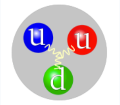
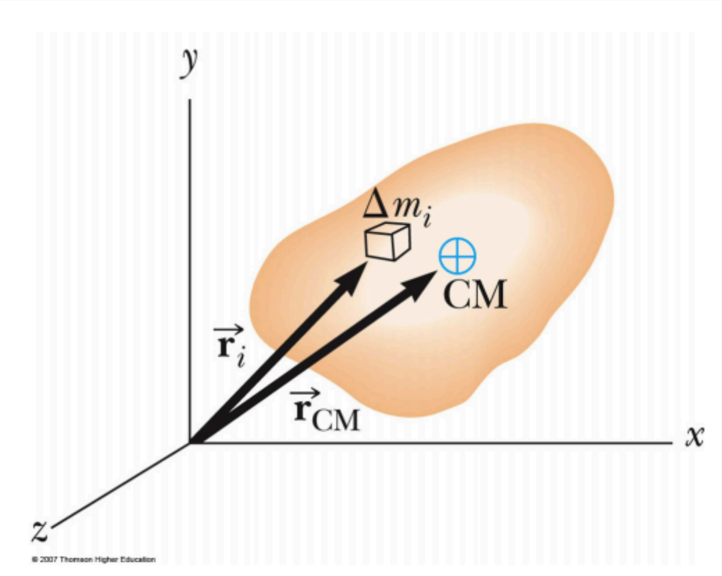
General Physics I

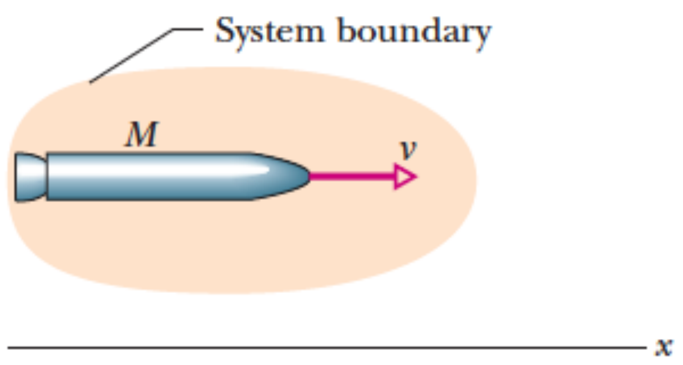
chapter 9

Sharif University of Technology
Mehr 1401 (2022-2023)

M. Reza Rahimi Tabar

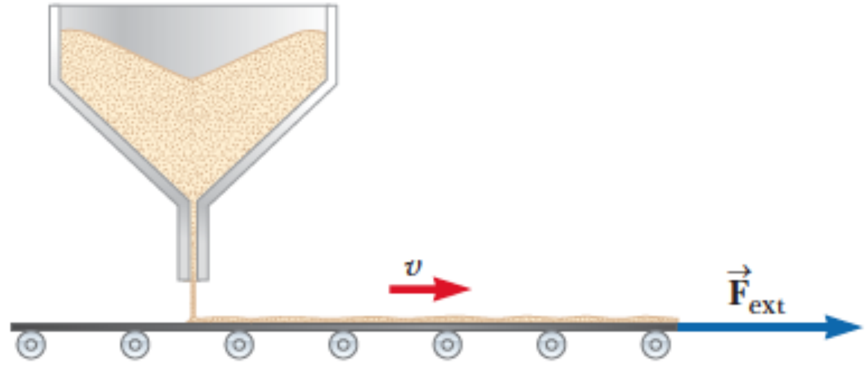

u
u
d

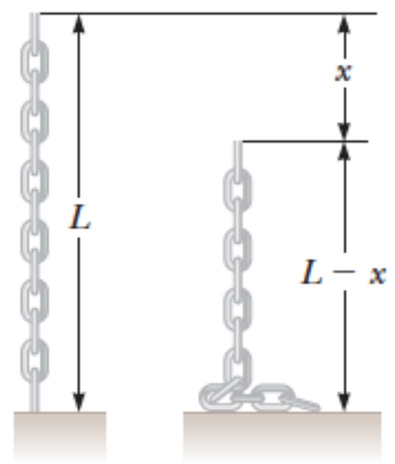

y
Δm_i
CM
$\vec{\mathbf{r}}_i$
$\vec{\mathbf{r}}_{CM}$
x
z
© 2007 Thomson Higher Education


System boundary
M
v
x
System boundary
−dM
M + dM
v + dv
U
x

(a)

(b)

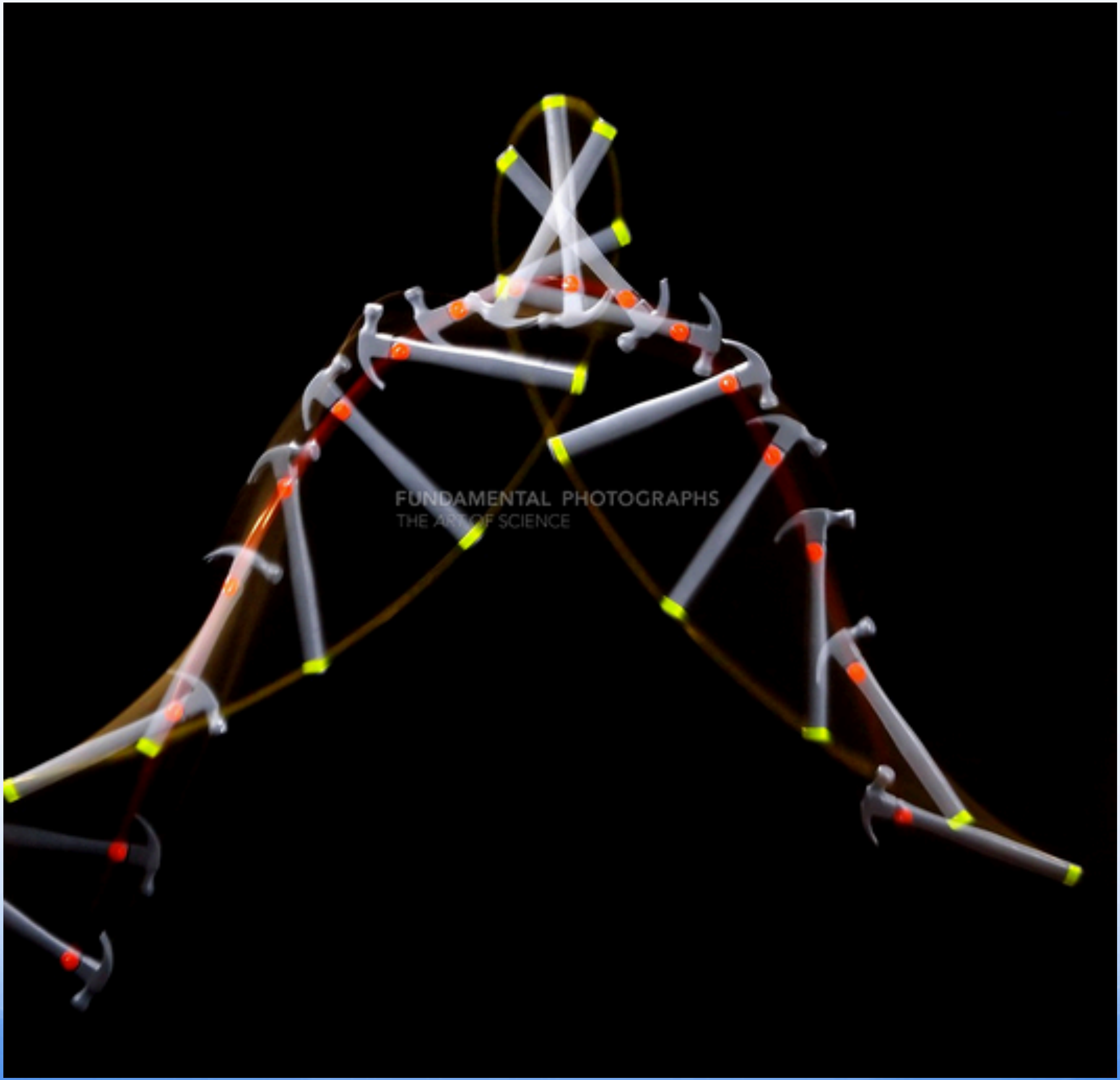

v
$\vec{F}_{ext}$

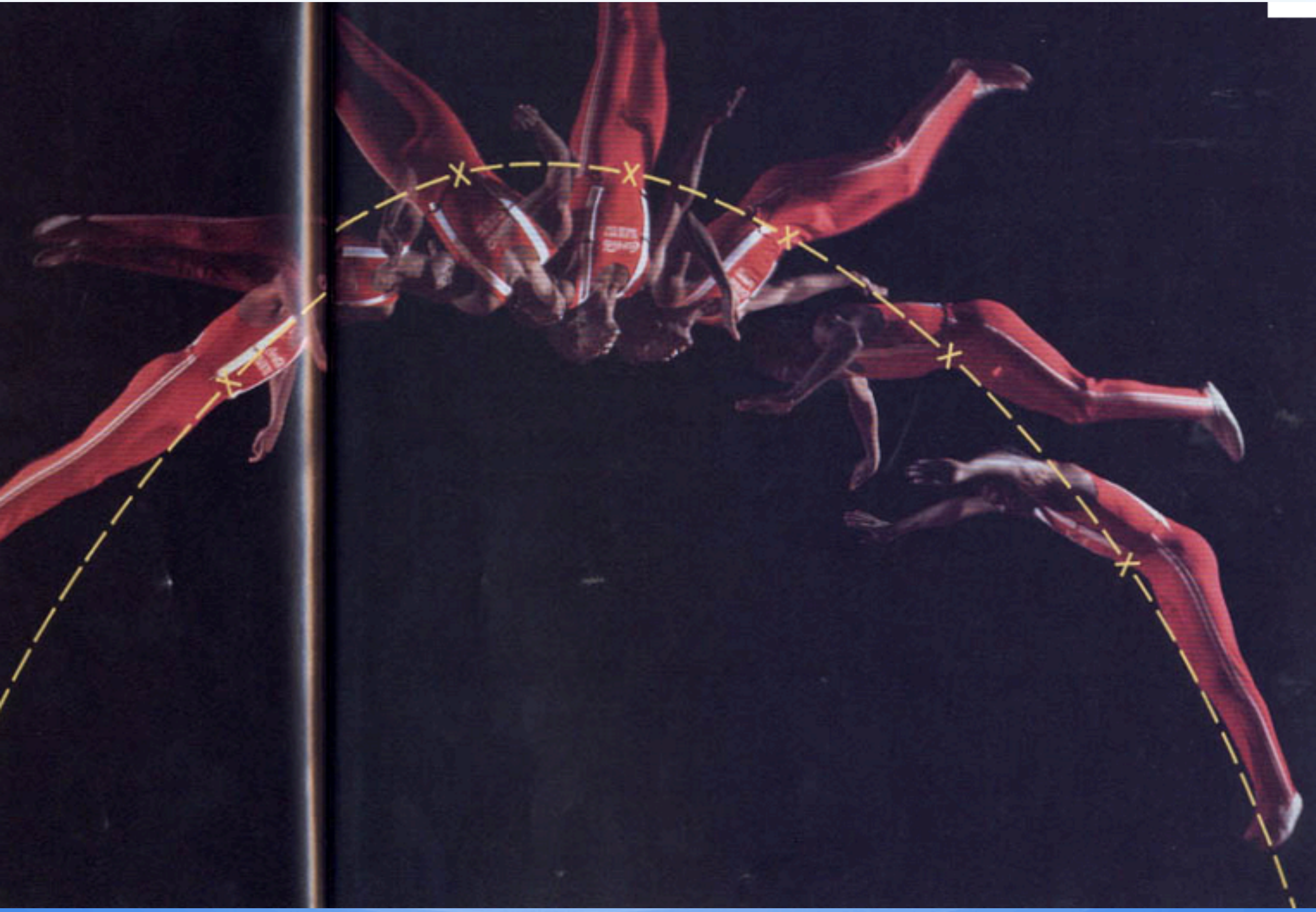

L
x
L − x

What Is Physics?

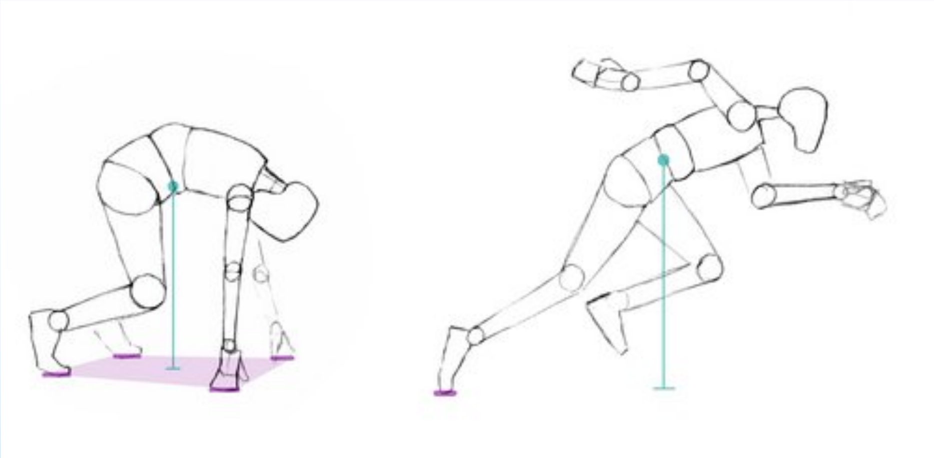
- How the complicated motion of a system of objects, such as a car or a ballerina, can be simplified if we determine a special point of the system - the center of mass of that system

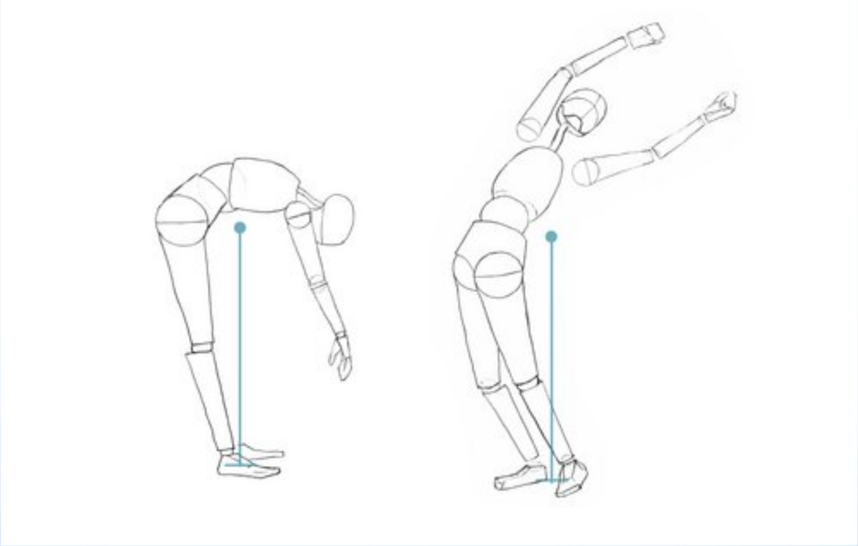
(b)

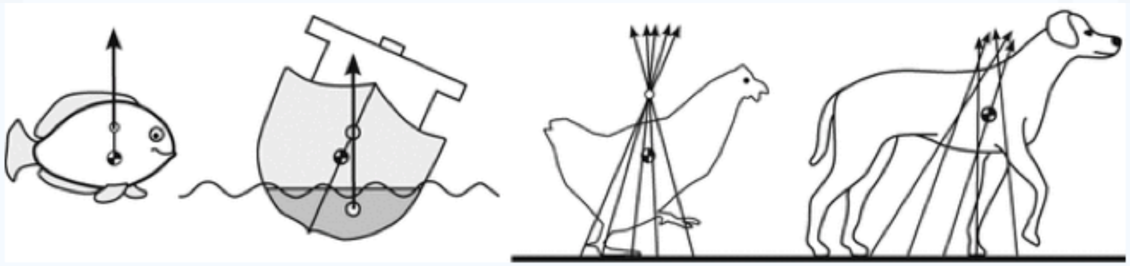

FUNDAMENTAL PHOTOGRAPHS
THE ART OF SCIENCE

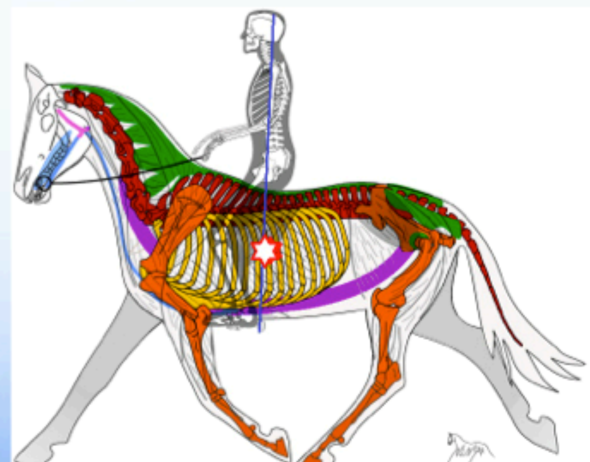


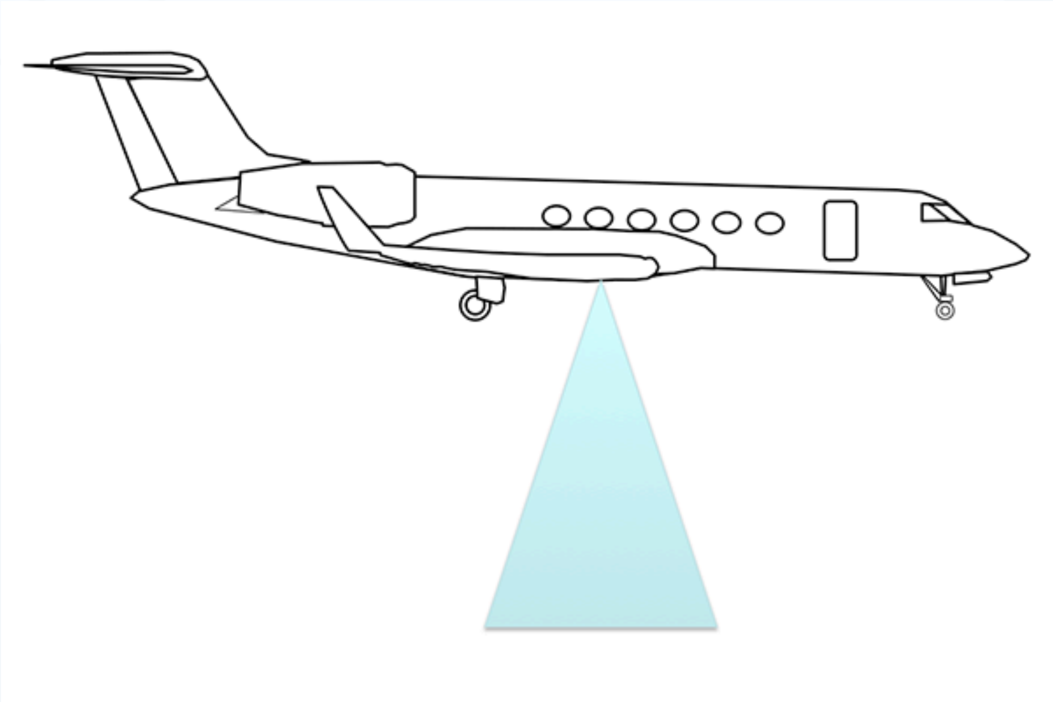
- Center of mass depends on the mass distribution

