance frequencies of the oscillators) produces localization (in agreement with Furstenberg's theorem [19]), while variations (up to 13%) in the sizes of the masses (off-diagonal disorder) result in localization lengths that are much larger than the size of the system. This is in agreement with our theoretical prediction.

To study the 1D system, we used the transfer-matrix (TM) method [12]. Discretizing Eq. (1) and writing down the result for site n of a linear chain yields $(\omega + \lambda_n)\psi_n + \lambda_{n+1}\psi_{n+2} - (\lambda_{n+1} - \lambda_n)\psi_{n+1} = 0$ or, in the recursive form, $\mathbf{M}_n(\psi_{n+2}, \psi_{n+1})^T = (\psi_{n+1}, \psi_n)^T$, with T representing the transpose operation, and

$$\mathbf{M}_n = \begin{pmatrix} -\frac{\omega^2 - \lambda_n + \lambda_{n-2}}{\lambda_n} & \frac{\lambda_{n-2}}{\lambda_n} \\ 1 & 0 \end{pmatrix}. \qquad (10)$$

$\xi(\omega)$ is then defined by $\xi(\omega)^{-1} = \lim_{N\to\infty} N^{-1}|\psi_N/\psi_0|$, where N is the chain's length. For every realization of the disorder we computed ψ_N and, hence, $\xi(\omega)$, using $\psi_0 = \psi_1 = 1/\sqrt{2}$, and averaging ξ over a large ensemble of realizations for a fixed N and frequency ω. The procedure was repeated for several values of N and ω. The extended states correspond to having $\lim_{N\to\infty}\xi/N = \text{const} > 1$. As ξ is a function of ω also, we chose $\omega = 2\pi\sqrt{\lambda_0}/N$, which is the system's smallest mode. Our RG analysis indicates that this mode most likely passes through the chain.

The TM computations indicate that the coupling constant $g(N)$ follows finite-size scaling, $g(N) = g_0 + 1/N$, with $g_0 = g(N \to \infty) \simeq 0.117$. Moreover, as $\xi \to N$, one has $N \propto (g - g_0)^{-\nu}$, with ν being the localization exponent. Our RG results indicate [20] that in d dimensions, $\nu = 1/d$, which agrees with the TM calculations that yield $\nu = 1$. Figure 3 shows the results for $\xi(\omega)$ which confirm the RG predictions.

We also solved Eq. (1) in 2D using the finite-difference (FD) method with second-order discretization for both the space and time variables. Such approximations are acceptable as we work in the limit of low frequencies or long wavelengths. For short wavelengths we should use higher-order discretizations for the spatial variables [20]. A $L_x \times L_y$ grid was used with $L_x = 8000$ and $L_y = 400$. The midpoint displacement method [21] was used to distribute in the grid $\lambda(\mathbf{x})$, the gridblock-scale elastic constant, with the power-law correlation function. Also simulated was the case in which $\lambda(\mathbf{x})$ was randomly and uniformly distributed, with the same variance as that of the power-law case.

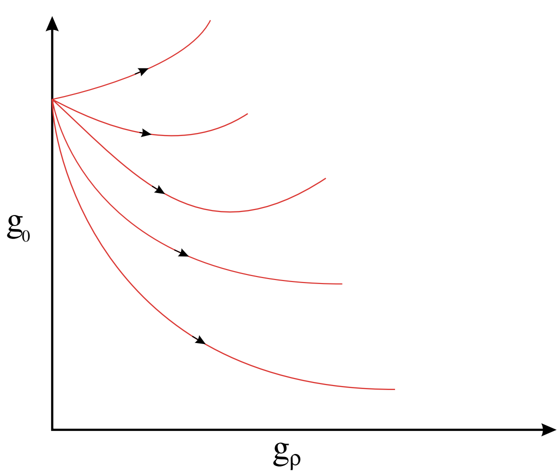

FIG. 2 (color online). Same as in Fig. 1, but for $\rho > d/2$.

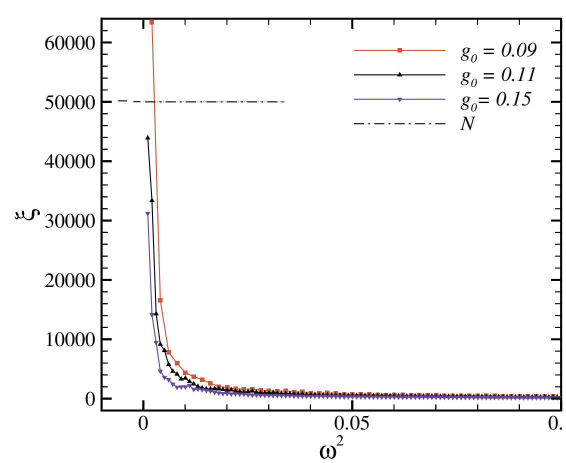

FIG. 3 (color online). Localization length ξ for a disordered chain with 50 000 site and its dependence on the frequency ω.

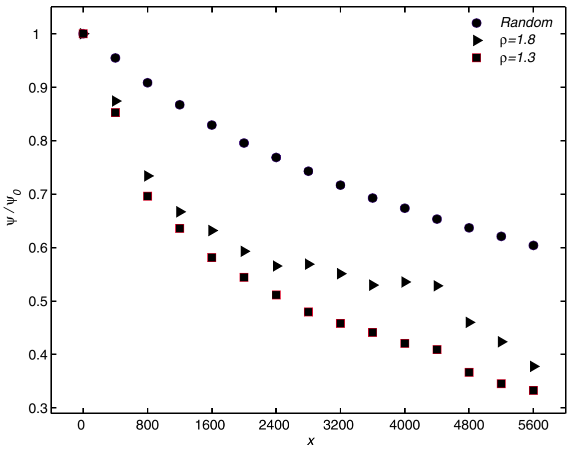

FIG. 4 (color online). Amplitude of the waves in 2D systems, for both randomly and uniformly distributed and power-law correlated elastic constants.

Periodic boundary conditions were imposed in the lateral direction, which did not distort the nature of the wave propagation, as large systems were used. By inserting a wave source on a line on one side of the grid, the wave equation was solved numerically. The decay in the wave's amplitude is caused by scattering from heterogeneities of the system generated by the distribution of $\lambda(\mathbf{x})$. The solution's accuracy was checked by considering the stability criterion and the source's wavelength [22], and using higher-order FD discretizations. The amplitude decay was computed by collecting the numerical results at 80 grid points along the direction of wave propagation, distributed evenly in the grid. The results were averaged over 32 realizations of the system.

Figure 4 presents the decay in the wave amplitude through the uniformly random medium, and that of one with a nondecaying power-law correlation function for $\lambda(\mathbf{x})$, with $\rho = 1.3$ and 1.8. The wave amplitudes for the correlated cases decline much faster than those in the uniformly random medium. In particular, for $\rho = 1.3$, which corresponds to negative correlations (that is, a large local elastic modulus is likely to be neighbor to a small one, and vice versa), the amplitude decreases rather sharply, hence confirming the RG predictions.

These results, which contradict the generally accepted view that off-diagonal disorder has a much weaker effect on localization than the diagonal disorder, have important practical implications [23]. For example, in order for seismic records to contain meaningful information on the geology and content of a natural porous formation of linear size L, the localization length ξ must be larger L. Otherwise, the seismic records would provide the information only up to ξ (which, itself, is a function of d, ρ, and other parameters), but not at larger length scales.

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*Corresponding author.
Electronic address: moe@iran.usc.edu

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