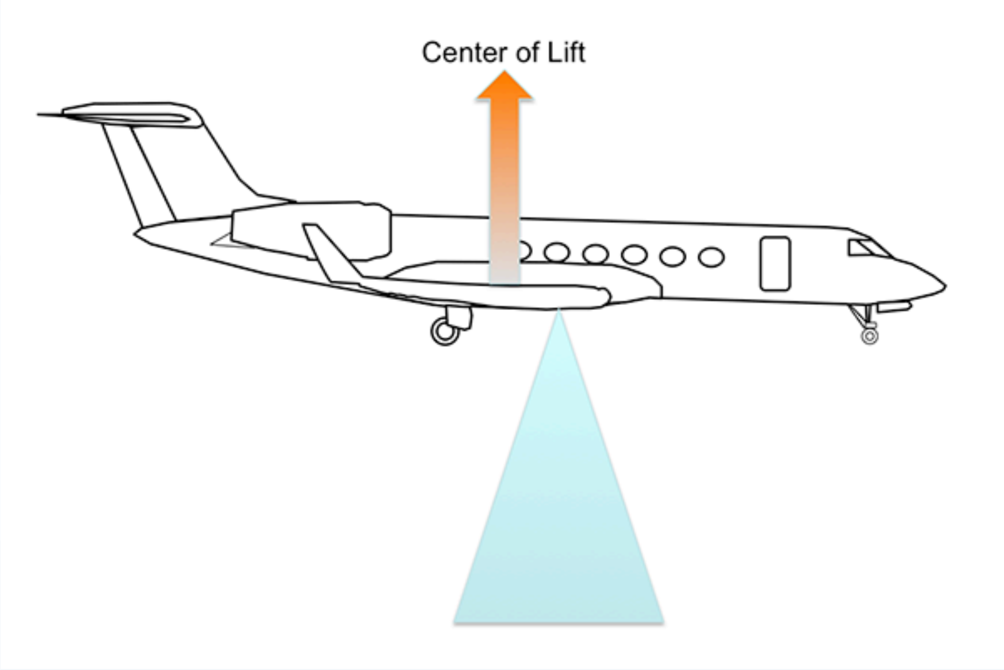


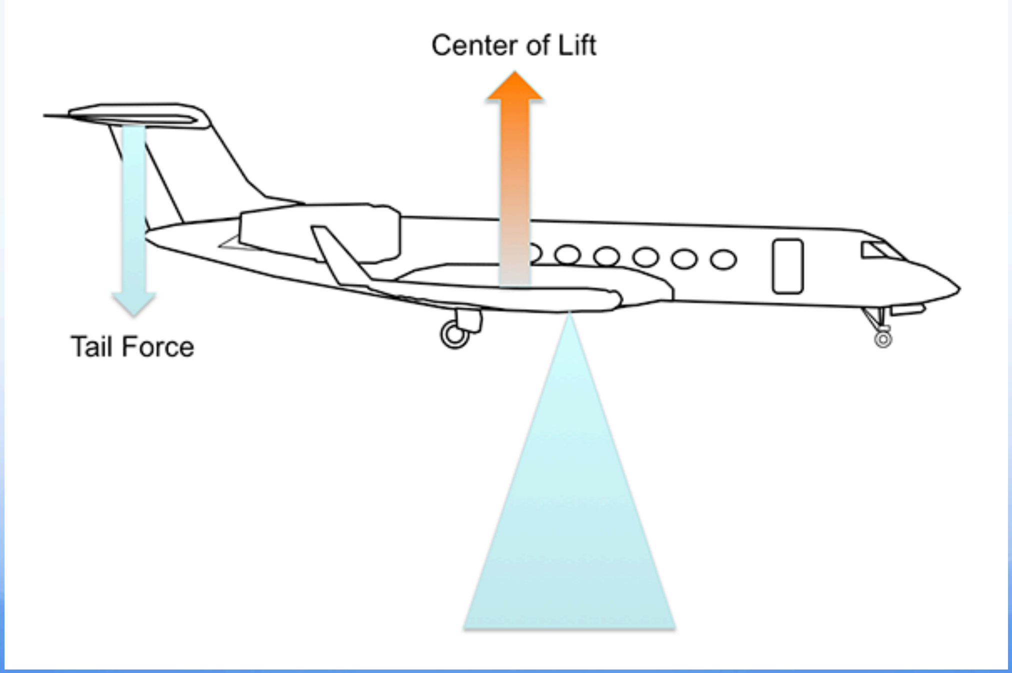


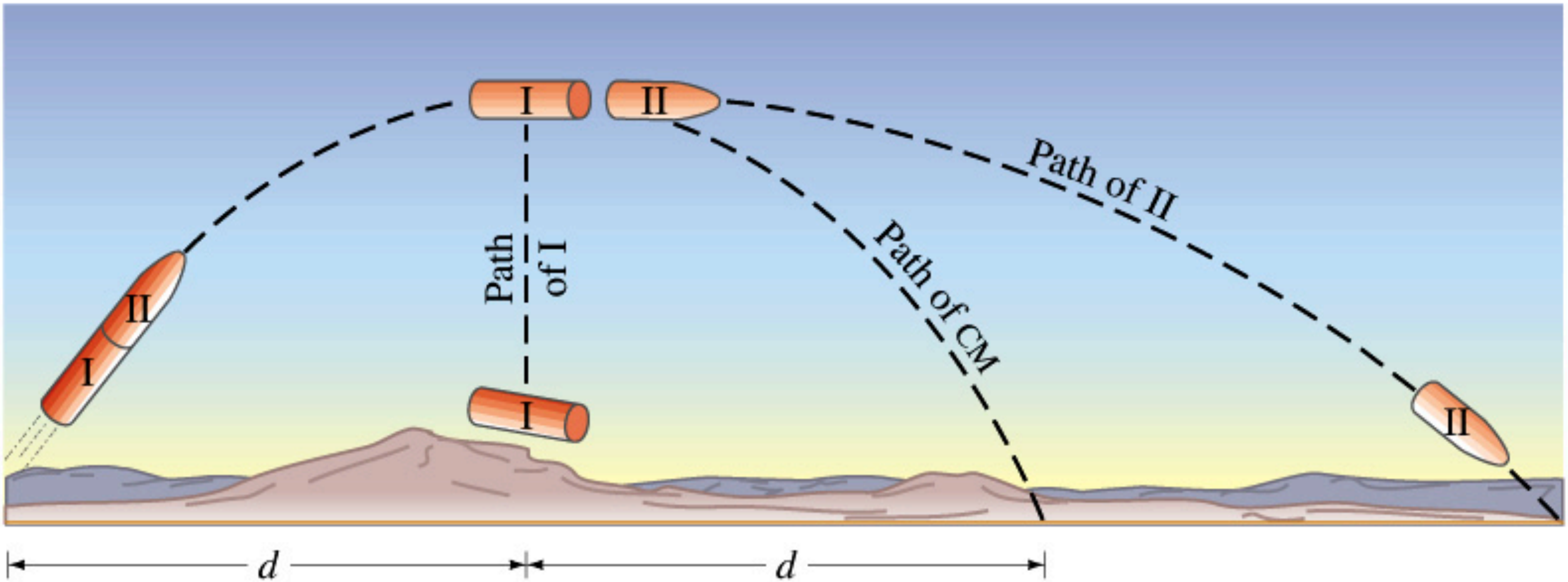


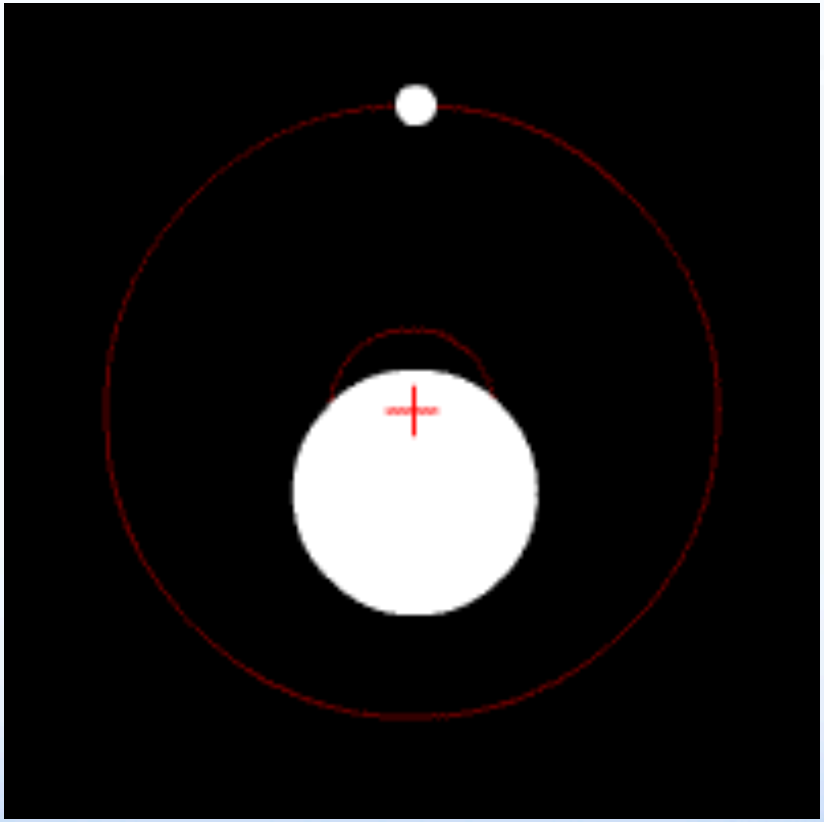


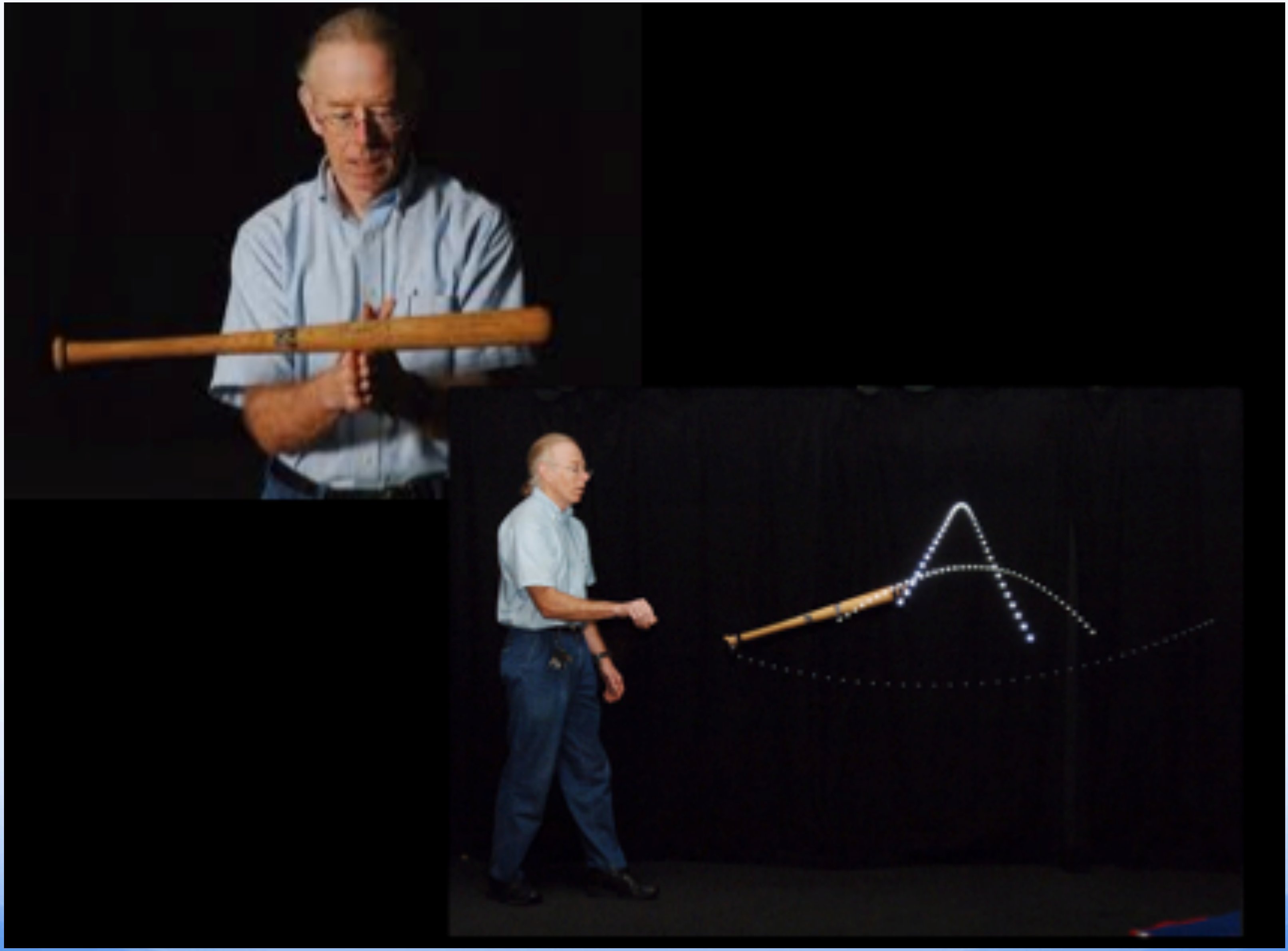



Center of Lift


Center of Lift
Tail Force


I
II
I
II
Path
of I
I
Path of CM
Path of II
II
d
d





Chapter 9
Center of Mass and Linear Momentum

9.01 Given the positions of several particles along an axis or a plane, determine the location of their center of mass.

9.02 Locate the center of mass of an extended, symmetric object by using the symmetry.

9.03 For a two-dimensional or three-dimensional extended object with a uniform distribution of mass, determine the center of mass by

(a) Mentally dividing the object into simple geometric figures, each of which can be replaced by a particle at its center

(b) Finding the center of mass of those particles.

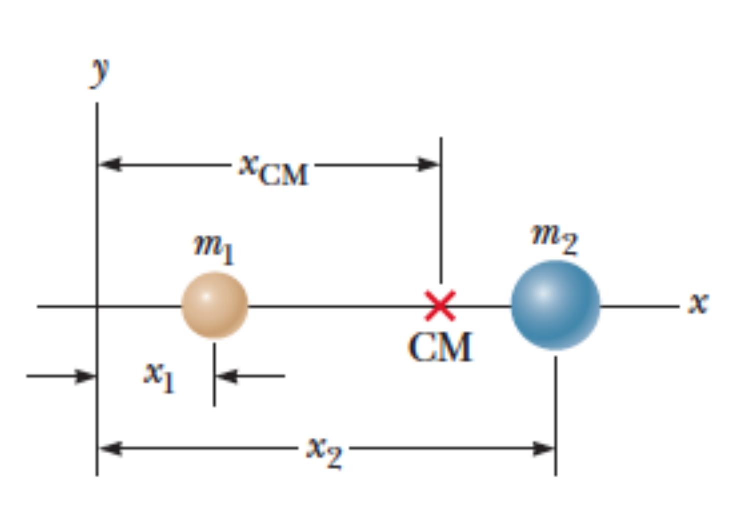
The Center of Mass

- The center of mass of a system of particles is the point that moves as though

(1) all of the system's mass were concentrated there and

(2) all external forces were applied there.

Center of Mass



$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

✔

N-Particles in one dimension:

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\sum_i m_i x_i}{M} = \frac{1}{M} \sum_i m_i x_i$$

$$y_{CM} \equiv \frac{1}{M}\sum_i m_i y_i \quad \text{and} \quad z_{CM} \equiv \frac{1}{M}\sum_i m_i z_i$$

$$\vec{\mathbf{r}}_{CM} = x_{CM}\hat{\mathbf{i}} + y_{CM}\hat{\mathbf{j}} + z_{CM}\hat{\mathbf{k}} = \frac{1}{M}\sum_i m_i x_i \hat{\mathbf{i}} + \frac{1}{M}\sum_i m_i y_i \hat{\mathbf{j}} + \frac{1}{M}\sum_i m_i z_i \hat{\mathbf{k}}$$

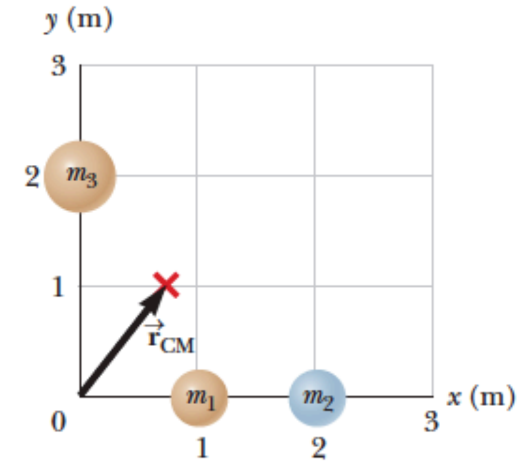
$$\vec{\mathbf{r}}_{CM} \equiv \frac{1}{M}\sum_i m_i \vec{\mathbf{r}}_i$$

$$\vec{\mathbf{r}}_i \equiv x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$$

Example 1:

The Center of Mass of Three Particles

A system consists of three particles located as shown in Figure 9.18. Find the center of mass of the system. The masses of the particles are $m_1 = m_2 = 1.0$ kg and $m_3 = 2.0$ kg.



Two particles are located on the x axis, and a single particle is located on the y axis as shown. The vector indicates the location of the system's center of mass.

continued

Categorize We categorize this example as a substitution problem because we will be using the equations for the center of mass developed in this section.

Use the defining equations for the coordinates of the center of mass and notice that $z_{CM} = 0$:

$$x_{CM} = \frac{1}{M}\sum_i m_i x_i = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.0\text{ kg})(1.0\text{ m}) + (1.0\text{ kg})(2.0\text{ m}) + (2.0\text{ kg})(0)}{1.0\text{ kg} + 1.0\text{ kg} + 2.0\text{ kg}} = \frac{3.0\text{ kg}\cdot\text{m}}{4.0\text{ kg}} = 0.75\text{ m}$$

$$y_{CM} = \frac{1}{M}\sum_i m_i y_i = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.0\text{ kg})(0) + (1.0\text{ kg})(0) + (2.0\text{ kg})(2.0\text{ m})}{4.0\text{ kg}} = \frac{4.0\text{ kg}\cdot\text{m}}{4.0\text{ kg}} = 1.0\text{ m}$$

Write the position vector of the center of mass:

$$\vec{\mathbf{r}}_{CM} \equiv x_{CM}\hat{\mathbf{i}} + y_{CM}\hat{\mathbf{j}} = \boxed{(0.75\hat{\mathbf{i}} + 1.0\hat{\mathbf{j}})\text{ m}}$$

Solid Bodies

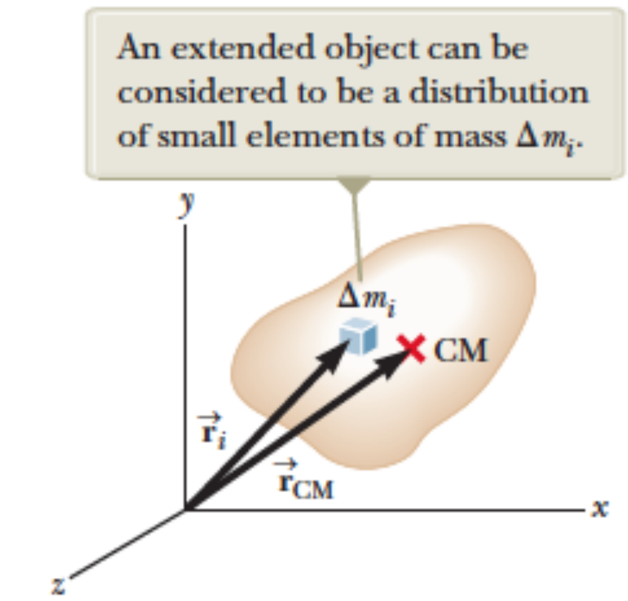
$$x_{CM} \approx \frac{1}{M} \sum_i x_i \Delta m_i$$

$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{1}{M} \sum_i x_i \Delta m_i = \frac{1}{M} \int x \, dm$$

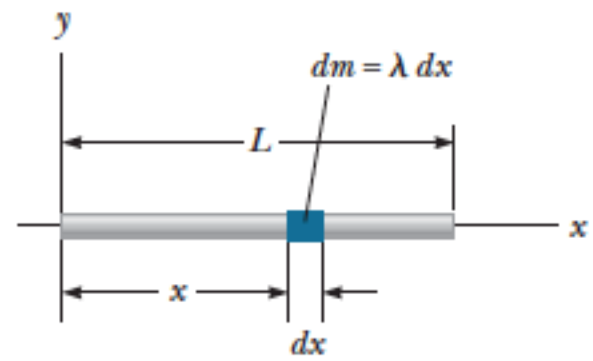
$$y_{CM} = \frac{1}{M} \int y \, dm \quad \text{and} \quad z_{CM} = \frac{1}{M} \int z \, dm$$

$$\vec{\mathbf{r}}_{CM} = \frac{1}{M} \int \vec{\mathbf{r}} \, dm$$

✔



Example 2:



The
geometry used to find the center
of mass of a uniform rod.

(A) Show that the center of mass of a rod of mass M and length L lies midway between its ends, assuming the rod has a uniform mass per unit length.

(B) Suppose a rod is *nonuniform* such that its mass per unit length varies linearly with x according to the expression $\lambda = \alpha x$, where α is a constant. Find the x coordinate of the center of mass as a fraction of L.

Analyze The mass per unit length (this quantity is called the *linear mass density*) can be written as $\lambda = M/L$ for the uniform rod. If the rod is divided into elements of length dx, the mass of each element is $dm = \lambda\, dx$.

an expression for x_{CM}:

$$x_{CM} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^L x\lambda\, dx = \frac{\lambda}{M}\left.\frac{x^2}{2}\right|_0^L = \frac{\lambda L^2}{2M}$$

Substitute $\lambda = M/L$:

$$x_{CM} = \frac{L^2}{2M}\left(\frac{M}{L}\right) = \boxed{\tfrac{1}{2}L}$$

One can also use symmetry arguments to obtain the same result.

Analyze In this case, we replace dm in Equation 9.32 by $\lambda\, dx$, where $\lambda = \alpha x$.

Use Equation 9.32 to find an expression for x_{CM}:

$$x_{CM} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^L x\lambda\, dx = \frac{1}{M}\int_0^L x\alpha x\, dx$$

$$= \frac{\alpha}{M}\int_0^L x^2\, dx = \frac{\alpha L^3}{3M}$$

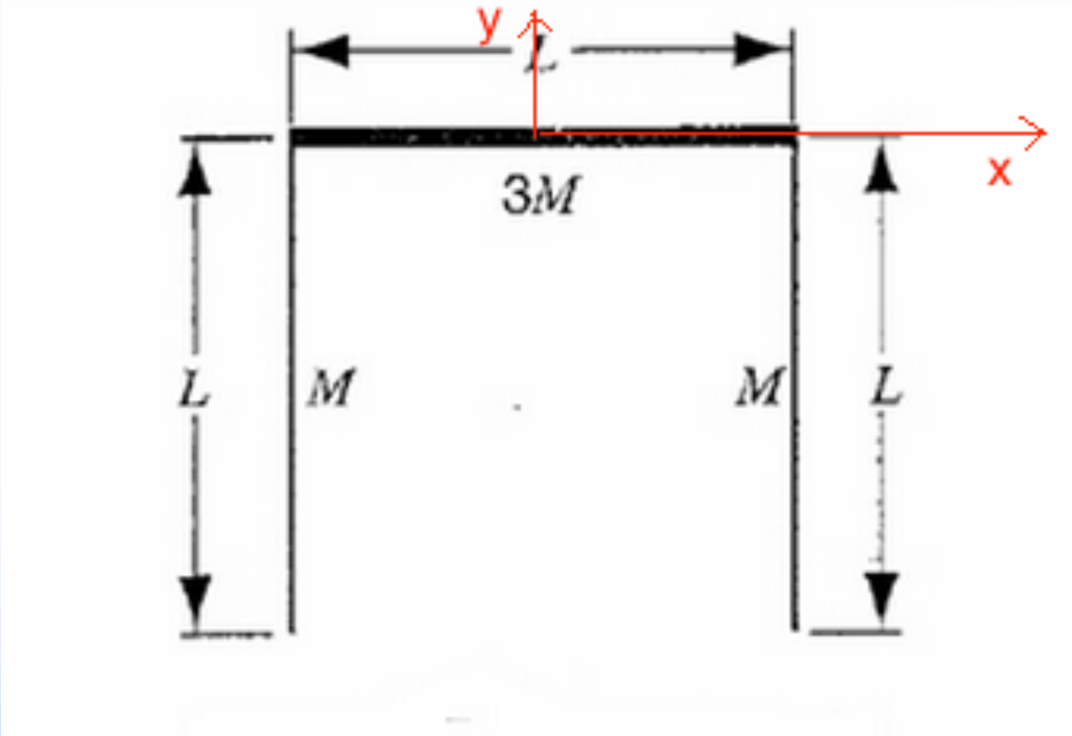
Find the total mass of the rod:

$$M = \int dm = \int_0^L \lambda\, dx = \int_0^L \alpha x\, dx = \frac{\alpha L^2}{2}$$

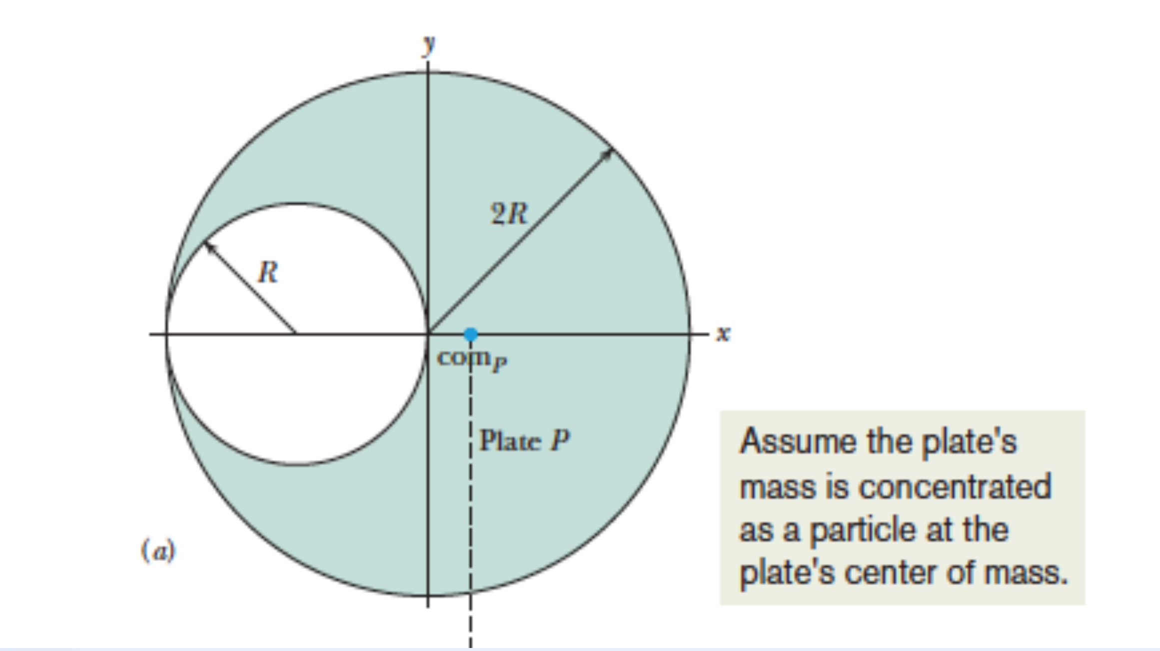
Substitute M into the expression for x_{CM}:

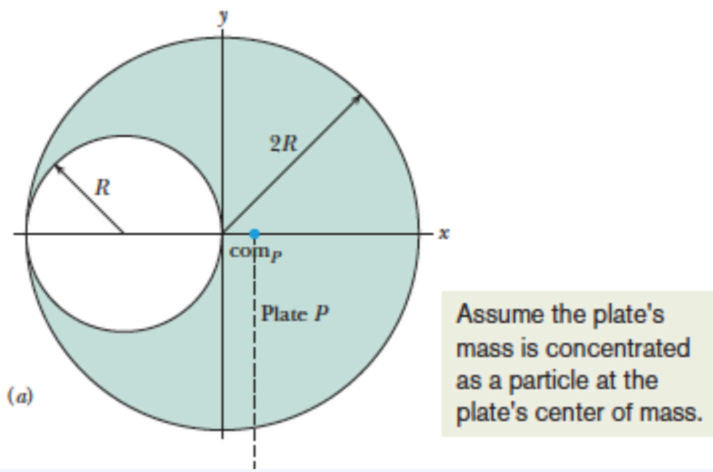
$$x_{CM} = \frac{\alpha L^3}{3\alpha L^2/2} = \boxed{\tfrac{2}{3}L}$$

Example 3:

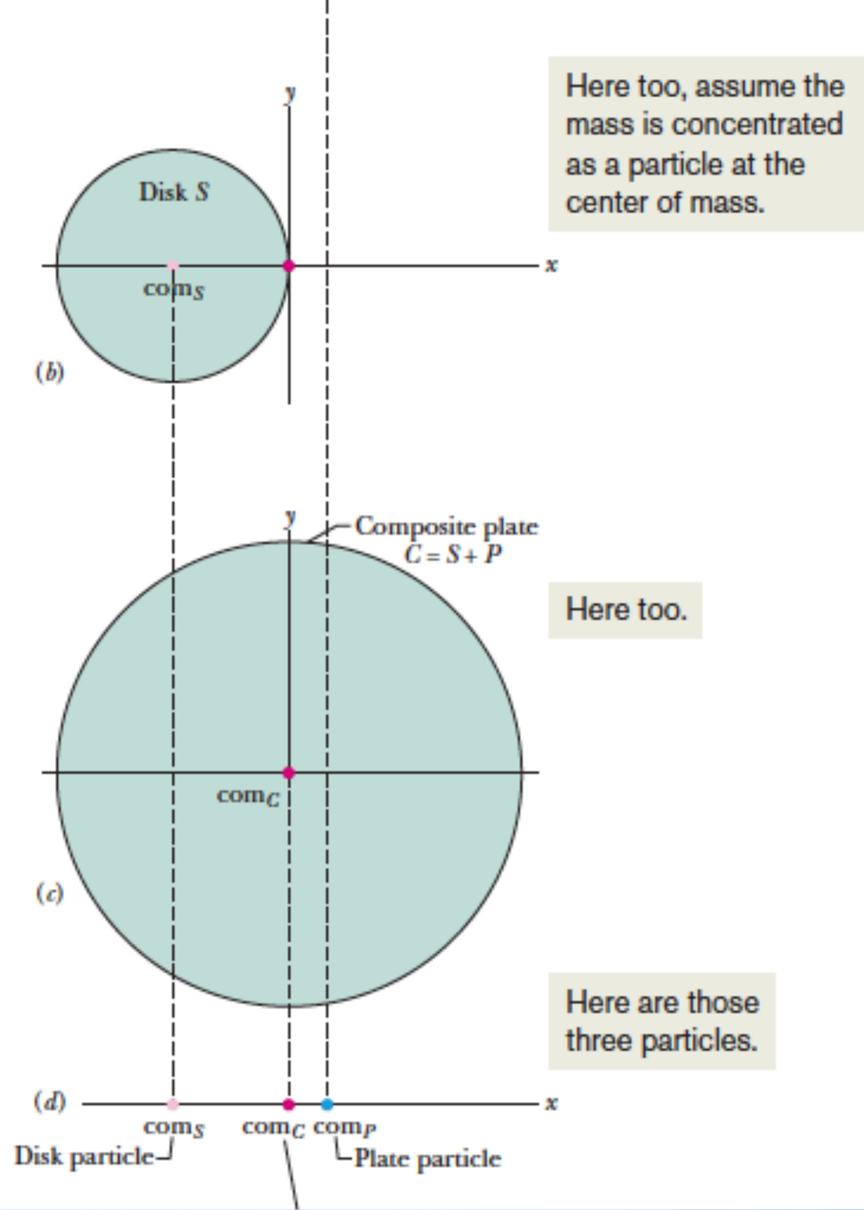


Example 4-1:


y
2R
R
x
com$_P$
Plate P
(a)
Assume the plate's mass is concentrated as a particle at the plate's center of mass.



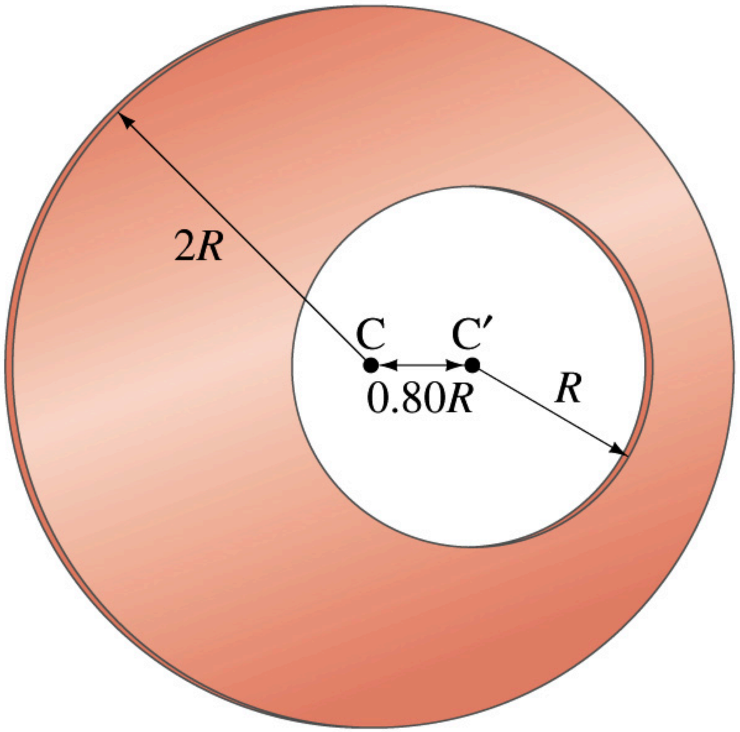
$$y_{CM} = 0$$



$$x_{CM} = \frac{0 \times \rho\pi(2R)^2 - (-R) \times \rho\pi(R)^2}{\rho\pi(2R)^2 - \rho\pi(R)^2} = \frac{1}{3}R$$

Example 4-2

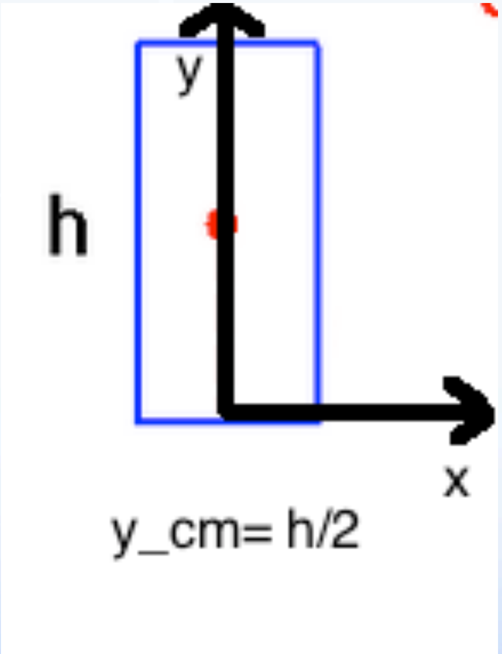
A uniform circular plate of radius $2R$ has a circular hole of radius R cut out of it. The center C′ of the smaller circle is a distance $0.80R$ from the center C of the larger circle, Fig. 9–45. What is the position of the center of mass of the plate?



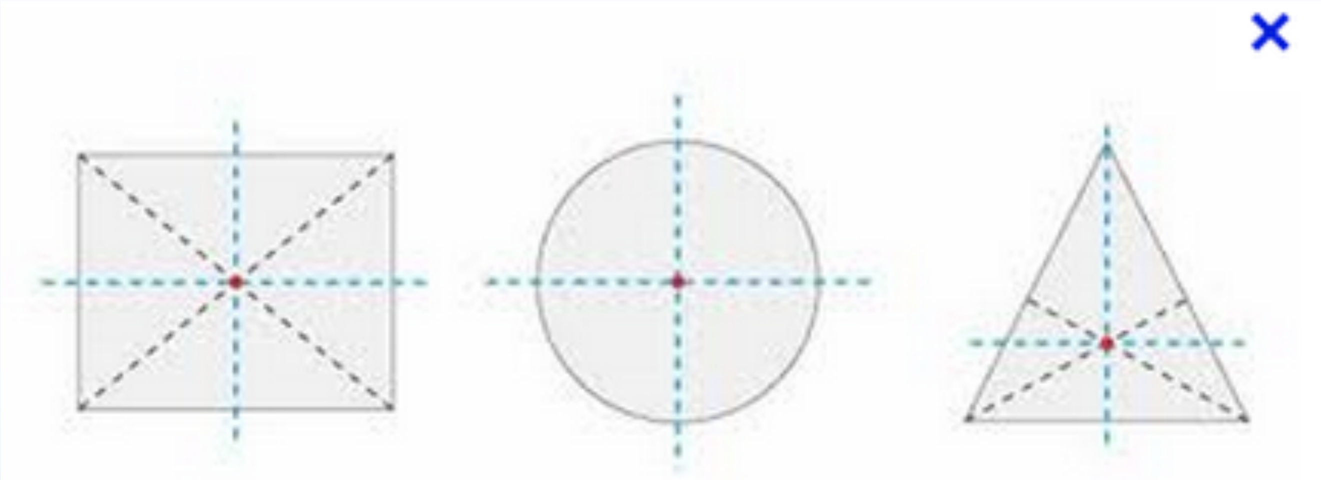
$$y_{com} = 0$$

$$x_{com} = -0.27R$$

Example 5:

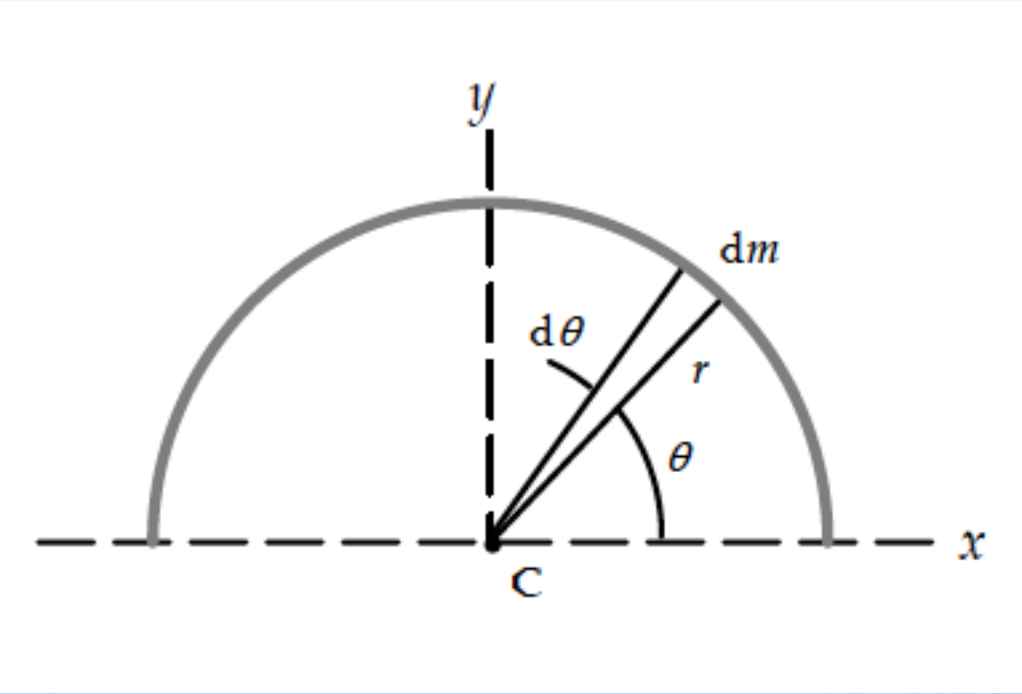


Example 6:

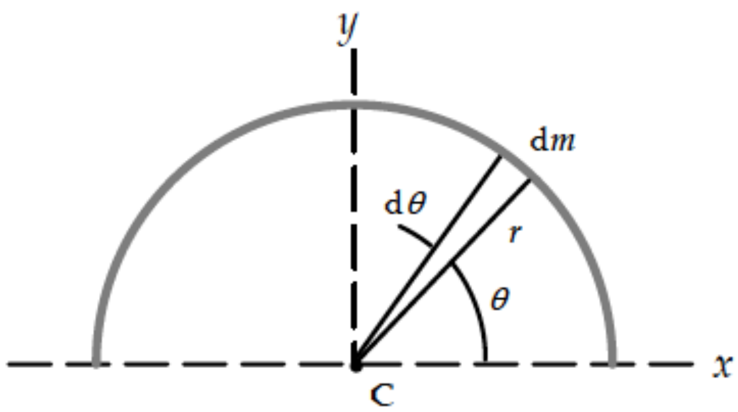


Example 7:

- Center of mass of semi circular wire:



From the symmetry of the wire, we know that $x_{CM} = 0$. Consider an infinitesimal piece of the wire, with mass dm, and coordinates $(x, y) = (r\cos\theta, r\sin\theta)$. If the length of that piece of wire is $d\ell$, then since the wire is uniform, we have $dm = \frac{M}{\pi r} d\ell$. And from the diagram and the definition of radian angle measure, we have $d\ell = r d\theta$.



Thus $dm = \frac{M}{\pi r} r d\theta = \frac{M}{\pi} d\theta$. Now apply Eq. 9-13.

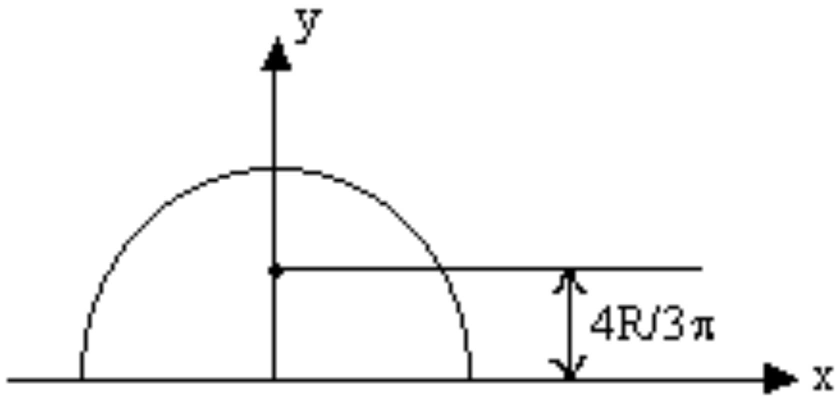
$$y_{CM} = \frac{1}{M}\int y\,dm = \frac{1}{M}\int_0^{\pi} r\sin\theta \frac{M}{\pi} d\theta = \frac{r}{\pi}\int_0^{\pi} \sin\theta d\theta = \frac{2r}{\pi}$$

Thus the coordinates of the center of mass are $(x_{CM}, y_{CM}) = \boxed{\left(0, \frac{2r}{\pi}\right)}$.

✔

Example 8:

C) Center of Mass of a uniform semi circular plate:

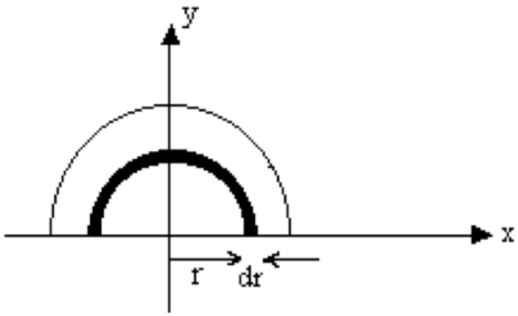


$$X_{COM} = 0 \quad \text{(By symmetry)}$$

$$Y_{COM} = \frac{4R}{3\pi}$$

✔

Derivation:



Here the element chosen is a thin wire (semi circular) of radius r.

As derived earlier, the Y_{COM} for this is at $\frac{2r}{\pi}$.

$$\text{Now, } dm = \frac{m}{\frac{\pi R^2}{2}} \times \pi r dr$$

$$= \frac{2mrdr}{R^2}$$

$$\text{So, } Y_{COM} = \int \frac{ydm}{m} = \frac{1}{m}\int_0^R \frac{2r}{\pi} \times \frac{2mr}{R^2} dr$$

$$= \frac{4}{\pi R^2} \cdot \left[\frac{r^3}{3}\right]_0^R$$

$$= \frac{4R}{3\pi}$$

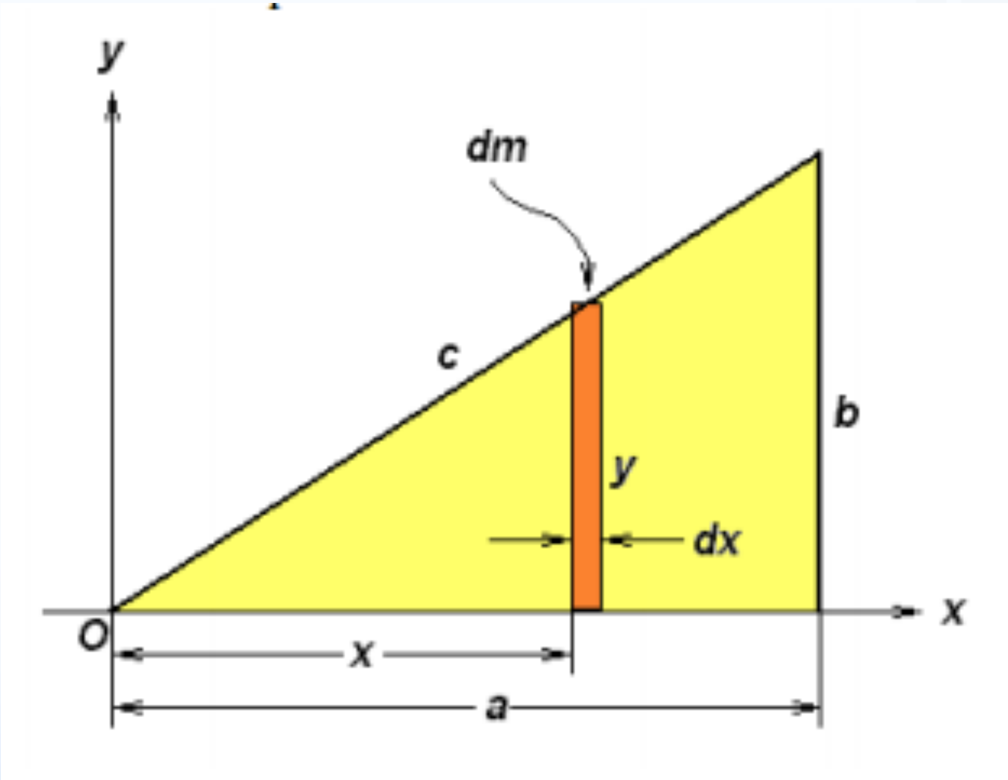
Dumb Question:

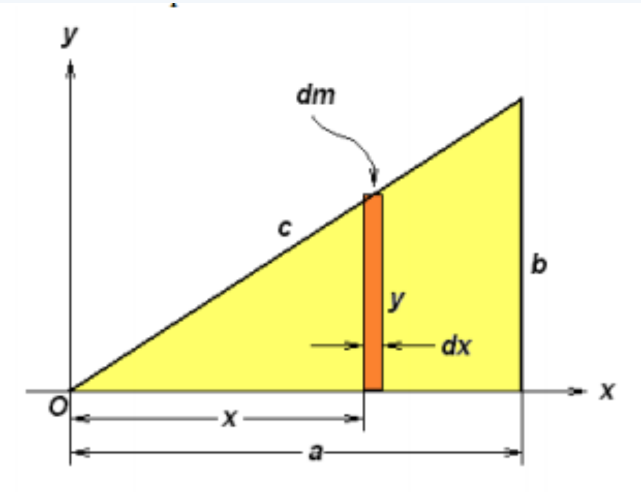
1) Why is $y = \frac{2r}{\pi}$ here?

Ans: The mass dm is a semi circular thin wire whose position is variable (y is not unique),

so we concentrate dm mass on the COM of this wire that is at $y = \frac{2r}{\pi}$.

Example 9:





$$dm = \frac{M}{A}\, dA$$

$$= \frac{M}{1/2\, ab}\,(y\, dx) = \left(\frac{2M}{ab}\right) y\, dx$$

$$x_{CM} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^a x\left(\frac{2M}{ab}\right) y\, dx = \frac{2}{ab}\int_0^a xy\, dx$$

$$\frac{y}{x} = \frac{b}{a} \qquad \text{or} \qquad y = \frac{b}{a}x$$

$$x_{CM} = \frac{2}{ab}\int_0^a x\left(\frac{b}{a}x\right) dx = \frac{2}{a^2}\int_0^a x^2\, dx = \frac{2}{a^2}\left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{2}{3}a$$

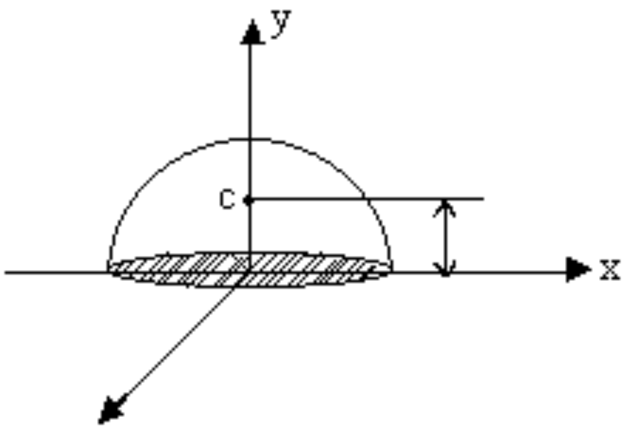
✔

$$x_{CM} = \left(\frac{2}{3}a, \frac{1}{3}b\right)$$

Example 10: Center of mass of hollow hemisphere

$X_{COM} = 0$ (by symmetry)

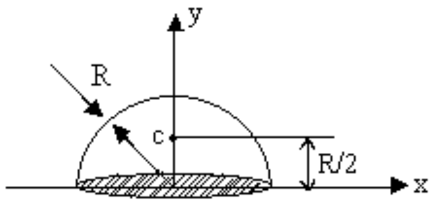
$Y_{COM} = \ ?$

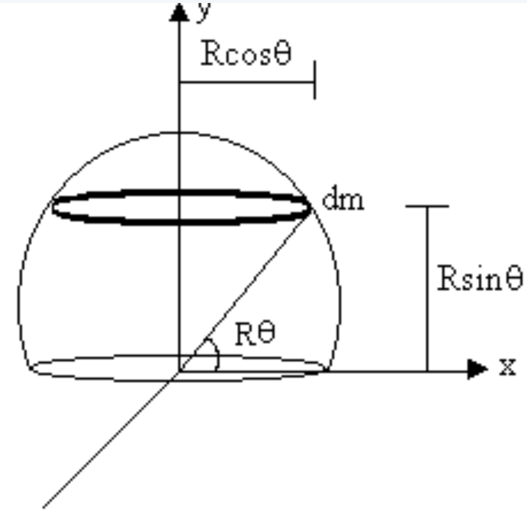


Center of Mass of hemispherical shell:

$X_{COM} = 0$ (by symmetry)

$Y_{COM} = \frac{R}{2}$





$$dm = (2\pi R\cos\theta)Rd\theta \times \frac{m}{2\pi R^2}$$

$$= \cos\theta\, m d\theta$$

$$y_{COM} = \frac{1}{m}\int y dm$$

$$= \frac{1}{m}\int_0^{\pi/2} R\sin\theta\cos\theta\, m d\theta$$

$$= R/2\int_0^{\pi/2} \sin 2\theta\, d\theta$$

$$= R/2\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= R/2\left[+\frac{1}{2}+\frac{1}{2}\right]$$

$$= \frac{R}{2}$$

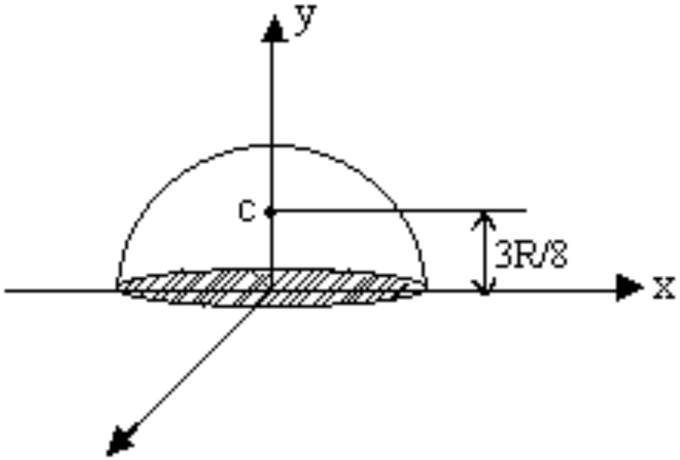
Example 11:

Center of Mass of a hemisphere:

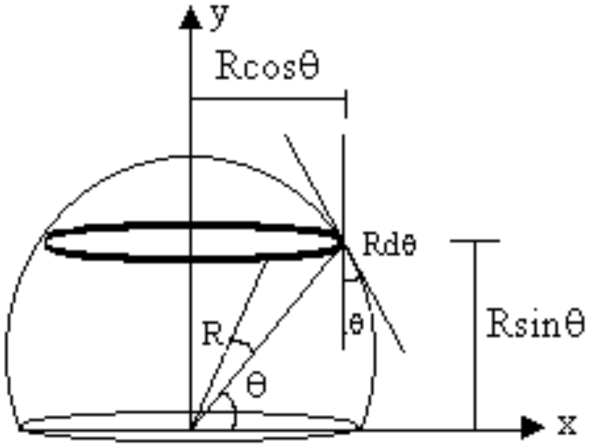
$$X_{COM} = 0 \quad \text{(by symmetry)}$$

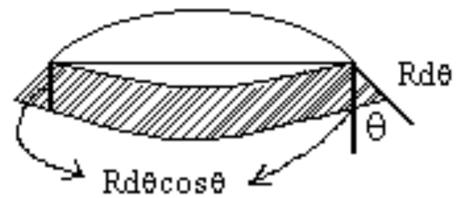
$$Y_{COM} = \frac{3R}{8}$$

✔



Derivation:





$$dm = \frac{m}{\frac{2}{3}\pi R^3}\pi(R\cos\theta)^2 R d\theta\cos\theta$$

$$y = R\sin\theta$$

$$Y_{COM} = \frac{1}{m}\int y\,dm$$

$$= \frac{1}{m}\int \frac{3m\,\pi R^3\cos^3\theta\,(R\sin\theta)d\theta}{2\pi R^3}$$

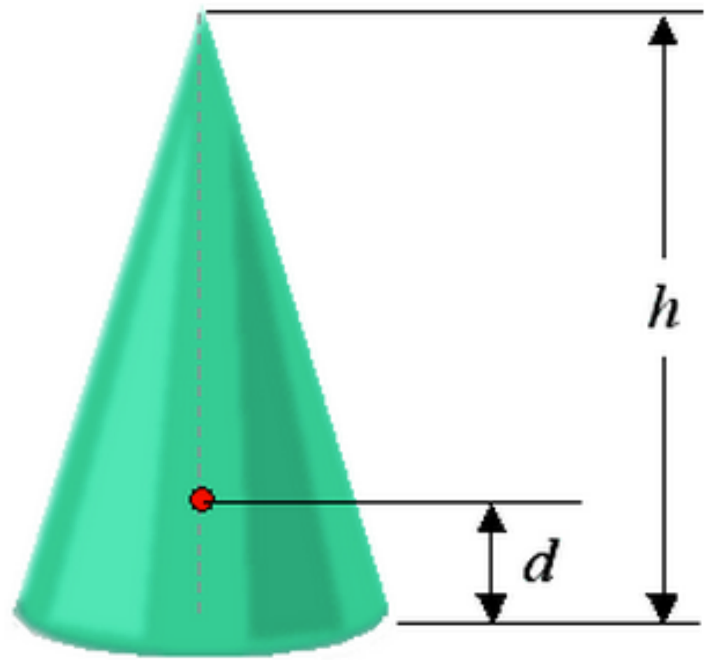
$$= \frac{3}{2}R\int_0^{\pi/2}\cos^3\theta\sin\theta\,d\theta$$

$$= \frac{3}{2}R\int_0^1 t^3 dt$$

$$= \frac{3R}{8}$$

Example 14-15:

Solid Cone or Pyramid of height h



$$d = \frac{h}{4}$$

$$d = \frac{h}{3} \quad (hollow)$$

For hollow cone,

Let (X_c, Y_c) are the coordinates of center of mass. Then,

$$Y_c = \frac{\int y\,dm}{\int dm} \text{ and } X_c = \frac{\int x\,dm}{\int dm}$$

For symmetry consider the center of mass at $X_c=0$.

For hollow cone,

$dm = \rho\, dA$

$$dA = 2\pi r\, dy$$

$$Y_c = \frac{\int y(2\pi r dy)}{\int 2\pi r dy} = \frac{\int_0^h y(2\pi R(1-\frac{y}{h})dy)}{\int_0^h 2\pi R(1-\frac{y}{h})dy} = \frac{\frac{h^2}{6}}{\frac{h}{2}} = \frac{1}{3}h$$

So, the center of mass of a hollow cone is at one-third of the height on the line joining from center of the base to the vertex.

For a solid cone,

Let (X_c, Y_c) are the coordinates of center of mass. Then,

$$Y_c = \frac{\int y\,dm}{\int dm} \text{ and } X_c = \frac{\int x\,dm}{\int dm}$$

For symmetry consider the center of mass at $X_c=0$.

Divide the cone in horizontal disks of mass then,

$dm = \rho dV$ where, ρ is the density of material. (constant)

In this case,

$dV = \frac{1}{3}\pi r^2 dy$ �where r is the radius of the cone at an arbitrary height *dy*.

radius of cone depends upon the height of the cone. So,

$y=0$; $r=R$ (R is the base radius of the cone)

$y=h$; $r=0$.

Using point-slope form:

$$r = -\frac{R}{h}y + R$$

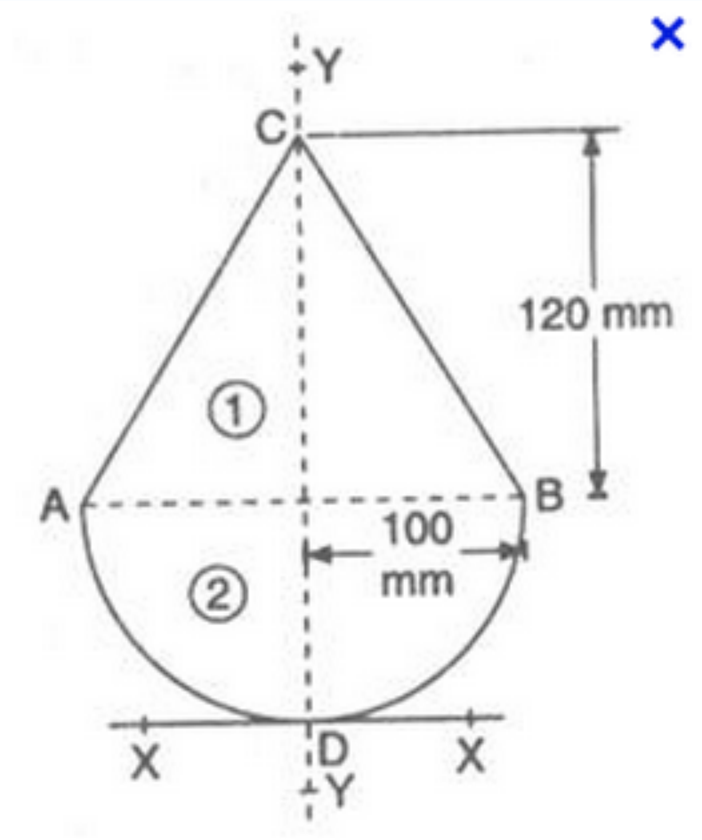
Now, the volume�

$$dV = \frac{1}{3}\pi r^2 dy$$

So,

$$Y_c = \frac{\int y(\frac{1}{3}\pi r^2 dy)}{\int \frac{1}{3}\pi r^2 dy} = \frac{\int_0^h y(\frac{1}{3}\pi(R(1-\frac{y}{h}))^2 dy)}{\int_0^h \frac{1}{3}\pi(R(1-\frac{y}{h}))^2 dy} = \frac{\frac{1}{3}\pi R^2 \frac{h^2}{12}}{\frac{1}{2}\pi R^2 \frac{h}{3}} = \frac{1}{4}h$$

Example 16:



9-2 NEWTON'S SECOND LAW FOR A SYSTEM OF PARTICLES

Key Idea

- The motion of the center of mass of any system of particles is governed by Newton's second law for a system of particles, which is

$$\vec{F}_{net} = M\vec{a}_{com}.$$

9-2 Newton's Second law for a System of Particles

- The center of mass of a system of particles having combined mass M moves like an equivalent particle of mass M would move under the influence of the net external force on the system.

$$\vec{\mathbf{r}}_{CM} \equiv \frac{1}{M}\sum_i m_i \vec{\mathbf{r}}_i$$

Velocity of the center of mass of a system of particles ▶

$$\vec{\mathbf{v}}_{CM} = \frac{d\vec{\mathbf{r}}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{\mathbf{r}}_i}{dt} = \frac{1}{M}\sum_i m_i \vec{\mathbf{v}}_i$$

Acceleration of the center of mass of a system of particles ▶

$$\vec{\mathbf{a}}_{CM} = \frac{d\vec{\mathbf{v}}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{\mathbf{v}}_i}{dt} = \frac{1}{M}\sum_i m_i\vec{\mathbf{a}}_i$$

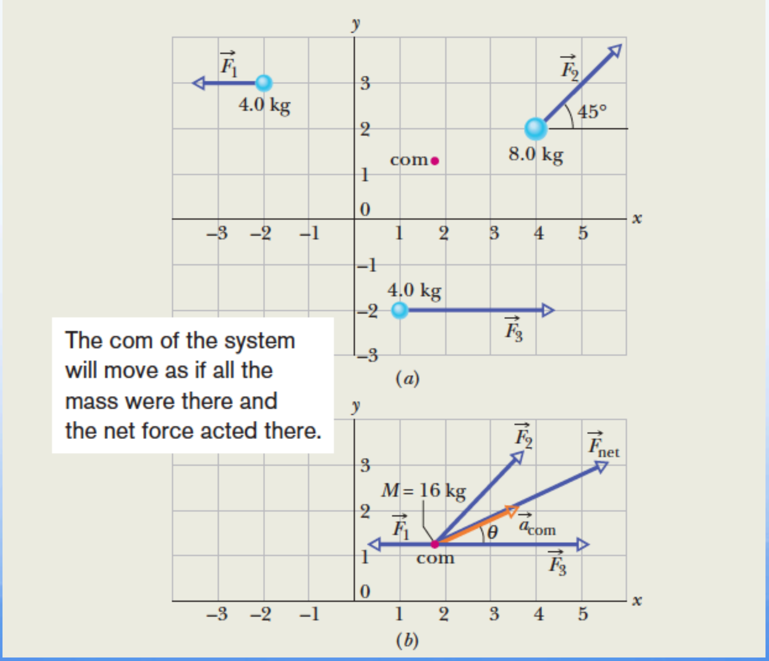
$$M\vec{\mathbf{a}}_{CM} = \sum_i m_i\vec{\mathbf{a}}_i = \sum_i \vec{\mathbf{F}}_i$$

$$\sum \vec{\mathbf{F}}_{ext} = M\vec{\mathbf{a}}_{CM}$$

✔

Sample Problem 9.03 Motion of the com of three particles

The three particles in Fig. 9-7*a* are initially at rest. Each experiences an *external* force due to bodies outside the three-particle system. The directions are indicated, and the magnitudes are $F_1 = 6.0$ N, $F_2 = 12$ N, and $F_3 = 14$ N. What is the acceleration of the center of mass of the system, and in what direction does it move?



Calculations: We can now apply Newton's second law ($\vec{F}_{net} = m\vec{a}$) to the center of mass, writing

$$\vec{F}_{net} = M\vec{a}_{com} \tag{9-20}$$

or

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = M\vec{a}_{com}$$

so

$$\vec{a}_{com} = \frac{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}{M}. \tag{9-21}$$

$$a_{com,x} = \frac{F_{1x} + F_{2x} + F_{3x}}{M} = \frac{-6.0\text{ N} + (12\text{ N})\cos 45° + 14\text{ N}}{16\text{ kg}} = 1.03\text{ m/s}^2.$$

Along the y axis, we have

$$a_{com,y} = \frac{F_{1y} + F_{2y} + F_{3y}}{M} = \frac{0 + (12\text{ N})\sin 45° + 0}{16\text{ kg}} = 0.530\text{ m/s}^2.$$

From these components, we find that $\vec{a}_{com}$ has the magnitude

$$a_{com} = \sqrt{(a_{com,x})^2 + (a_{com,y})^2} = 1.16\text{ m/s}^2 \approx 1.2\text{ m/s}^2 \qquad \text{(Answer)}$$

and the angle (from the positive direction of the x axis)

$$\theta = \tan^{-1}\frac{a_{com,y}}{a_{com,x}} = 27°. \qquad \text{(Answer)}$$

9-3 Linear Momentum

The **linear momentum** of a particle or an object that can be modeled as a particle of mass m moving with a velocity $\vec{\mathbf{v}}$ is defined to be the product of the mass and velocity of the particle:

$$\vec{\mathbf{p}} \equiv m\vec{\mathbf{v}}$$

✔

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z$$

$$\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}} = m\frac{d\vec{\mathbf{v}}}{dt}$$

$$\sum \vec{\mathbf{F}} = \frac{d(m\vec{\mathbf{v}})}{dt} = \frac{d\vec{\mathbf{p}}}{dt}$$

✔

The Linear Momentum of a System of Particles

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_n$$
$$= m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n.$$

Velocity of the center of mass of a system of particles ▶

$$\vec{\mathbf{v}}_{\mathrm{CM}} = \frac{d\vec{\mathbf{r}}_{\mathrm{CM}}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{\mathbf{r}}_i}{dt} = \frac{1}{M}\sum_i m_i\vec{\mathbf{v}}_i$$

Total momentum of a system of particles ▶

$$M\vec{\mathbf{v}}_{\mathrm{CM}} = \sum_i m_i\vec{\mathbf{v}}_i = \sum_i \vec{\mathbf{p}}_i = \vec{\mathbf{p}}_{\mathrm{tot}}$$

$$\vec{P} = M\vec{v}_{\mathrm{com}} \qquad \text{(linear momentum, system of particles)},$$

✔

The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the center of mass.

Force and Momentum
Conservation of Linear momentum

$$\vec{P} = M\vec{v}_{\text{com}} \qquad \text{(linear momentum, system of particles),}$$

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}_{\text{com}}}{dt} = M\vec{a}_{\text{com}}.$$

$$\vec{F}_{\text{net}} = \frac{d\vec{P}}{dt} \qquad \text{(system of particles),}$$

If $\vec{F}_{net} = 0 \quad \Longrightarrow \quad \vec{P} = c\vec{o}n.$ ✔

$$\sum \vec{\mathbf{F}} = \frac{d(m\vec{\mathbf{v}})}{dt} = \frac{d\vec{\mathbf{p}}}{dt}$$

$$M\vec{\mathbf{v}}_{\mathrm{CM}} = \vec{\mathbf{p}}_{\mathrm{tot}} = \text{constant} \quad \left(\text{when } \sum \vec{\mathbf{F}}_{\mathrm{ext}} = 0\right)$$

$$\vec{P} = \text{constant} \qquad \text{(closed, isolated system).}$$

✔

$$\vec{P}_i = \vec{P}_f \qquad \text{(closed, isolated system).}$$

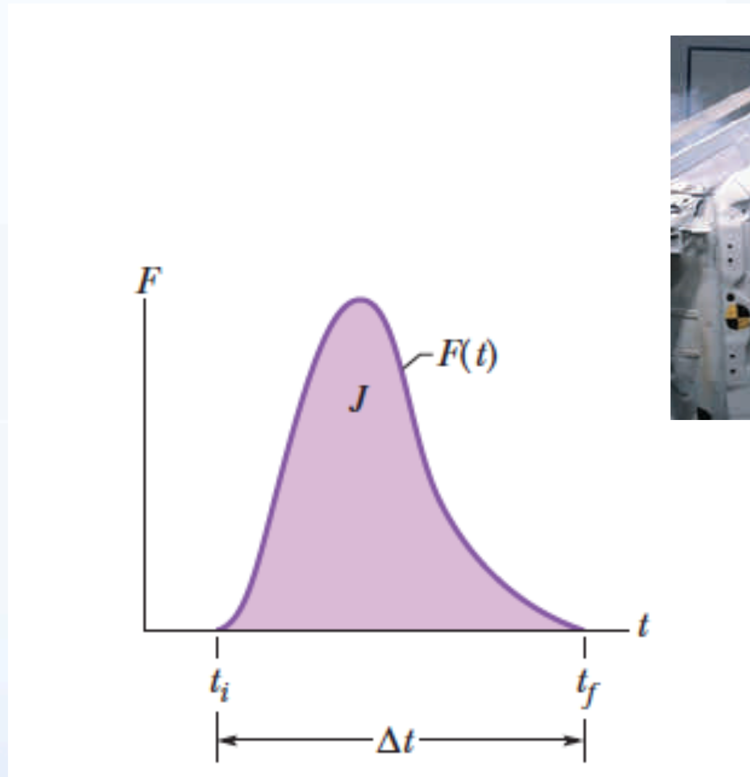
$$\begin{pmatrix}\text{total linear momentum}\\ \text{at some initial time } t_i\end{pmatrix} = \begin{pmatrix}\text{total linear momentum}\\ \text{at some later time } t_f\end{pmatrix}.$$

9-4 Collision and Impulse

$$d\vec{p} = \vec{F}(t)\,dt.$$

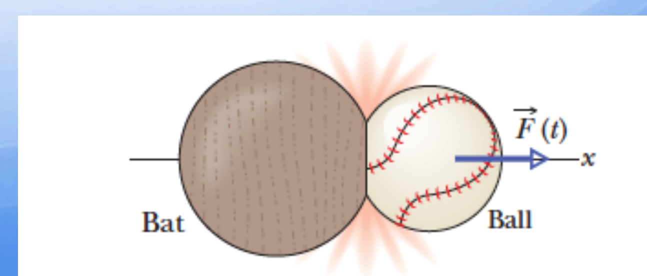
$$\int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)\,dt.$$

$$\vec{p}_f - \vec{p}_i = \vec{J}$$



$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)\,dt \quad \text{(impulse defined).}$$

✔



Impulse Vector

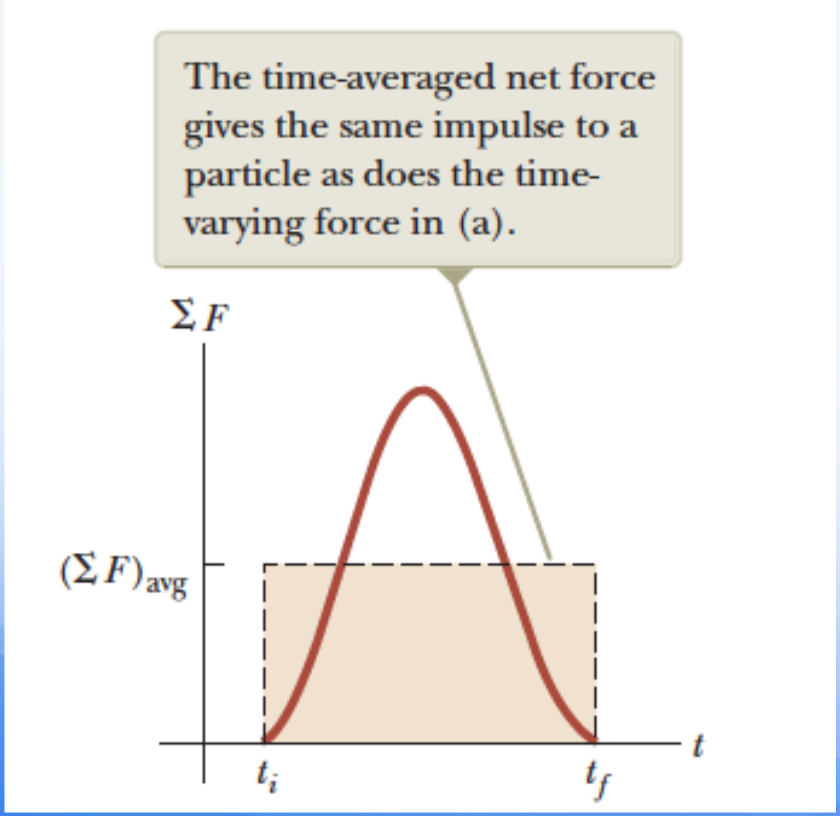
$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)\, dt \quad \text{(impulse defined).}$$

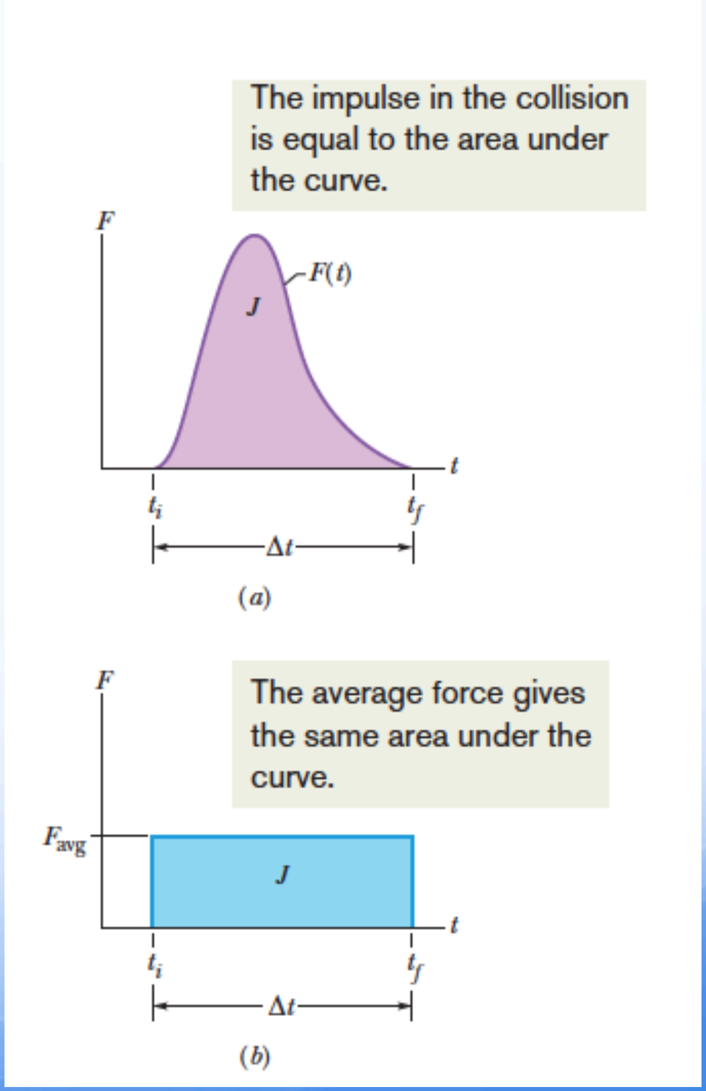
$$\Delta p_x = J_x$$

$$p_{fx} - p_{ix} = \int_{t_i}^{t_f} F_x\, dt.$$

Average Impulse

$$J = F_{avg}\,\Delta t.$$






ADAC
test

Example:

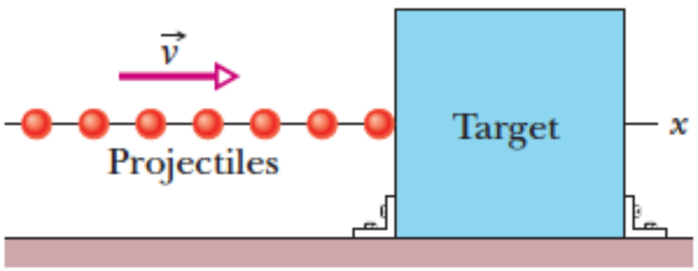


Figure 9-10 A steady stream of projectiles, with identical linear momenta, collides with a target, which is fixed in place. The average force F_{avg} on the target is to the right and has a magnitude that depends on the rate at which the projectiles collide with the target or, equivalently, the rate at which mass collides with the target.

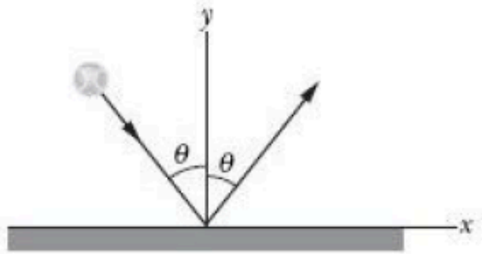
$$J = -n\,\Delta p,$$

$$F_{avg} = \frac{J}{\Delta t} = -\frac{n}{\Delta t}\Delta p = -\frac{n}{\Delta t} m\,\Delta v.$$

✔ $$\Delta v = v_f - v_i = 0 - v = -v,$$

✔ $$\Delta v = v_f - v_i = -v - v = -2v.$$

$$F_{avg} = -\frac{\Delta m}{\Delta t}\Delta v.$$

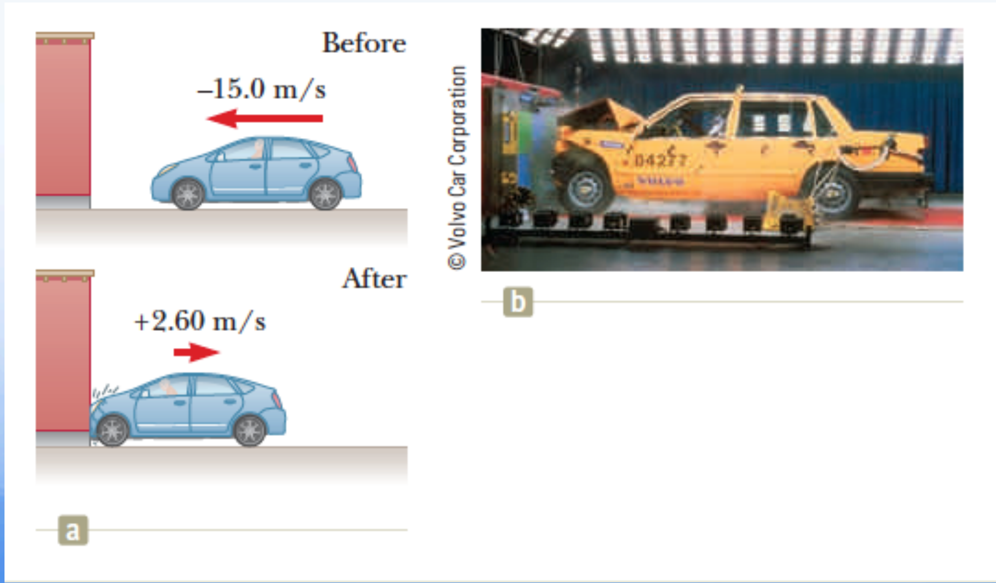


01

CHECKPOINT 5 The figure shows an overhead view of a ball bouncing from a vertical wall without any change in its speed. Consider the change $\Delta\vec{p}$ in the ball's linear momentum. (a) Is Δp_x positive, negative, or zero? (b) Is Δp_y positive, negative, or zero? (c) What is the direction of $\Delta\vec{p}$?

- Example:

In a particular crash test, a car of mass 1 500 kg collides with a wall as shown in Figure. The initial and final velocities of the car are $\vec{\mathbf{v}}_i = -15.0\hat{\mathbf{i}}$ m/s and $\vec{\mathbf{v}}_f = 2.60\hat{\mathbf{i}}$ m/s, respectively. If the collision lasts 0.150 s, find the impulse caused by the collision and the average net force exerted on the car.



$$\vec{\mathbf{p}}_i = m\vec{\mathbf{v}}_i = (1\,500\text{ kg})(-15.0\hat{\mathbf{i}}\text{ m/s}) = -2.25 \times 10^4\hat{\mathbf{i}}\text{ kg}\cdot\text{m/s}$$

$$\vec{\mathbf{p}}_f = m\vec{\mathbf{v}}_f = (1\,500\text{ kg})(2.60\hat{\mathbf{i}}\text{ m/s}) = 0.39 \times 10^4\hat{\mathbf{i}}\text{ kg}\cdot\text{m/s}$$

$$\vec{\mathbf{I}} = \Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i = 0.39 \times 10^4\hat{\mathbf{i}}\text{ kg}\cdot\text{m/s} - (-2.25 \times 10^4\hat{\mathbf{i}}\text{ kg}\cdot\text{m/s})$$

$$= \boxed{2.64 \times 10^4\hat{\mathbf{i}}\text{ kg}\cdot\text{m/s}}$$

$$\left(\sum\vec{\mathbf{F}}\right)_{\text{avg}} = \frac{\vec{\mathbf{I}}}{\Delta t} = \frac{2.64 \times 10^4\hat{\mathbf{i}}\text{ kg}\cdot\text{m/s}}{0.150\text{ s}} = \boxed{1.76 \times 10^5\hat{\mathbf{i}}\text{ N}}$$


UNITED MARTIAL ARTS

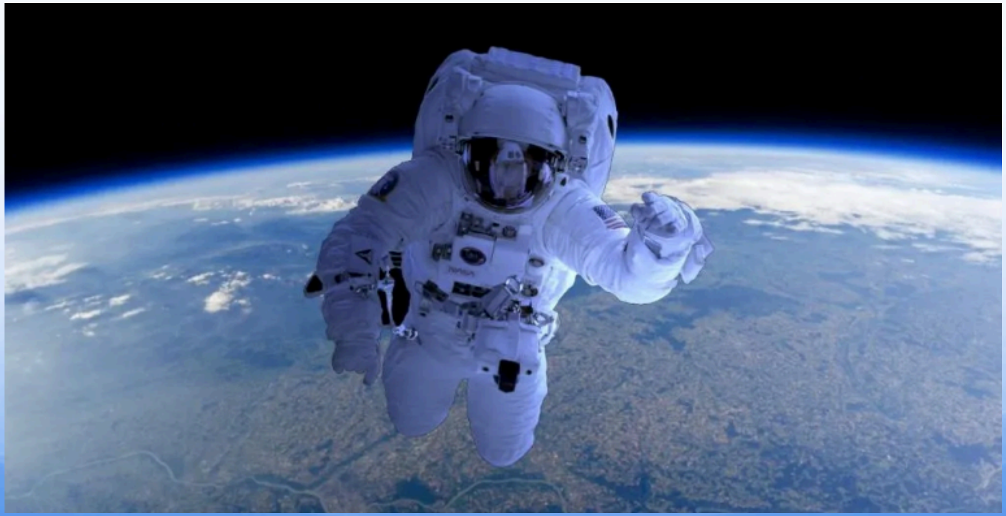
Example:

Figure 9.2 (Example 9.1) An archer fires an arrow horizontally to the right. Because he is standing on frictionless ice, he will begin to slide to the left across the ice.

Let us consider the situation proposed at the beginning of Section 9.1. A 60-kg archer stands at rest on frictionless ice and fires a 0.50-kg arrow horizontally at 50 m/s (Fig. 9.2). With what velocity does the archer move across the ice after firing the arrow?

$$m_1\vec{\mathbf{v}}_{1f} + m_2\vec{\mathbf{v}}_{2f} = 0$$

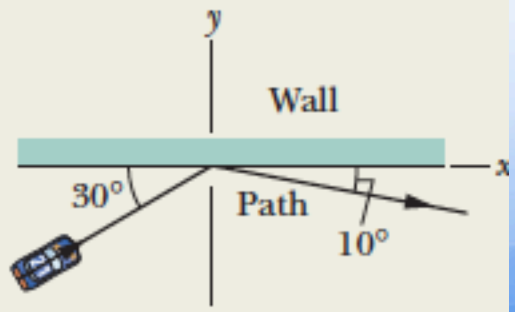
$$\vec{\mathbf{v}}_{1f} = -\frac{m_2}{m_1}\vec{\mathbf{v}}_{2f} = -\left(\frac{0.50\text{ kg}}{60\text{ kg}}\right)(50\hat{\mathbf{i}}\text{ m/s}) = \boxed{-0.42\hat{\mathbf{i}}\text{ m/s}}$$

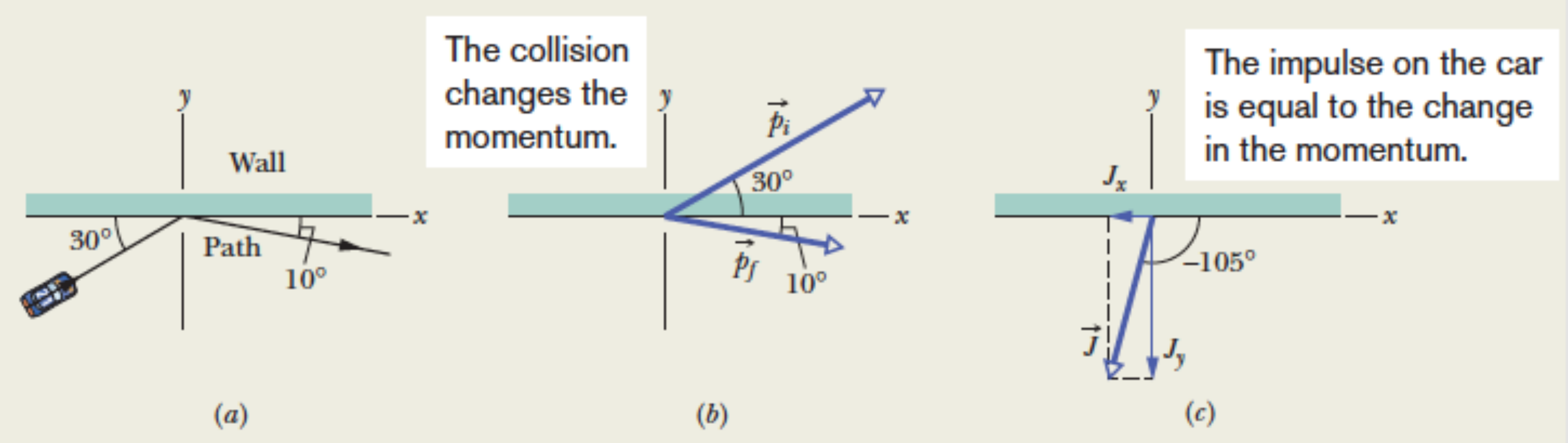


Example:

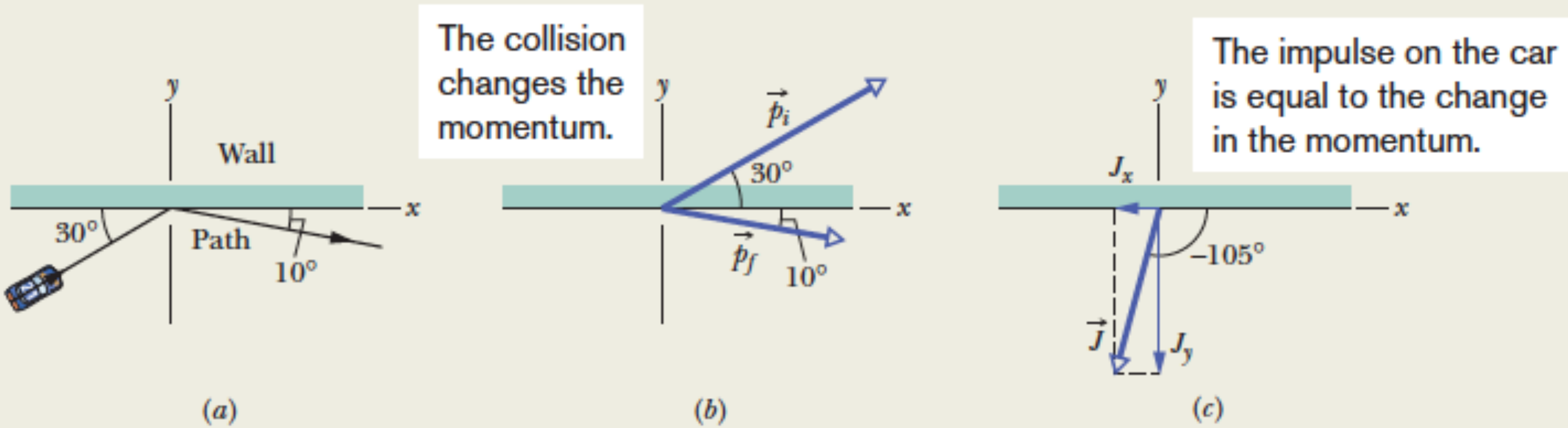
Race car–wall collision. Figure 9-11*a* is an overhead view of the path taken by a race car driver as his car collides with the racetrack wall. Just before the collision, he is traveling at speed $v_i = 70$ m/s along a straight line at 30° from the wall. Just after the collision, he is traveling at speed $v_f = 50$ m/s along a straight line at 10° from the wall. His mass m is 80 kg.

(a) What is the impulse $\vec{J}$ on the driver due to the collision?





$$\vec{J} = \vec{p}_f - \vec{p}_i = m\vec{v}_f - m\vec{v}_i = m(\vec{v}_f - \vec{v}_i).$$



***x* component:** Along the x axis we have

$$J_x = m(v_{fx} - v_{ix})$$
$$= (80\text{ kg})[(50\text{ m/s})\cos(-10°) - (70\text{ m/s})\cos 30°]$$
$$= -910\text{ kg}\cdot\text{m/s}.$$

***y* component:** Along the y axis,

$$J_y = m(v_{fy} - v_{iy})$$
$$= (80\text{ kg})[(50\text{ m/s})\sin(-10°) - (70\text{ m/s})\sin 30°]$$
$$= -3495\text{ kg}\cdot\text{m/s} \approx -3500\text{ kg}\cdot\text{m/s}.$$

$$\vec{J} = (-910\hat{\mathrm{i}} - 3500\hat{\mathrm{j}})\ \mathrm{kg \cdot m/s},$$

$$\theta = \tan^{-1}\frac{J_y}{J_x},$$

$$J = \sqrt{J_x^2 + J_y^2} = 3616\ \mathrm{kg \cdot m/s} \approx 3600\ \mathrm{kg \cdot m/s}.$$

$$\theta = -105^\circ.$$

(b) The collision lasts for 14 ms. What is the magnitude of the average force on the driver during the collision?

Calculations: We have

$$F_{avg} = \frac{J}{\Delta t} = \frac{3616\ \mathrm{kg \cdot m/s}}{0.014\ \mathrm{s}}$$

$$= 2.583 \times 10^5\ \mathrm{N} \approx 2.6 \times 10^5\ \mathrm{N}. \qquad \text{(Answer)}$$

Using $F = ma$ with $m = 80$ kg, you can show that the magnitude of the driver's average acceleration during the collision is about $3.22 \times 10^3\ \mathrm{m/s^2} = 329g$, which is fatal.

9-5 Conservation of Linear Momentum

$$\sum \vec{\mathbf{F}} = \frac{d(m\vec{\mathbf{v}})}{dt} = \frac{d\vec{\mathbf{p}}}{dt}$$

$$M\vec{\mathbf{v}}_{\mathrm{CM}} = \vec{\mathbf{p}}_{\mathrm{tot}} = \text{constant} \quad \left(\text{when } \sum \vec{\mathbf{F}}_{\mathrm{ext}} = 0\right)$$

$$\vec{P} = \text{constant} \quad \text{(closed, isolated system).}$$

✔

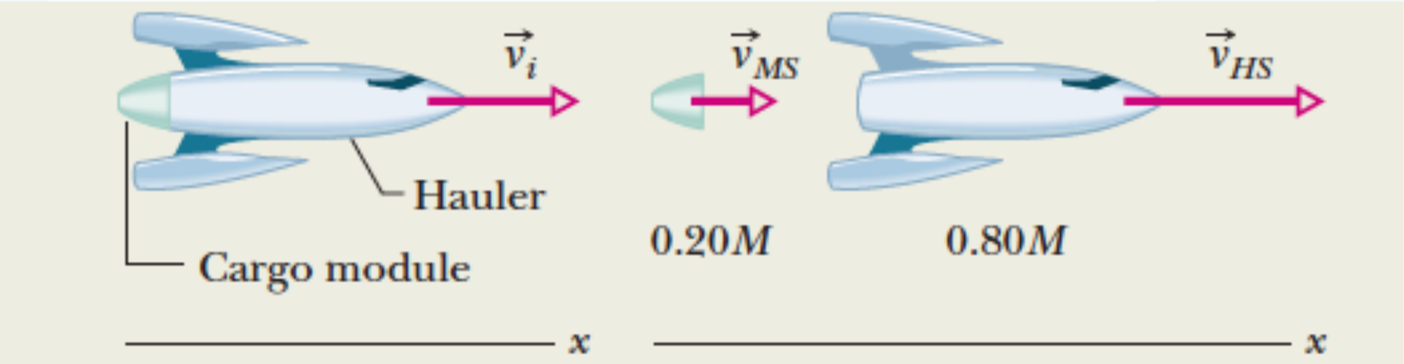
$$\vec{P}_i = \vec{P}_f \quad \text{(closed, isolated system).}$$

$$\begin{pmatrix}\text{total linear momentum} \\ \text{at some initial time } t_i\end{pmatrix} = \begin{pmatrix}\text{total linear momentum} \\ \text{at some later time } t_f\end{pmatrix}.$$

Conservation of Linear Momentum

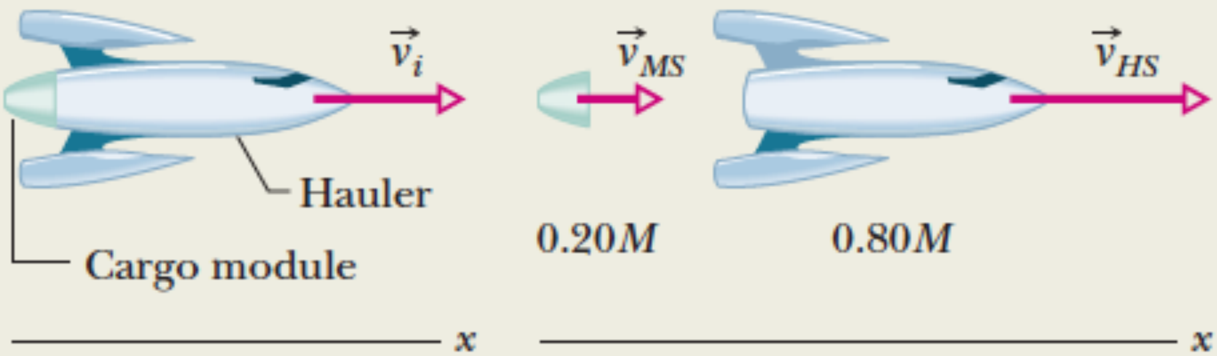
- If the component of the net external force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.

Example:



One-dimensional explosion: Figure 9-12*a* shows a space hauler and cargo module, of total mass M, traveling along an x axis in deep space. They have an initial velocity $\vec{v}_i$ of magnitude 2100 km/h relative to the Sun. With a small explosion, the hauler ejects the cargo module, of mass $0.20M$ (Fig. 9-12*b*). The hauler then travels 500 km/h faster than the module along the x axis; that is, the relative speed v_{rel} between the hauler and the module is 500 km/h. What then is the velocity $\vec{v}_{HS}$ of the hauler relative to the Sun?

The explosive separation can change the momentum of the parts but not the momentum of the system.

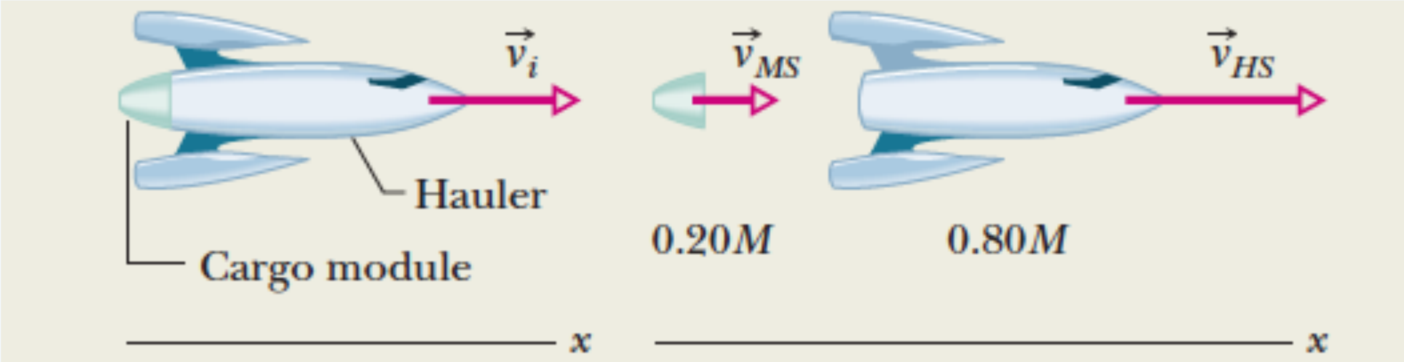


(a) (b)

$$\vec{P}_i = \vec{P}_f,$$

$$P_i = Mv_i.$$

$$P_f = (0.20M)v_{MS} + (0.80M)v_{HS},$$



$$\begin{pmatrix}\text{velocity of}\\ \text{hauler relative}\\ \text{to Sun}\end{pmatrix} = \begin{pmatrix}\text{velocity of}\\ \text{hauler relative}\\ \text{to module}\end{pmatrix} + \begin{pmatrix}\text{velocity of}\\ \text{module relative}\\ \text{to Sun}\end{pmatrix}.$$

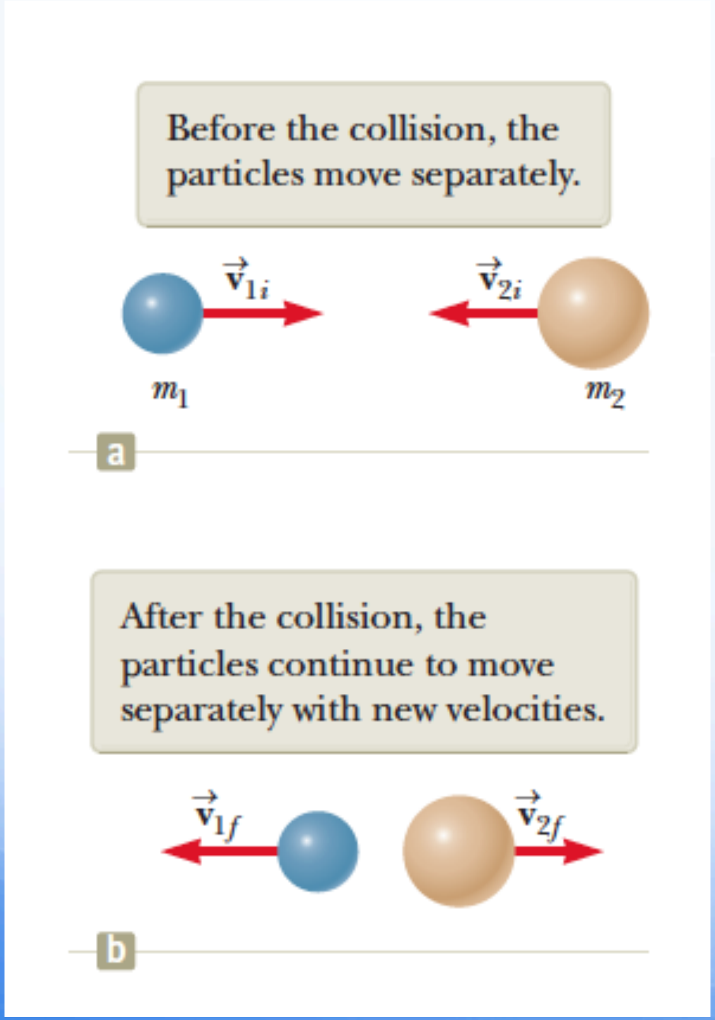
$$v_{HS} = v_{\text{rel}} + v_{MS}$$
$$v_{MS} = v_{HS} - v_{\text{rel}}.$$

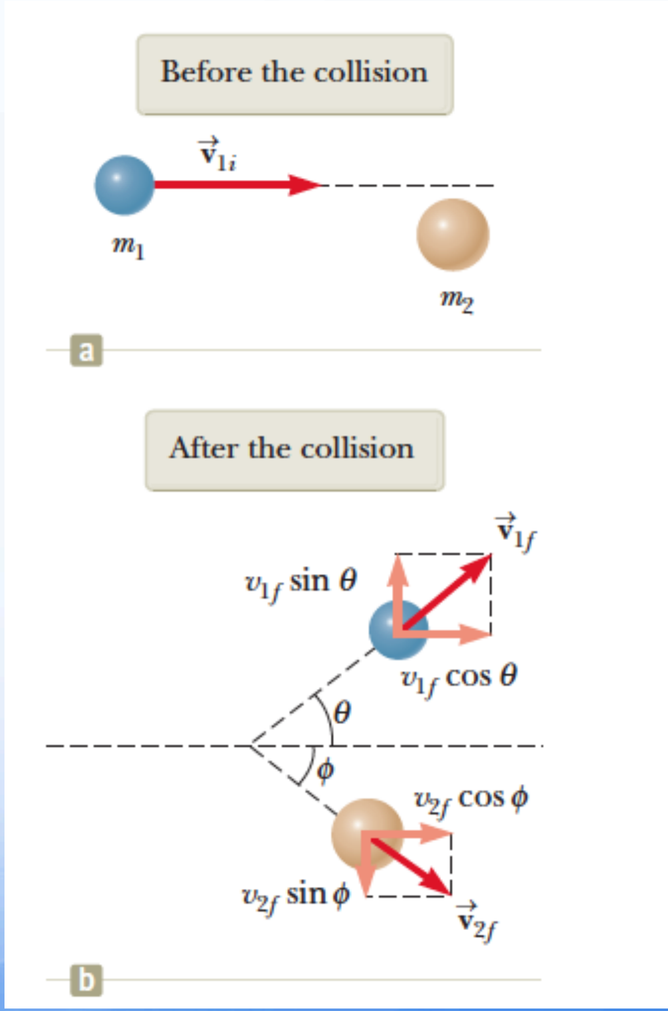
$$Mv_i = 0.20M(v_{HS} - v_{\text{rel}}) + 0.80Mv_{HS},$$

$$\begin{aligned} v_{HS} &= v_i + 0.20v_{\text{rel}}, \\ v_{HS} &= 2100\ \text{km/h} + (0.20)(500\ \text{km/h}) \\ &= 2200\ \text{km/h}. \end{aligned}$$

Collision

9-6 Momentum and Kinetic Energy in Collisions

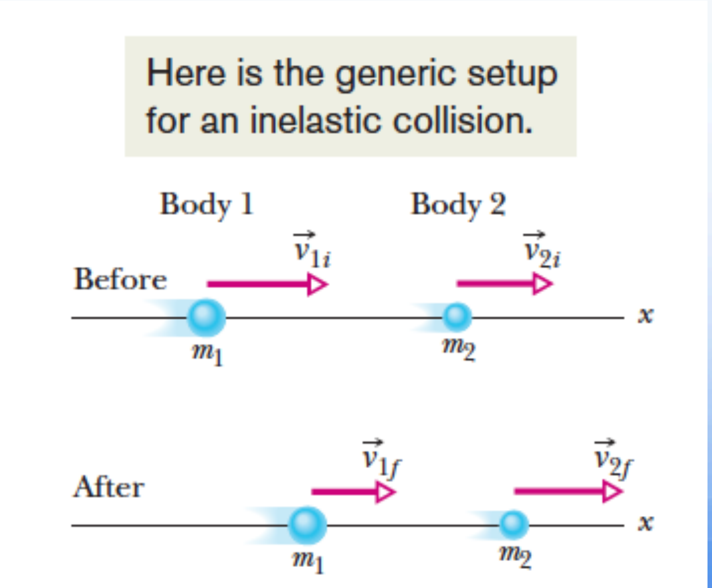


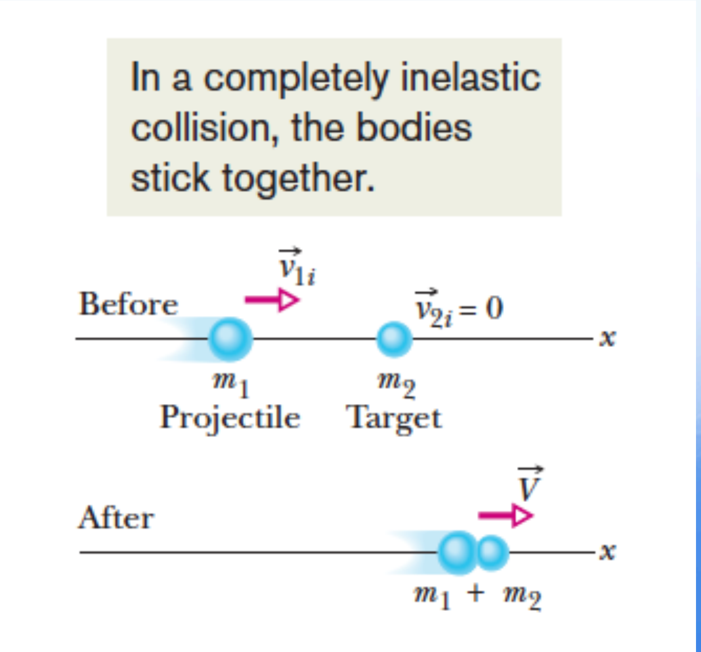


Collisions

- Elastic Collision, P and K are conserved
- Inelastic Collision, K is not conserved
- Completely Inelastic Collision, K is not conserved

✔





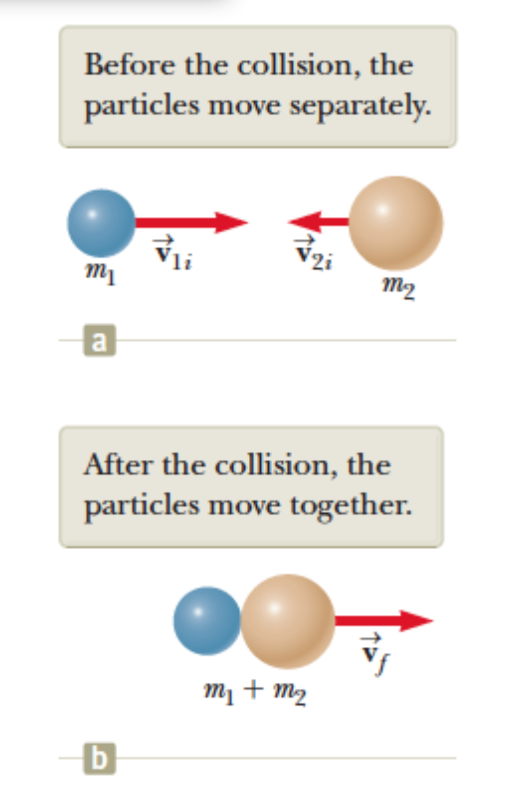
Collisions in One Dimension

- Completely inelastic collision

$$m_1\vec{\mathbf{v}}_{1i} + m_2\vec{\mathbf{v}}_{2i} = (m_1 + m_2)\vec{\mathbf{v}}_f$$

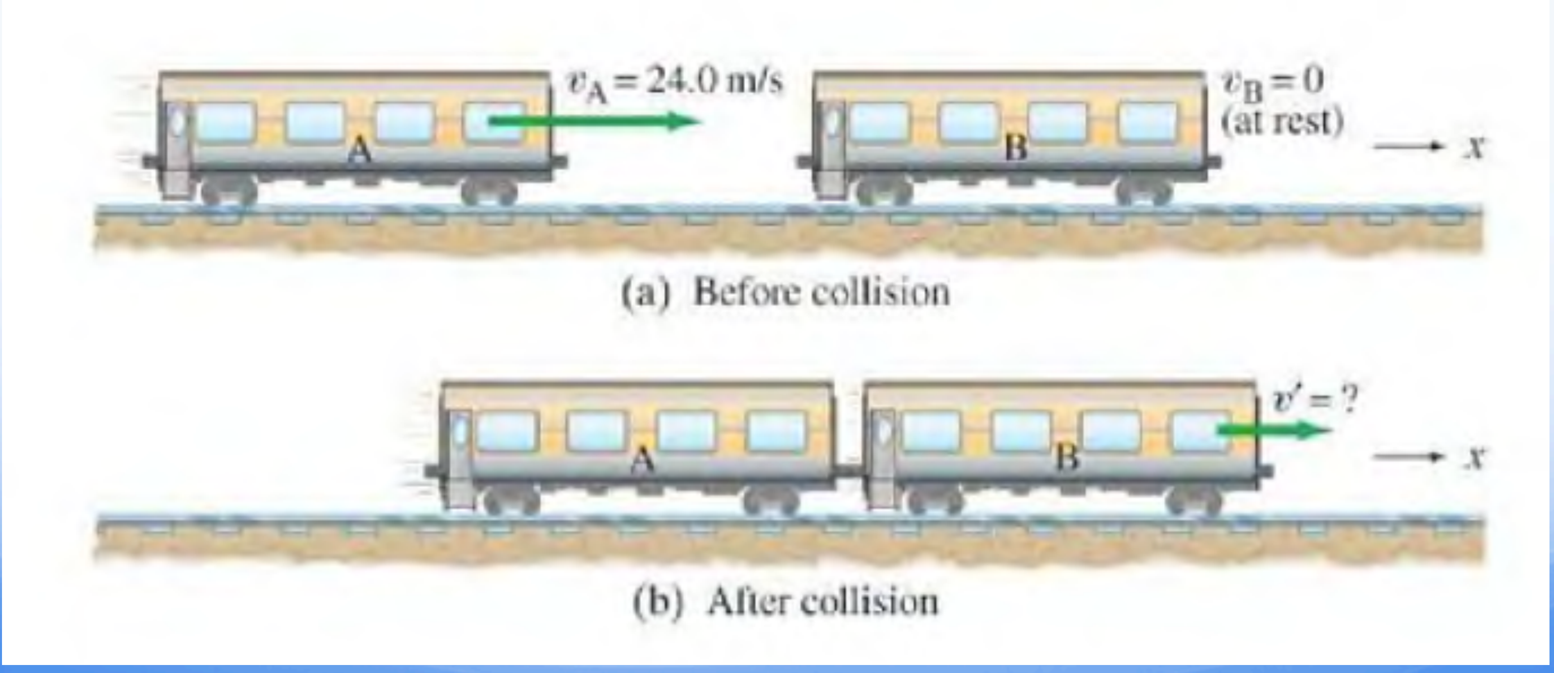
$$\vec{\mathbf{v}}_f = \frac{m_1\vec{\mathbf{v}}_{1i} + m_2\vec{\mathbf{v}}_{2i}}{m_1 + m_2}$$

✔



Example:

EXAMPLE 9–3 **Railroad cars collide: momentum conserved.** A 10,000-kg railroad car, A, traveling at a speed of 24.0 m/s strikes an identical car, B, at rest. If the cars lock together as a result of the collision, what is their common speed immediately after the collision? See Fig. 9–5.



(a) Before collision

(b) After collision

$$P_{\text{initial}} = P_{\text{final}}.$$

SOLUTION The initial total momentum is

$$P_{\text{initial}} = m_A v_A + m_B v_B = m_A v_A$$

because car B is at rest initially $(v_B = 0)$. The direction is to the right in the $+x$ direction. After the collision, the two cars become attached, so they will have the same speed, call it v'. Then the total momentum after the collision is

$$P_{\text{final}} = (m_A + m_B)v'.$$

We have assumed there are no external forces, so momentum is conserved:

$$P_{\text{initial}} = P_{\text{final}}$$

$$m_A v_A = (m_A + m_B)v'.$$

Solving for v', we obtain

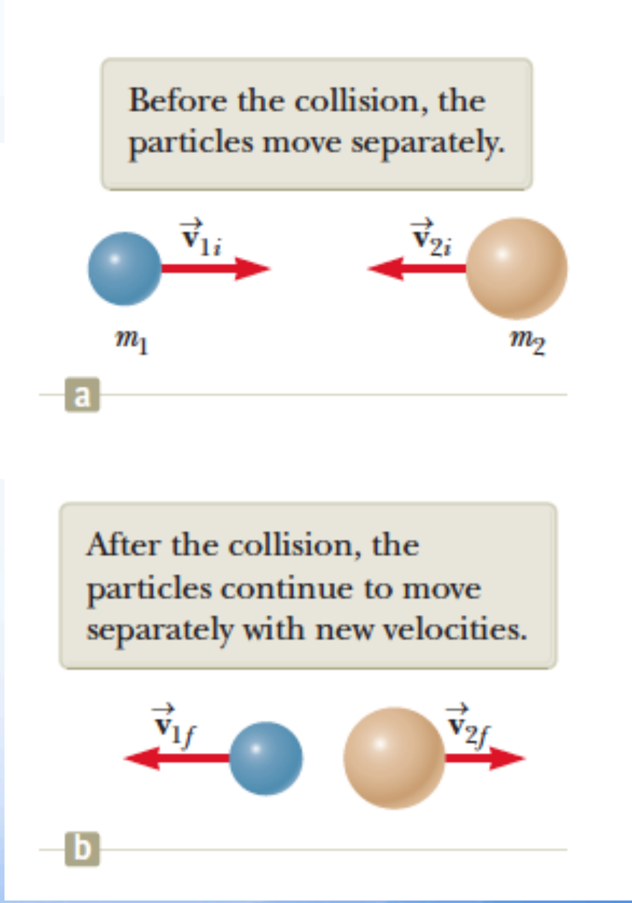
$$v' = \frac{m_A}{m_A + m_B} v_A = \left(\frac{10{,}000\text{ kg}}{10{,}000\text{ kg} + 10{,}000\text{ kg}}\right)(24.0\text{ m/s}) = 12.0\text{ m/s},$$

to the right. Their mutual speed after collision is half the initial speed of car A because their masses are equal.

Elastic Collision

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\frac{1}{2} m_1 v_{1i}{}^2 + \frac{1}{2} m_2 v_{2i}{}^2 = \frac{1}{2} m_1 v_{1f}{}^2 + \frac{1}{2} m_2 v_{2f}{}^2$$



$$m_1(v_{1i}^2 - v_{1f}^2) = m_2(v_{2f}^2 - v_{2i}^2)$$

ı sides of this equation gives

$$m_1(v_{1i} - v_{1f})\ (v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i})$$

separate the terms containing m_1 and m_2 in Equation 9.1

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i})$$

final result, we divide Equation 9.18 by Equation 9.19 and

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$ ✔

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i}$$

✔

$$v_{2i} = 0$$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}.$$

- Special cases:

1. ***Equal masses*** If $m_1 = m_2$, Eqs. 9-67 and 9-68 reduce to

$$v_{1f} = 0 \quad \text{and} \quad v_{2f} = v_{1i},$$

2. ***A massive target*** In Fig. 9-18, a massive target means that $m_2 \gg m_1$. For example, we might fire a golf ball at a stationary cannonball. Equations 9-67 and 9-68 then reduce to

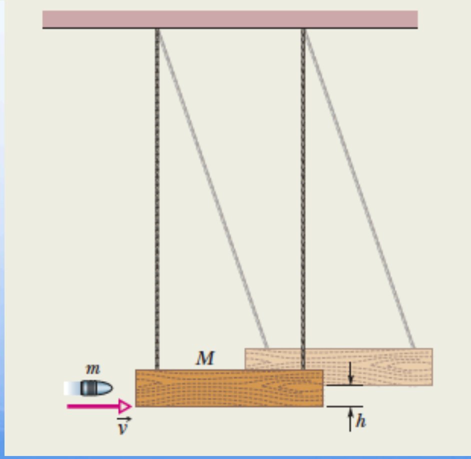
$$v_{1f} \approx -v_{1i} \quad \text{and} \quad v_{2f} \approx \left(\frac{2m_1}{m_2}\right) v_{1i}. \tag{9-69}$$

3. ***A massive projectile*** This is the opposite case; that is, $m_1 \gg m_2$. This time, we fire a cannonball at a stationary golf ball. Equations 9-67 and 9-68 reduce to

$$v_{1f} \approx v_{1i} \quad \text{and} \quad v_{2f} \approx 2v_{1i}. \tag{9-70}$$

Example:

The *ballistic pendulum* was used to measure the speeds of bullets before electronic timing devices were developed. The version shown in Fig. 9-17 consists of a large block of wood of mass $M = 5.4$ kg, hanging from two long cords. A bullet of mass $m = 9.5$ g is fired into the block, coming quickly to rest. The *block + bullet* then swing upward, their center of mass rising a vertical distance $h = 6.3$ cm before the pendulum comes momentarily to rest at the end of its arc. What is the speed of the bullet just prior to the collision?



$$\begin{pmatrix}\text{total momentum}\\ \text{before the collision}\end{pmatrix} = \begin{pmatrix}\text{total momentum}\\ \text{after the collision}\end{pmatrix}.$$

$$\begin{pmatrix}\text{mechanical energy}\\ \text{at bottom}\end{pmatrix} = \begin{pmatrix}\text{mechanical energy}\\ \text{at top}\end{pmatrix}.$$

$$V = \frac{m}{m + M} v.$$

$$\tfrac{1}{2}(m + M)V^2 = (m + M)gh.$$

$$v = \frac{m + M}{m} \sqrt{2gh} \tag{9-61}$$

$$= \left(\frac{0.0095 \text{ kg} + 5.4 \text{ kg}}{0.0095 \text{ kg}}\right) \sqrt{(2)(9.8 \text{ m/s}^2)(0.063 \text{ m})}$$

$$= 630 \text{ m/s.} \qquad \text{(Answer)}$$

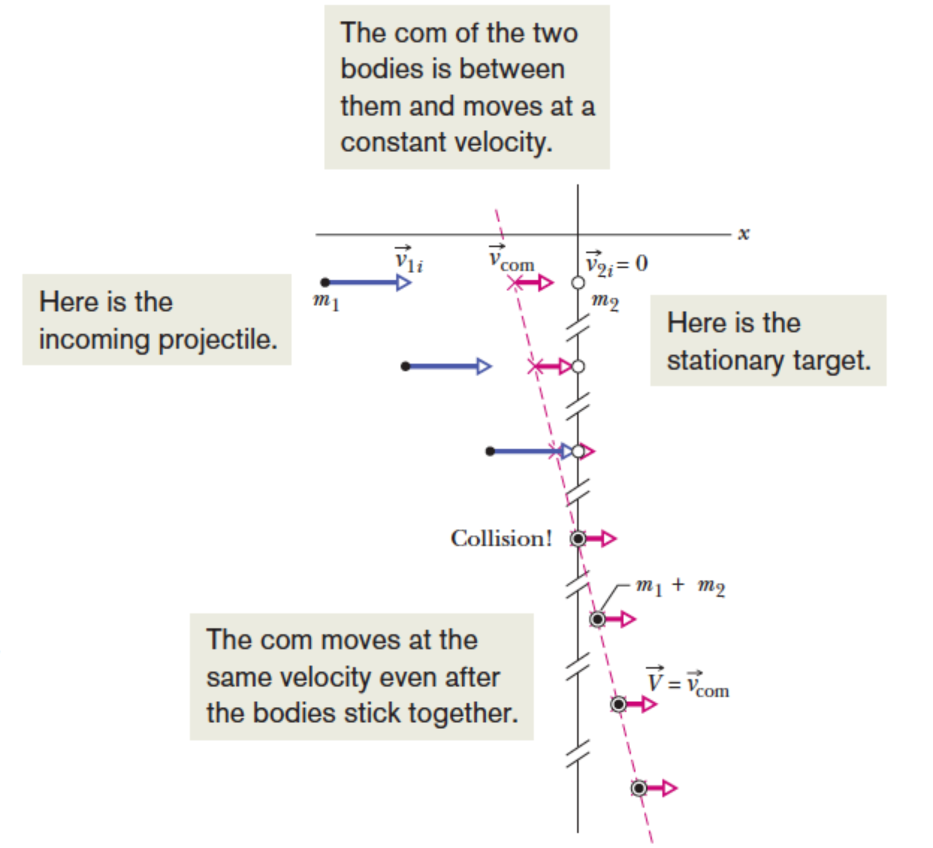
Example:

- Velocity of the Center of Mass

$$\vec{P} = M\vec{v}_{com} = (m_1 + m_2)\vec{v}_{com}.$$

$$\vec{P} = \vec{p}_{1i} + \vec{p}_{2i}.$$

$$\vec{v}_{com} = \frac{\vec{P}}{m_1 + m_2} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2}.$$



Example:

In Fig. 9-20*a*, block 1 approaches a line of two stationary blocks with a velocity of $v_{1i} = 10$ m/s. It collides with block 2, which then collides with block 3, which has mass $m_3 = 6.0$ kg. After the second collision, block 2 is again stationary and block 3 has velocity $v_{3f} = 5.0$ m/s (Fig. 9-20*b*). Assume that the collisions are elastic. What are the masses of blocks 1 and 2? What is the final velocity v_{1f} of block 1?

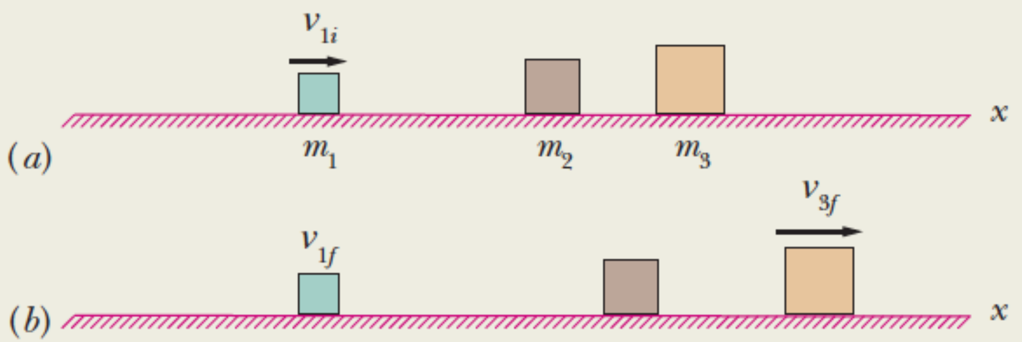


Figure 9-20 Block 1 collides with stationary block 2, which then collides with stationary block 3.

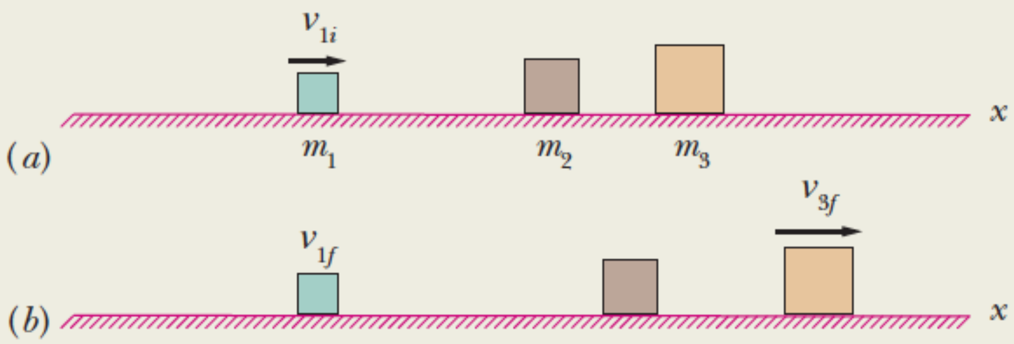


Figure 9-20 Block 1 collides with stationary block 2, which then collides with stationary block 3.

$$v_{2f} = \frac{m_2 - m_3}{m_2 + m_3} v_{2i},$$

where v_{2i} is the velocity of block 2 just before the collision and v_{2f} is the velocity just afterward. Substituting $v_{2f} = 0$ (block 2 stops) and then $m_3 = 6.0$ kg gives us

$$m_2 = m_3 = 6.00 \text{ kg}. \qquad \text{(Answer)}$$

$$v_{3f} = \frac{2m_2}{m_2 + m_3} v_{2i},$$

$$v_{2i} = v_{3f} = 5.0 \text{ m/s}.$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i},$$

$$5.0 \text{ m/s} = \frac{2m_1}{m_1 + m_2} (10 \text{ m/s}),$$

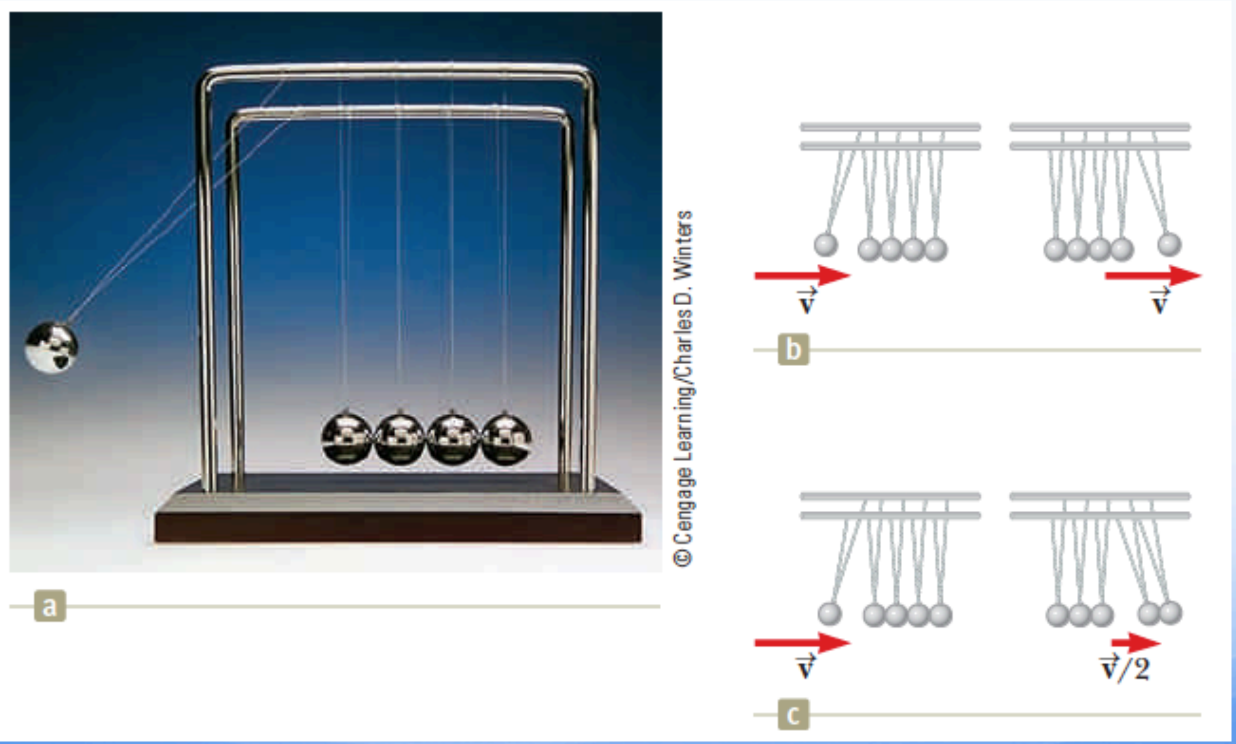
$$m_1 = \tfrac{1}{3} m_2 = \tfrac{1}{3}(6.0 \text{ kg}) = 2.0 \text{ kg}.$$

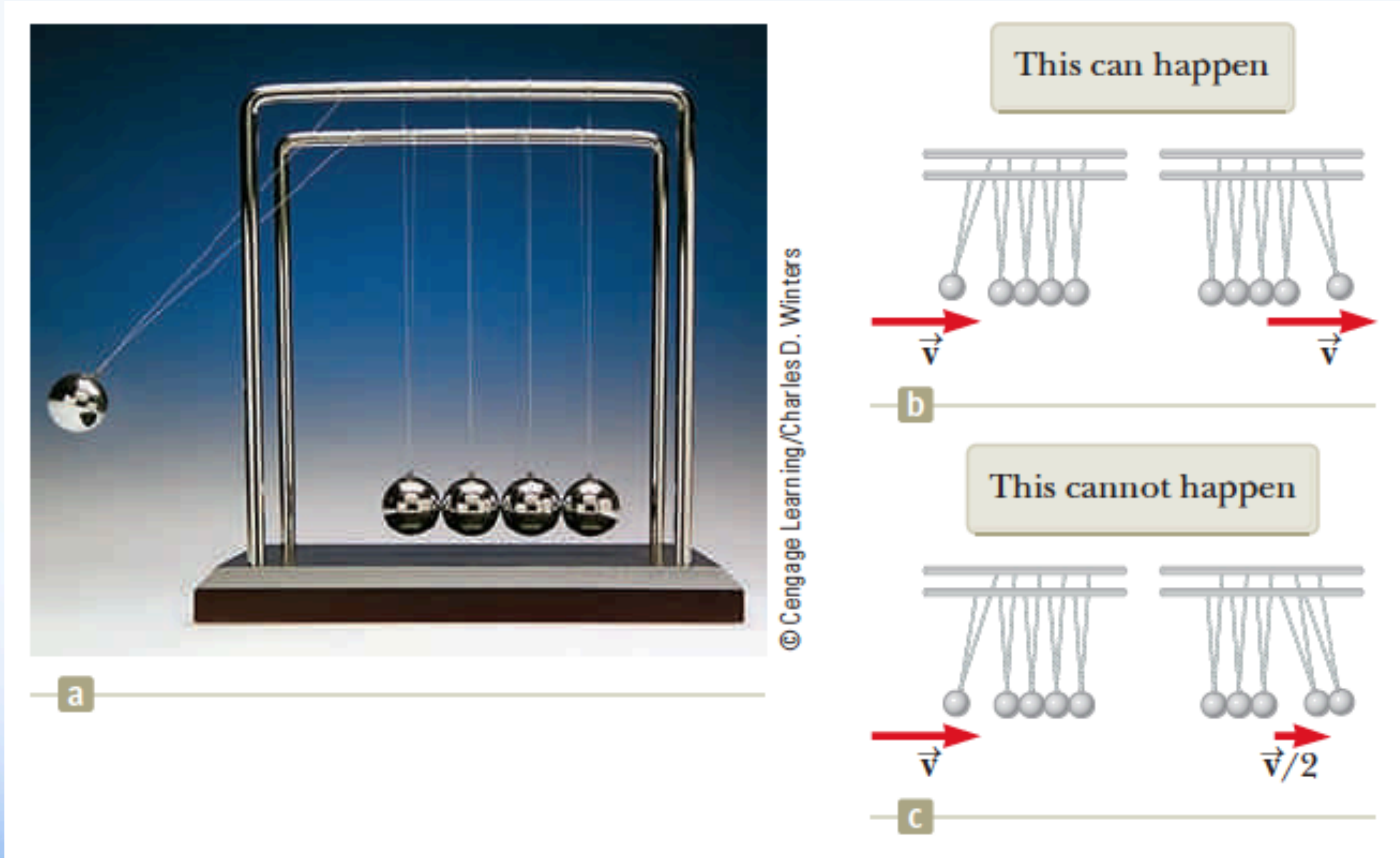
$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i},$$

$$= \frac{\frac{1}{3}m_2 - m_2}{\frac{1}{3}m_2 + m_2}(10 \text{ m/s}) = -5.0 \text{ m/s}.$$

Example: Newton Ball

- http://www.youtube.com/watch?v=JadO3RuOJGU&feature=related



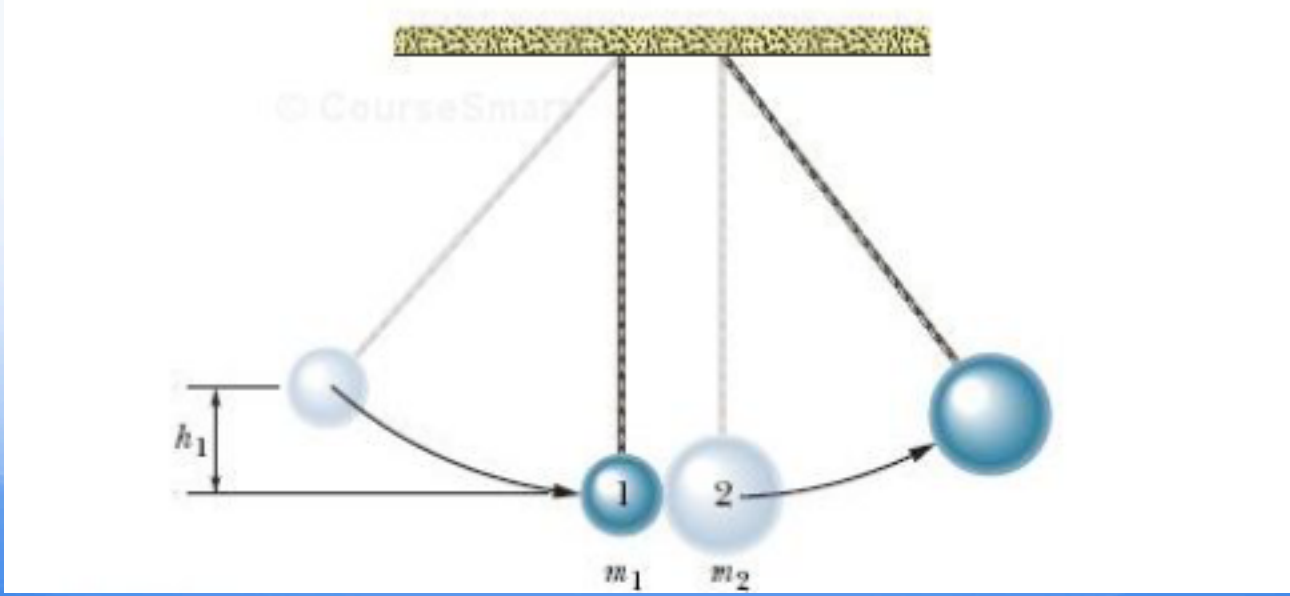


© Cengage Learning/Charles D. Winters
a
This can happen
$\vec{v}$
$\vec{v}$
b
This cannot happen
$\vec{v}$
$\vec{v}/2$
c

Newton Ball

Example:

Two metal spheres, suspended by vertical cords, initially just touch, as shown in Fig. 9-22. Sphere 1, with mass $m_1 = 30$ g, is pulled to the left to height $h_1 = 8.0$ cm, and then released from rest. After swinging down, it undergoes an elastic collision with sphere 2, whose mass $m_2 = 75$ g. What is the velocity v_{1f} of sphere 1 just after the collision?



when sphere 1 is at height h_1. Thus,

$$\tfrac{1}{2}m_1 v_{1i}^2 = m_1 g h_1,$$

which we solve for the speed v_{1i} of sphere 1 just before the collision:

$$v_{1i} = \sqrt{2gh_1} = \sqrt{(2)(9.8\ \text{m/s}^2)(0.080\ \text{m})}$$
$$= 1.252\ \text{m/s}.$$

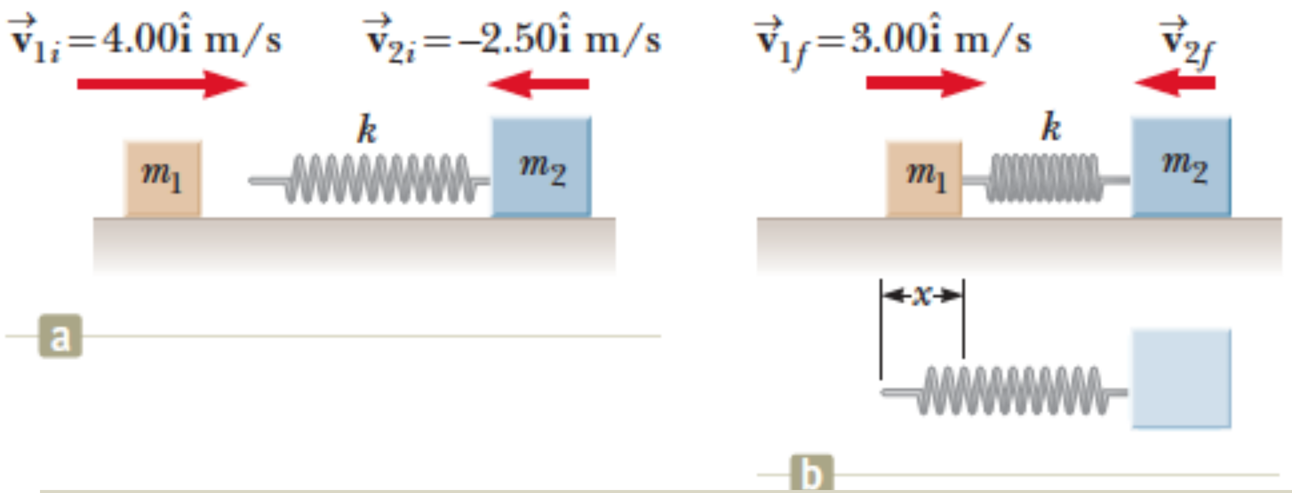
Step 2: Here we can make two assumptions in addition to the assumption that the collision is elastic. First, we can assume that the collision is one-dimensional because the motions of the spheres are approximately horizontal from just before the collision to just after it. Second, because the collision is so brief, we can assume that the two-sphere system is closed and isolated. This means that the total linear momentum of the system is conserved.

Calculation: Thus, we can use Eq. 9-67 to find the velocity of sphere 1 just after the collision:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$
$$= \frac{0.030\ \text{kg} - 0.075\ \text{kg}}{0.030\ \text{kg} + 0.075\ \text{kg}} (1.252\ \text{m/s})$$
$$= -0.537\ \text{m/s} \approx -0.54\ \text{m/s}. \qquad \text{(Answer)}$$

The minus sign tells us that sphere 1 moves to the left just after the collision.

Example:



A block of mass $m_1 = 1.60$ kg initially moving to the right with a speed of 4.00 m/s on a frictionless, horizontal track collides with a light spring attached to a second block of mass $m_2 = 2.10$ kg initially moving to the left with a speed of 2.50 m/s as shown in Figure 9.10a. The spring constant is 600 N/m.

(A) Find the velocities of the two blocks after the collision.

$$(1)\ m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$(2)\ v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$$

$$(3)\ m_1 v_{1i} - m_1 v_{2i} = -m_1 v_{1f} + m_1 v_{2f}$$

$$2m_1 v_{1i} + (m_2 - m_1) v_{2i} = (m_1 + m_2) v_{2f}$$

$$v_{2f} = \frac{2m_1 v_{1i} + (m_2 - m_1) v_{2i}}{m_1 + m_2}$$

$$v_{2f} = \frac{2(1.60\text{ kg})(4.00\text{ m/s}) + (2.10\text{ kg} - 1.60\text{ kg})(-2.50\text{ m/s})}{1.60\text{ kg} + 2.10\text{ kg}} = \boxed{3.12\text{ m/s}}$$

$$v_{1f} = v_{2f} - v_{1i} + v_{2i} = 3.12\text{ m/s} - 4.00\text{ m/s} + (-2.50\text{ m/s}) = \boxed{-3.38\text{ m/s}}$$

(B) Determine the velocity of block 2 during the collision, at the instant block 1 is moving to the right with a velocity of +3.00 m/s as in Figure 9.10b.

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$v_{2f} = \frac{m_1 v_{1i} + m_2 v_{2i} - m_1 v_{1f}}{m_2}$$

$$v_{2f} = \frac{(1.60\ \text{kg})(4.00\ \text{m/s}) + (2.10\ \text{kg})(-2.50\ \text{m/s}) - (1.60\ \text{kg})(3.00\ \text{m/s})}{2.10\ \text{kg}}$$

$$= \boxed{-1.74\ \text{m/s}}$$

(C) Determine the distance the spring is compressed at that instant.

$$K_i + U_i = K_f + U_f$$

$$\tfrac{1}{2}m_1 v_{1i}^{\ 2} + \tfrac{1}{2}m_2 v_{2i}^{\ 2} + 0 = \tfrac{1}{2}m_1 v_{1f}^{\ 2} + \tfrac{1}{2}m_2 v_{2f}^{\ 2} + \tfrac{1}{2}kx^2$$

$$\tfrac{1}{2}(1.60\text{ kg})(4.00\text{ m/s})^2 + \tfrac{1}{2}(2.10\text{ kg})(2.50\text{ m/s})^2 + 0$$
$$= \tfrac{1}{2}(1.60\text{ kg})(3.00\text{ m/s})^2 + \tfrac{1}{2}(2.10\text{ kg})(1.74\text{ m/s})^2 + \tfrac{1}{2}(600\text{ N/m})x^2$$

$$x = \boxed{0.173\text{ m}}$$

9-8 Collisions in Two Dimensions

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta + m_2 v_{2f} \cos\phi$$

$$0 = m_1 v_{1f} \sin\theta - m_2 v_{2f} \sin\phi$$

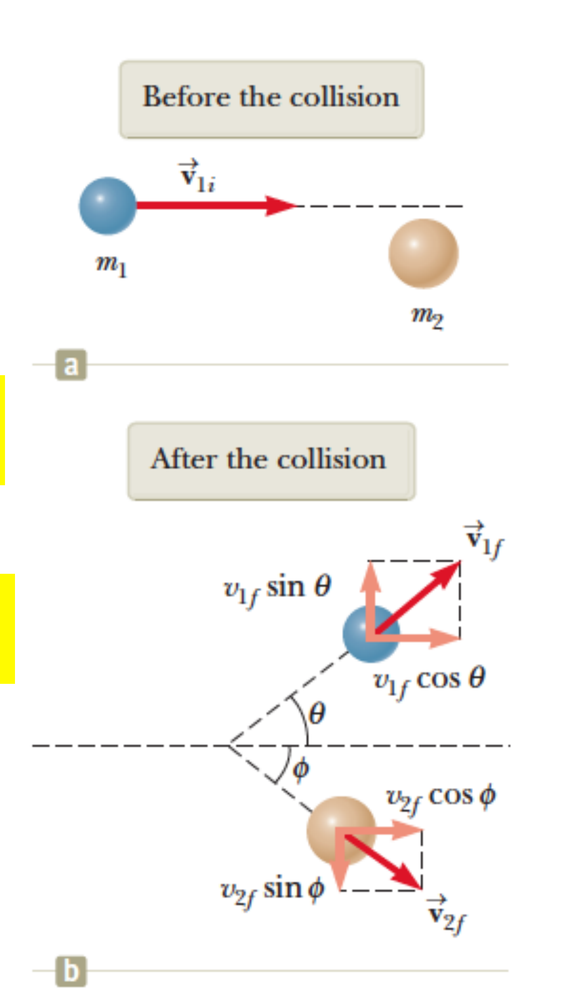
✔

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}.$$

✔

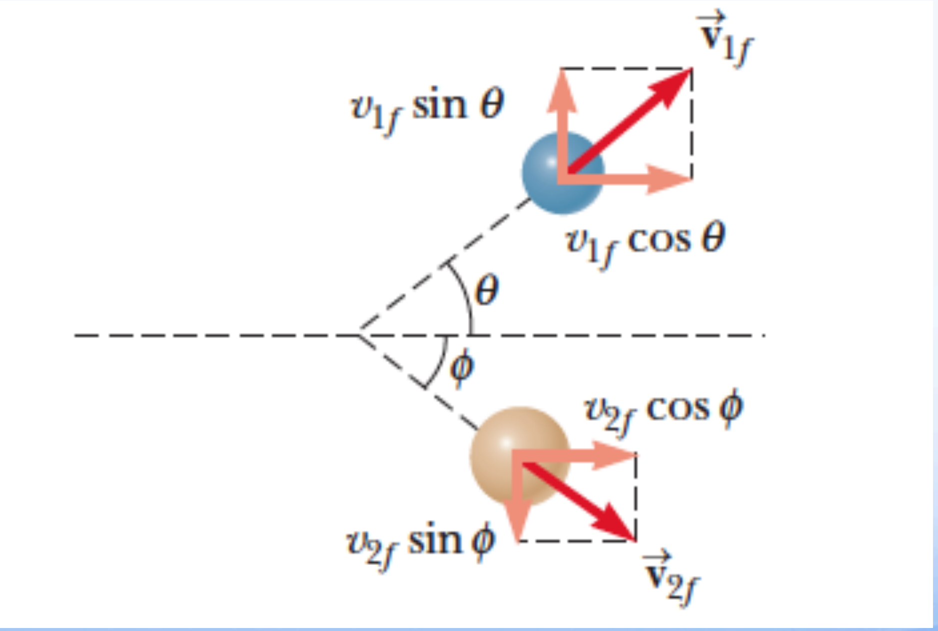
with $v_{2i} = 0$:

$$\boxed{\tfrac{1}{2} m_1 {v_{1i}}^2 = \tfrac{1}{2} m_1 {v_{1f}}^2 + \tfrac{1}{2} m_2 {v_{2f}}^2}$$



Example:

A proton collides elastically with another proton that is initially at rest. The incoming proton has an initial speed of 3.50×10^5 m/s and makes a glancing collision with the second proton as in Active Figure 9.11. (At close separations, the protons exert a repulsive electrostatic force on each other.) After the collision, one proton moves off at an angle of 37.0° to the original direction of motion and the second deflects at an angle of ϕ to the same axis. Find the final speeds of the two protons and the angle ϕ.



$$(1)\quad v_{1f}\cos\theta + v_{2f}\cos\phi = v_{1i}$$

$$(2)\quad v_{1f}\sin\theta - v_{2f}\sin\phi = 0$$ ✔

$$(3)\quad {v_{1f}}^2 + {v_{2f}}^2 = {v_{1i}}^2$$ ✔

$$v_{2f}\cos\phi = v_{1i} - v_{1f}\cos\theta$$

$$v_{2f}\sin\phi = v_{1f}\sin\theta$$

$$\begin{aligned}&{v_{2f}}^2\cos^2\phi + {v_{2f}}^2\sin^2\phi = \\ &\quad {v_{1i}}^2 - 2v_{1i}v_{1f}\cos\theta + {v_{1f}}^2\cos^2\theta + {v_{1f}}^2\sin^2\theta\end{aligned}$$

$$(4)\quad {v_{2f}}^2 = {v_{1i}}^2 - 2v_{1i}v_{1f}\cos\theta + {v_{1f}}^2$$

$${v_{1f}}^2 + ({v_{1i}}^2 - 2v_{1i}v_{1f}\cos\theta + {v_{1f}}^2) = {v_{1i}}^2$$

$$(5)\quad {v_{1f}}^2 - v_{1i}v_{1f}\cos\theta = 0$$ ✔

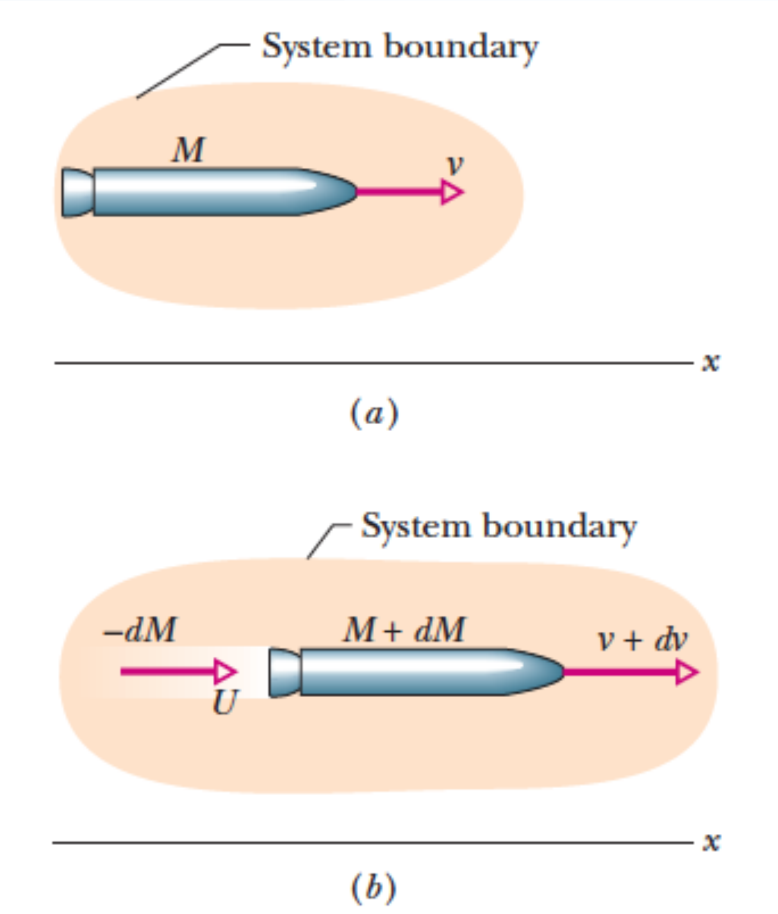
$$v_{1f} = v_{1i} \cos\theta = (3.50 \times 10^5 \text{ m/s}) \cos 37.0° = \boxed{2.80 \times 10^5 \text{ m/s}}$$

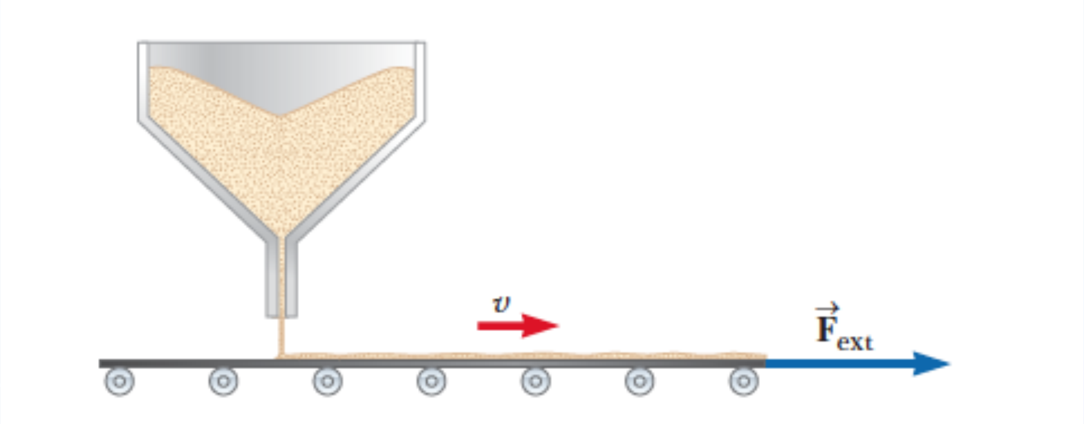
$$v_{2f} = \sqrt{v_{1i}^2 - v_{1f}^2} = \sqrt{(3.50 \times 10^5 \text{ m/s})^2 - (2.80 \times 10^5 \text{ m/s})^2}$$
$$= \boxed{2.11 \times 10^5 \text{ m/s}}$$

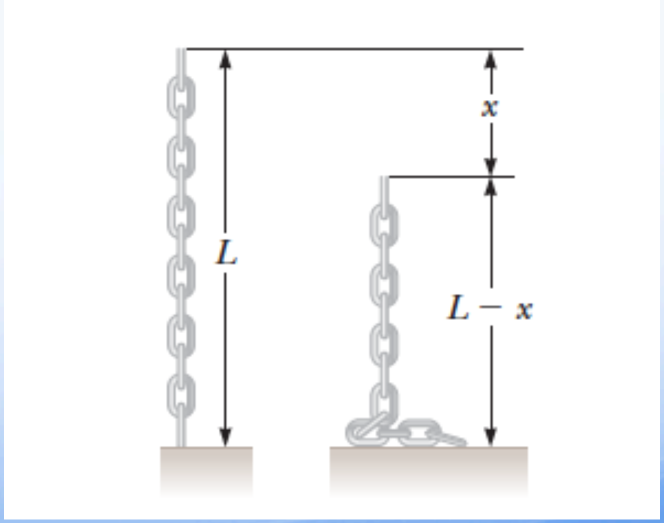
$$(2)\ \phi = \sin^{-1}\left(\frac{v_{1f} \sin\theta}{v_{2f}}\right) = \sin^{-1}\left[\frac{(2.80 \times 10^5 \text{ m/s}) \sin 37.0°}{(2.11 \times 10^5 \text{ m/s})}\right]$$
$$= \boxed{53.0°}$$

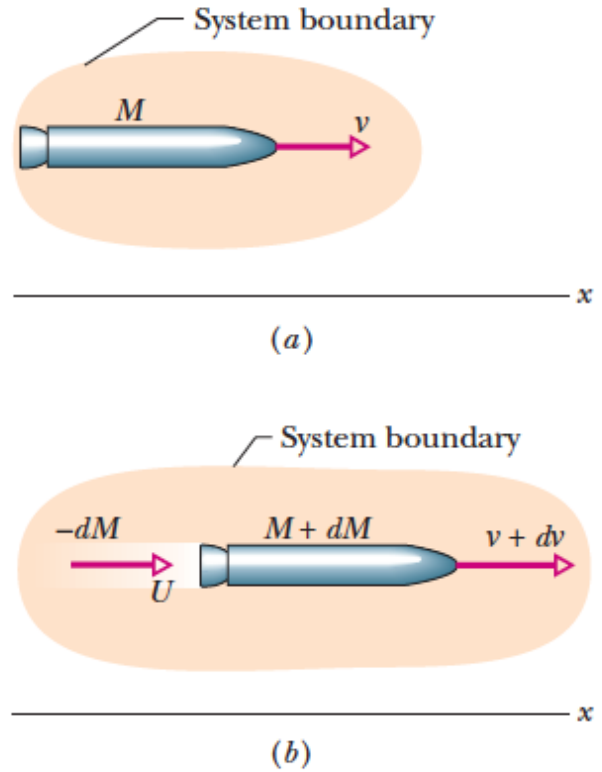
Finalize It is interesting that $\theta + \phi = 90°$. This result is *not* accidental. Whenever two objects of equal mass collide elastically in a glancing collision and one of them is initially at rest, their final velocities are perpendicular to each other.

9-9 Systems with Varying Mass









Rocket

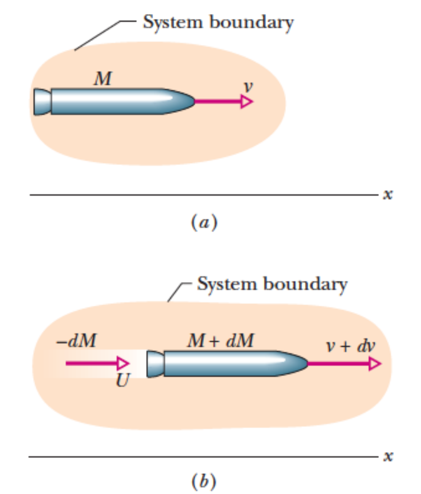
Conserve Momentum.

$$P_i = P_f,$$

$$Mv = -dM\,U + (M + dM)(v + dv),$$

Use Relative Speed.

$$\begin{pmatrix}\text{velocity of rocket}\\ \text{relative to frame}\end{pmatrix} = \begin{pmatrix}\text{velocity of rocket}\\ \text{relative to products}\end{pmatrix} + \begin{pmatrix}\text{velocity of products}\\ \text{relative to frame}\end{pmatrix}.$$



$$(v + dv) = v_{\text{rel}} + U,$$

$$U = v + dv - v_{\text{rel}}.$$

$$Mv = -dM\,U + (M + dM)(v + dv),$$

$$-dM\,v_{\text{rel}} = M\,dv.$$

$$-\frac{dM}{dt}v_{\text{rel}} = M\frac{dv}{dt}.$$

$$\frac{dM}{dt} = -R$$

$$Rv_{\text{rel}} = Ma \qquad \text{(first rocket equation).}$$

Thrust $T = Rv_{\text{rel}}$

$$dv = -v_{\text{rel}}\frac{dM}{M}.$$

$$\int_{v_i}^{v_f} dv = -v_{\text{rel}}\int_{M_i}^{M_f}\frac{dM}{M},$$

$$v_f - v_i = v_{\text{rel}} \ln \frac{M_i}{M_f} \quad \text{(second rocket equation)}$$

$$\text{Thrust} = M\frac{dv}{dt} = \left| v_e \frac{dM}{dt} \right|$$

Example:

Two firefighters must apply a total force of 600 N to steady a hose that is discharging water at the rate of 3 600 L/min. Estimate the speed of the water as it exits the nozzle.

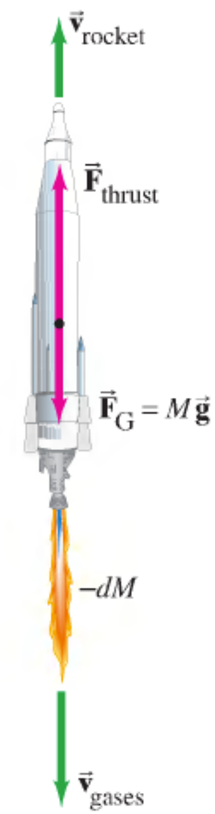
$$\text{Thrust} = \left| v_e \frac{dM}{dt} \right|$$

$$600\ \text{N} = |v_e(60\ \text{kg/s})|$$

$$v_e = \boxed{10\ \text{m/s}}$$

Example:

EXAMPLE 9–20 **Rocket propulsion.** A fully fueled rocket has a mass of 21,000 kg, of which 15,000 kg is fuel. The burned fuel is spewed out the rear at a rate of 190 kg/s with a speed of 2800 m/s relative to the rocket. If the rocket is fired vertically upward (Fig. 9–35) calculate: (*a*) the thrust of the rocket; (*b*) the net force on the rocket at blastoff, and just before burnout (when all the fuel has been used up); (*c*) the rocket's velocity as a function of time, and (*d*) its final velocity at burnout. Ignore air resistance and assume the acceleration due to gravity is constant at $g = 9.80\ \text{m/s}^2$.



SOLUTION (*a*) The thrust is:

$$F_{\text{thrust}} = v_{\text{rel}} \frac{dM}{dt} = (-2800\,\text{m/s})(-190\,\text{kg/s}) = 5.3 \times 10^5\,\text{N},$$

where we have taken upward as positive so v_{rel} is negative because it is downward, and dM/dt is negative because the rocket's mass is diminishing.
(*b*) $F_{\text{ext}} = Mg = (2.1 \times 10^4\,\text{kg})(9.80\,\text{m/s}^2) = 2.1 \times 10^5\,\text{N}$ initially, and at burnout $F_{\text{ext}} = (6.0 \times 10^3\,\text{kg})(9.80\,\text{m/s}^2) = 5.9 \times 10^4\,\text{N}$. Hence, the net force on the rocket at blastoff is

$$F_{\text{net}} = 5.3 \times 10^5\,\text{N} - 2.1 \times 10^5\,\text{N} = 3.2 \times 10^5\,\text{N}, \qquad [\text{blastoff}]$$

and just before burnout it is

$$F_{\text{net}} = 5.3 \times 10^5\,\text{N} - 5.9 \times 10^4\,\text{N} = 4.7 \times 10^5\,\text{N}. \qquad [\text{burnout}]$$

After burnout, of course, the net force is that of gravity, $-5.9 \times 10^4\,\text{N}$.
(*c*) From Eq. 9–19b we have

$$dv = \frac{F_{\text{ext}}}{M}\,dt + v_{\text{rel}} \frac{dM}{M},$$

where $F_{\text{ext}} = -Mg$, and M is the mass of the rocket and is a function of time. Since v_{rel} is constant, we can integrate this easily:

$$\int_{v_0}^{v} dv = -\int_0^t g\,dt + v_{\text{rel}} \int_{M_0}^{M} \frac{dM}{M}$$

or

$$v(t) = v_0 - gt + v_{\text{rel}} \ln \frac{M}{M_0},$$

where $v(t)$ is the rocket's velocity and M its mass at any time t. Note that v_{rel} is negative ($-2800\,\mathrm{m/s}$ in our case) because it is opposite to the motion, and that $\ln\left(M/M_0\right)$ is also negative because $M_0 > M$. Hence, the last term—which represents the thrust—is positive and acts to increase the velocity.

(*d*) The time required to reach burnout is the time needed to use up all the fuel (15,000 kg) at a rate of 190 kg/s; so at burnout,

$$t = \frac{1.50 \times 10^4\,\mathrm{kg}}{190\,\mathrm{kg/s}} = 79\,\mathrm{s}.$$

If we take $v_0 = 0$, then using the result of part (*c*):

$$v = -(9.80\,\mathrm{m/s^2})(79\,\mathrm{s}) + (-2800\,\mathrm{m/s})\left(\ln\frac{6000\,\mathrm{kg}}{21{,}000\,\mathrm{kg}}\right) = 2700\,\mathrm{m/s}.$$

Example:

•••17 GO In Fig. 9-45*a*, a 4.5 kg dog stands on an 18 kg flatboat at distance $D = 6.1$ m from the shore. It walks 2.4 m along the boat toward shore and then stops. Assuming no friction between the boat and the water, find how far the dog is then from the shore. (*Hint:* See Fig. 9-45*b*.)

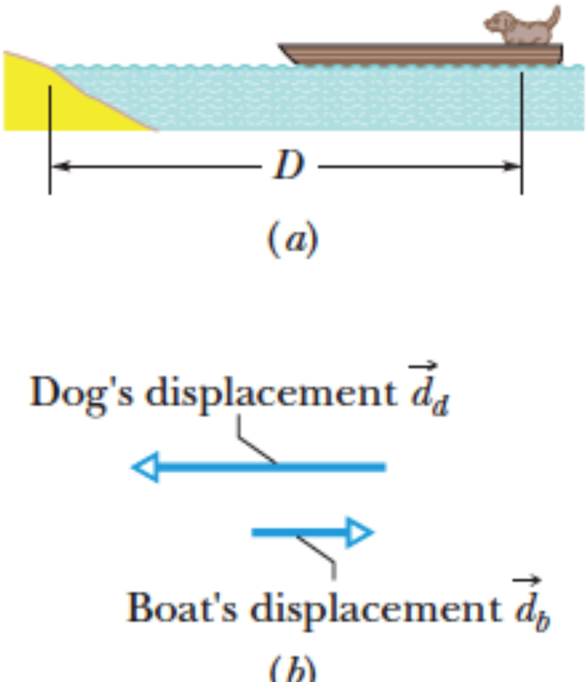


Figure 9-45 Problem 17.

Example:

••22 Figure 9-47 gives an overhead view of the path taken by a 0.165 kg cue ball as it bounces from a rail of a pool table. The ball's initial speed is 2.00 m/s, and the angle θ_1 is 30.0°. The bounce reverses the y component of the ball's velocity but does not alter the x component. What are (a) angle θ_2 and (b) the change in the ball's linear momentum in unit-vector notation? (The fact that the ball rolls is irrelevant to the problem.)

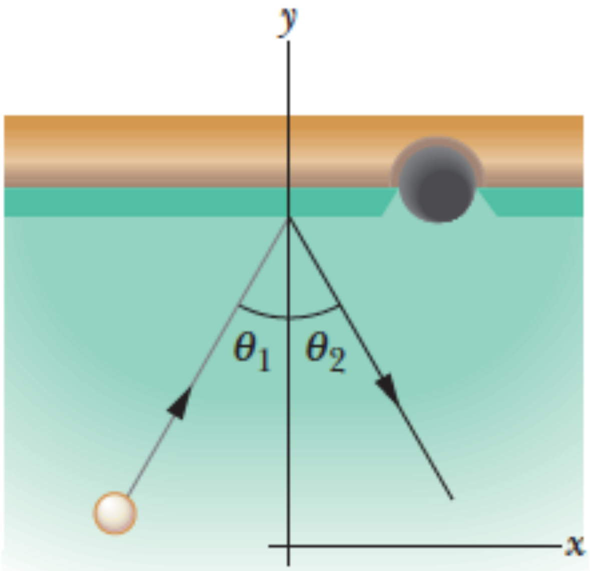


Figure 9-47 Problem 22.

Example:

••57 GO In Fig. 9-61, a ball of mass $m = 60$ g is shot with speed $v_i = 22$ m/s into the barrel of a spring gun of mass $M = 240$ g initially at rest on a frictionless surface. The ball sticks in the barrel at the point of maximum compression of the spring. Assume that the increase in thermal energy due to friction between the ball and the barrel is negligible. (a) What is the speed of the spring gun after the ball stops in the barrel? (b) What fraction of the initial kinetic energy of the ball is stored in the spring?

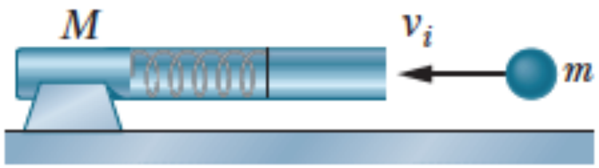


Figure 9-61 Problem 57.

Example:

•••58 In Fig. 9-62, block 2 (mass 1.0 kg) is at rest on a frictionless surface and touching the end of an unstretched spring of spring constant 200 N/m. The other end of the spring is fixed to a wall. Block 1 (mass 2.0 kg), traveling at speed $v_1 = 4.0$ m/s, collides with block 2, and the two blocks stick together. When the blocks momentarily stop, by what distance is the spring compressed?

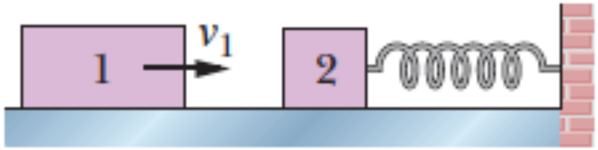


Figure 9-62 Problem 58.

Example:

••67 In Fig. 9-66, particle 1 of mass $m_1 = 0.30$ kg slides rightward along an x axis on a frictionless floor with a speed of 2.0 m/s. When it reaches $x = 0$, it undergoes a one-dimensional elastic collision with stationary particle 2 of mass $m_2 = 0.40$ kg. When particle 2 then reaches a wall at $x_w = 70$ cm, it bounces from the wall with no loss of speed. At what position on the x axis does particle 2 then collide with particle 1?

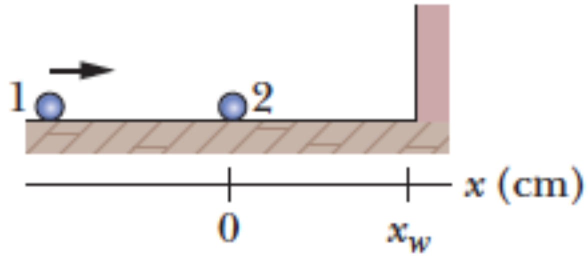


Figure 9-66 Problem 67.

Example:

••74 Two 2.0 kg bodies, A and B, collide. The velocities before the collision are $\vec{v}_A = (15\hat{i} + 30\hat{j})$ m/s and $\vec{v}_B = (-10\hat{i} + 5.0\hat{j})$ m/s. After the collision, $\vec{v}'_A = (-5.0\hat{i} + 20\hat{j})$ m/s. What are (a) the final velocity of B and (b) the change in the total kinetic energy (including sign)?

Example:

102 In Fig. 9-79, an 80 kg man is on a ladder hanging from a balloon that has a total mass of 320 kg (including the basket passenger). The balloon is initially stationary relative to the ground. If the man on the ladder begins to climb at 2.5 m/s relative to the ladder, (a) in what direction and (b) at what speed does the balloon move? (c) If the man then stops climbing, what is the speed of the balloon?

Figure 9-79 Problem 102.

Example:

123 An unmanned space probe (of mass m and speed v relative to the Sun) approaches the planet Jupiter (of mass M and speed V_J relative to the Sun) as shown in Fig. 9-84. The spacecraft rounds the planet and departs in the opposite direction. What is its speed (in kilometers per second), relative to the Sun, after this slingshot encounter, which can be analyzed as a collision? Assume $v = 10.5$ km/s and $V_J = 13.0$ km/s (the orbital speed of Jupiter). The mass of Jupiter is very much greater than the mass of the spacecraft ($M \gg m$).

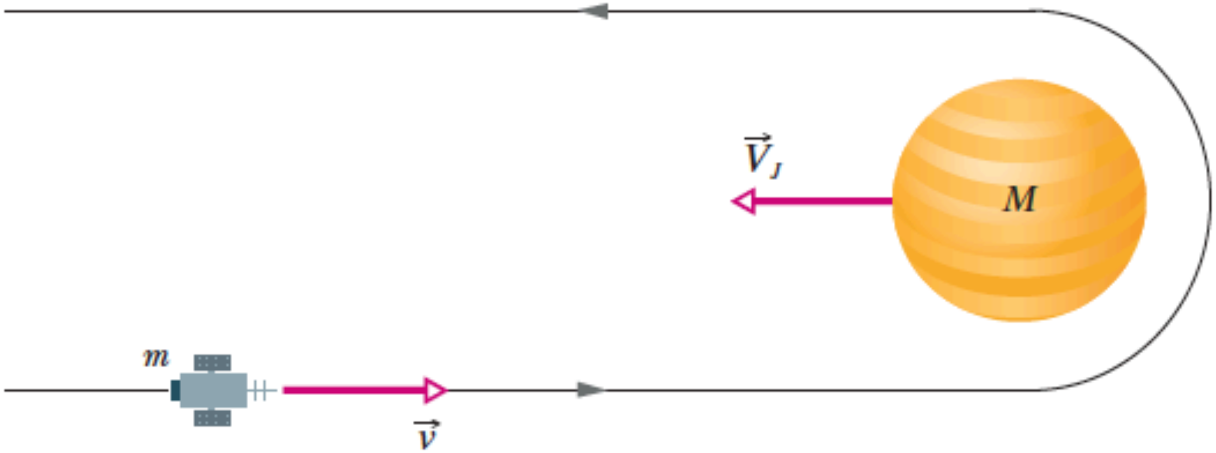


Figure 9-84 Problem 123.

Example:

•77 SSM In Fig. 9-70, two long barges are moving in the same direction in still water, one with a speed of 10 km/h and the other with a speed of 20 km/h. While they are passing each other, coal is shoveled from the slower to the faster one at a rate of 1000 kg/min. How much additional force must be provided by the driving engines of (a) the faster barge and (b) the slower barge if neither is to change speed? Assume that the shoveling is always perfectly sideways and that the frictional forces between the barges and the water do not depend on the mass of the barges.

Figure 9-70 Problem 77.

Example:

34. The mass of the blue puck in Figure P9.34 is 20.0% greater than the mass of the green puck. Before colliding, the pucks approach each other with momenta of equal magnitudes and opposite directions, and the green puck has an initial speed of 10.0 m/s. Find the speeds the pucks have after the collision if half the kinetic energy of the system becomes internal energy during the collision.

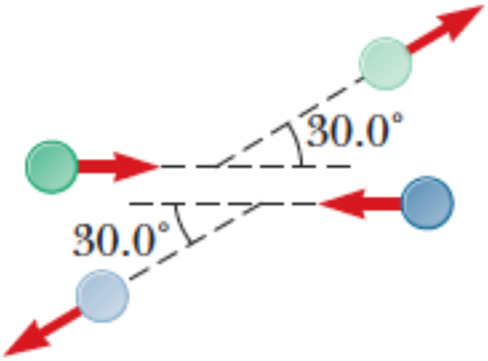


Figure P9.34

Example:

46. **S** Figure P9.46a shows an overhead view of the initial configuration of two pucks of mass m on frictionless ice. The pucks are tied together with a string of length ℓ and negligible mass. At time $t = 0$, a constant force of magnitude F begins to pull to the right on the center point of the string. At time t, the moving pucks strike each other and stick together. At this time, the force has moved through a distance d, and the pucks have attained a speed v (Fig. P9.46b). (a) What is v in terms of F, d, ℓ, and m? (b) How much of the energy transferred into the system by work done by the force has been transformed to internal energy?

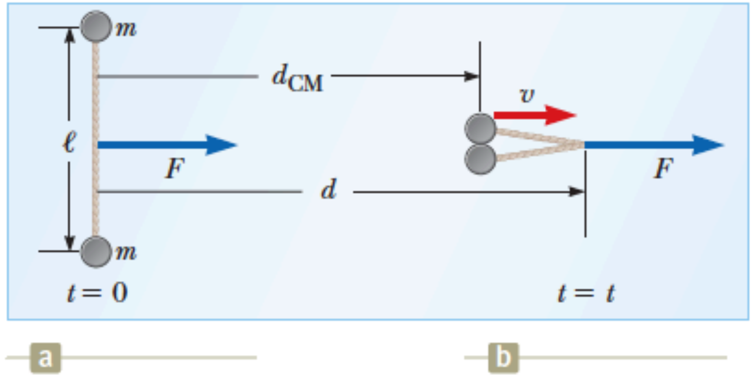


Figure P9.46

Example:

47. Q|C S A particle is suspended from a post on top of a cart by a light string of length L as shown in Figure P9.47a. The cart and particle are initially moving to the right at constant speed v_i, with the string vertical. The cart suddenly comes to rest when it runs into and sticks to a bumper as shown in Figure P9.47b. The suspended particle swings through an angle θ. (a) Show that the original speed of the cart can be computed from $v_i = \sqrt{2gL(1 - \cos\theta)}$. (b) If the bumper is still exerting a horizontal force on the cart when the hanging particle is at its maximum angle *forward* from the vertical, at what moment does the bumper *stop* exerting a horizontal force?

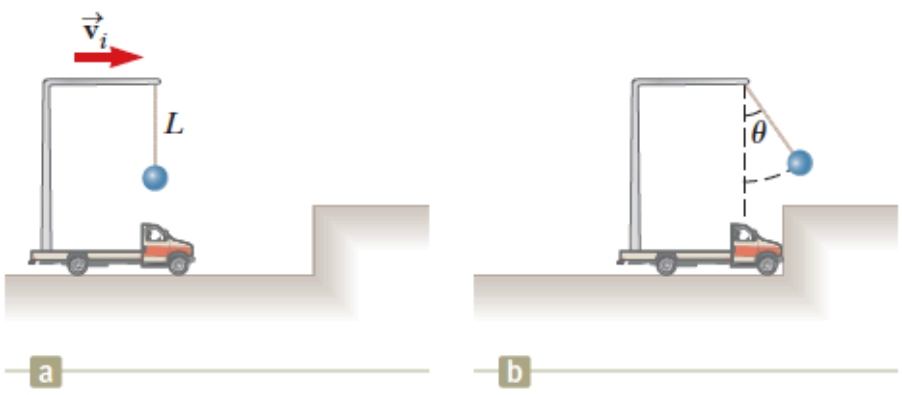


Figure P9.47

Example:

52. A rocket has total mass $M_i = 360$ kg, including $M_f = 330$ kg of fuel and oxidizer. In interstellar space, it starts from rest at the position $x = 0$, turns on its engine at time $t = 0$, and puts out exhaust with relative speed $v_e = 1\,500$ m/s at the constant rate $k = 2.50$ kg/s. The fuel will last for a burn time of $T_b = M_f/k = 330\text{ kg}/(2.5\text{ kg/s}) = 132$ s. (a) Show that during the burn the velocity of the rocket as a function of time is given by

$$v(t) = -v_e \ln\left(1 - \frac{kt}{M_i}\right)$$

(b) Make a graph of the velocity of the rocket as a function of time for times running from 0 to 132 s. (c) Show that the acceleration of the rocket is

$$a(t) = \frac{kv_e}{M_i - kt}$$

(d) Graph the acceleration as a function of time. (e) Show that the position of the rocket is

$$x(t) = v_e\left(\frac{M_i}{k} - t\right)\ln\left(1 - \frac{kt}{M_i}\right) + v_e t$$

(f) Graph the position during the burn as a function of time.

Example:

65. **S** **Review.** A bullet of mass m is fired into a block of mass M initially at rest at the edge of a frictionless table of height h (Fig. P9.65). The bullet remains in the block, and after

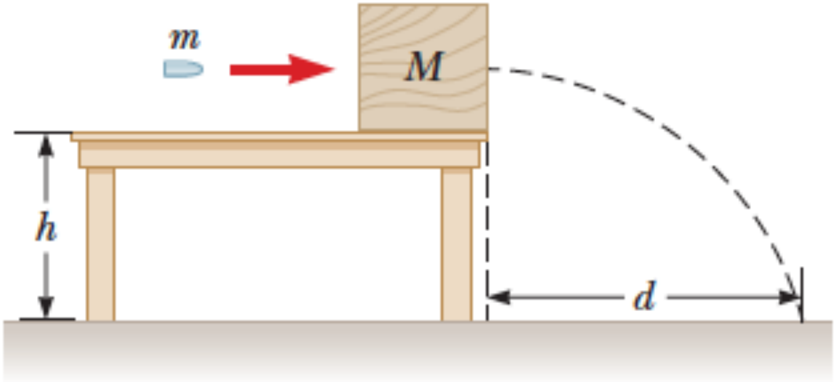


Figure P9.65

impact the block lands a distance d from the bottom of the table. Determine the initial speed of the bullet.

Example:

66. A small block of mass $m_1 = 0.500$ kg is released from rest at the top of a frictionless, curve-shaped wedge of mass $m_2 = 3.00$ kg, which sits on a frictionless, horizontal surface as shown in Figure P9.66a. When the block leaves the wedge, its velocity is measured to be 4.00 m/s to the right as shown in Figure P9.66b. (a) What is the velocity of the wedge after the block reaches the horizontal surface? (b) What is the height h of the wedge?

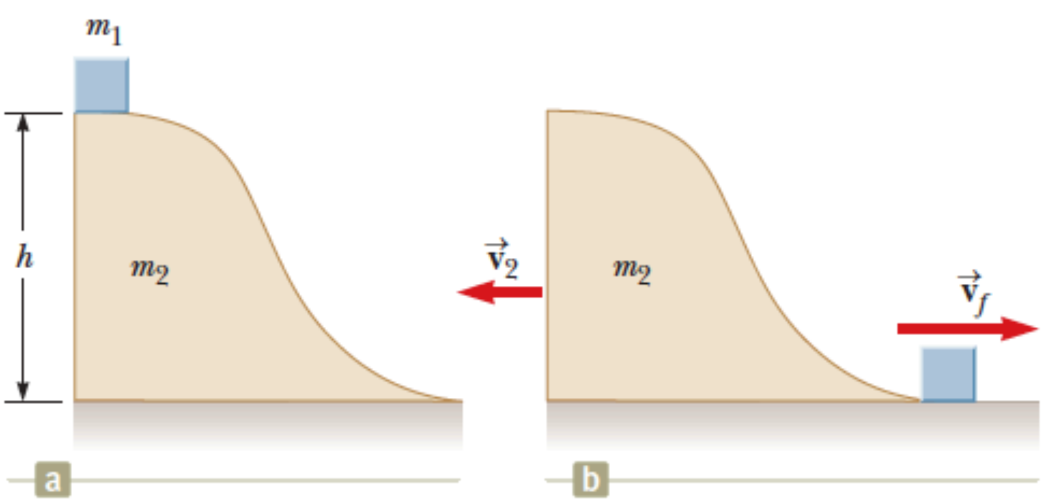


Figure P9.66

Example:

73. **M** A 5.00-g bullet moving with an initial speed of v_i = 400 m/s is fired into and passes through a 1.00-kg block as shown in Figure P9.73. The block, initially at rest on a frictionless, horizontal surface, is connected to a spring with force constant 900 N/m. The block moves d = 5.00 cm to the right after impact before being brought to rest by the spring. Find (a) the speed at which the bullet emerges from the block and (b) the amount of initial kinetic energy of the bullet that is converted into internal energy in the bullet–block system during the collision.

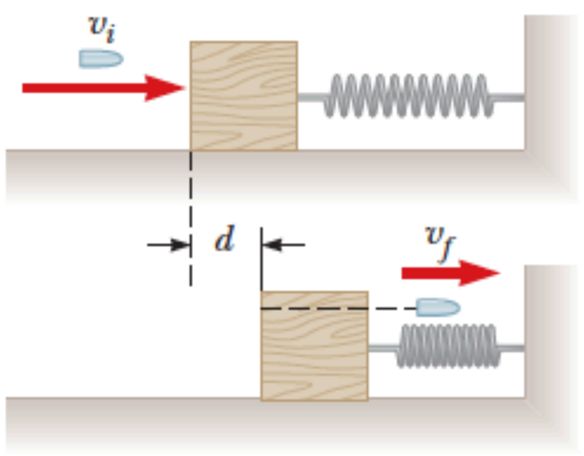


Figure P9.73

Example:

78. Q|C Sand from a stationary hopper falls onto a moving conveyor belt at the rate of 5.00 kg/s as shown in Figure P9.78. The conveyor belt is supported by frictionless rollers and moves at a constant speed of $v = 0.750$ m/s under the action of a constant horizontal external force $\vec{\mathbf{F}}_{\text{ext}}$ supplied by the motor that drives the belt. Find (a) the sand's rate of change of momentum in the horizontal direction, (b) the force of friction exerted by the belt on the sand, (c) the external force $\vec{\mathbf{F}}_{\text{ext}}$, (d) the work done by $\vec{\mathbf{F}}_{\text{ext}}$ in 1 s, and (e) the kinetic energy acquired by the falling sand each second due to the change in its horizontal motion. (f) Why are the answers to parts (d) and (e) different?

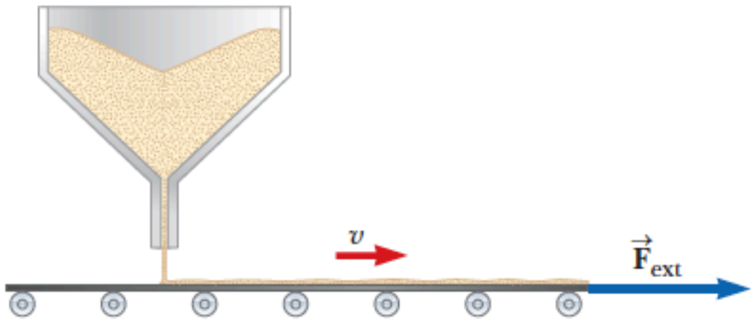


Figure P9.78

Example:

79. **S** **Review.** A chain of length L and total mass M is released from rest with its lower end just touching the top of a table as shown in Figure P9.79a. Find the force exerted by the table on the chain after the chain has fallen through a distance x as shown in Figure P9.79b. (Assume each link comes to rest the instant it reaches the table.)

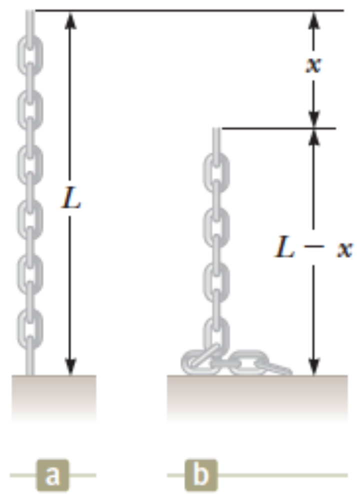


Figure P9.79

Example:

49. (II) A measure of inelasticity in a head-on collision of two objects is the *coefficient of restitution*, e, defined as

$$e = \frac{v'_A - v'_B}{v_B - v_A},$$

where $v'_A - v'_B$ is the relative velocity of the two objects after the collision and $v_B - v_A$ is their relative velocity before it. (*a*) Show that $e = 1$ for a perfectly elastic collision, and $e = 0$ for a completely inelastic collision. (*b*) A simple method for measuring the coefficient of restitution for an object colliding with a very hard surface like steel is to drop the object onto a heavy steel plate, as shown in Fig. 9–41. Determine a formula for e in terms of the original height h and the maximum height h' reached after collision.

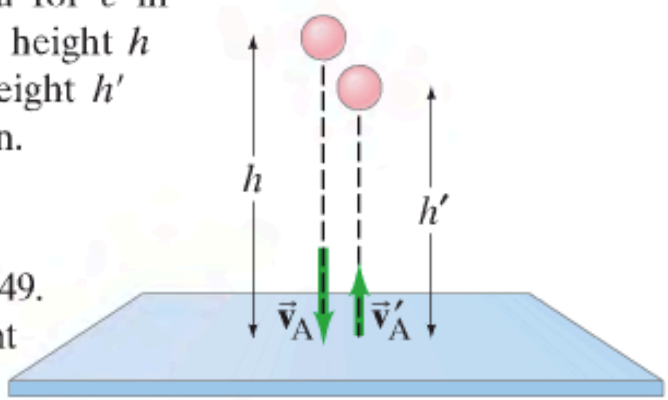


FIGURE 9–41 Problem 49. Measurement of coefficient of restitution.

Example:

50. (II) A pendulum consists of a mass M hanging at the bottom end of a massless rod of length ℓ, which has a frictionless pivot at its top end. A mass m, moving as shown in Fig. 9–42 with velocity v, impacts M and becomes embedded. What is the smallest value of v sufficient to cause the pendulum (with embedded mass m) to swing clear over the top of its arc?

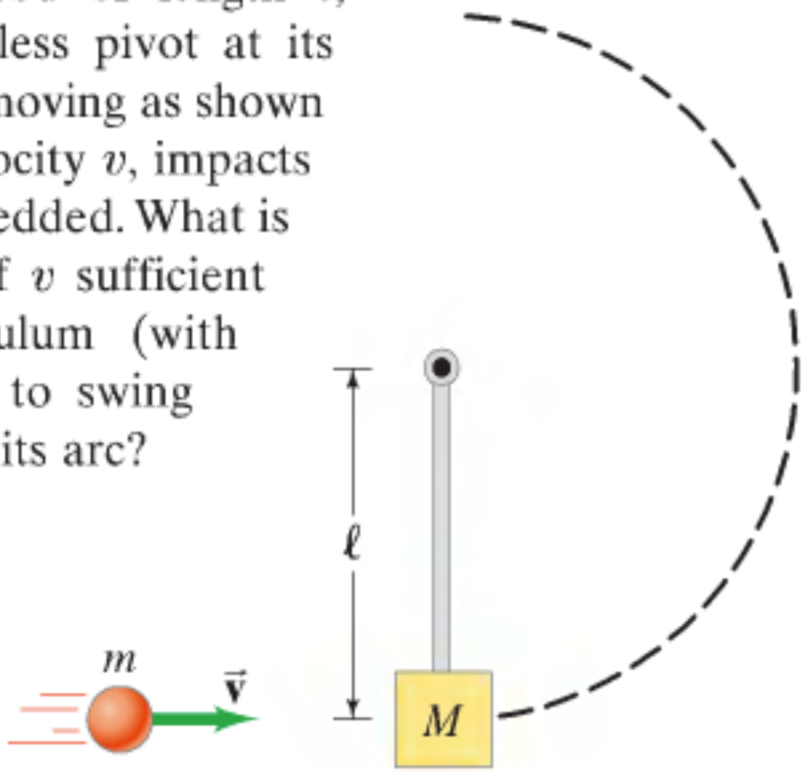


FIGURE 9–42
Problem 50.

Example:

94. Two blocks of mass m_A and m_B, resting on a frictionless table, are connected by a stretched spring and then released (Fig. 9–51). (*a*) Is there a net external force on the system? (*b*) Determine the ratio of their speeds, v_A / v_B. (*c*) What is the ratio of their kinetic energies? (*d*) Describe the motion of the CM of this system. (*e*) How would the presence of friction alter the above results?

FIGURE 9–51 Problem 94.

Example:

89. A gun fires a bullet vertically into a 1.40-kg block of wood at rest on a thin horizontal sheet, Fig. 9–50. If the bullet has a mass of 24.0 g and a speed of 310 m/s, how high will the block rise into the air after the bullet becomes embedded in it?

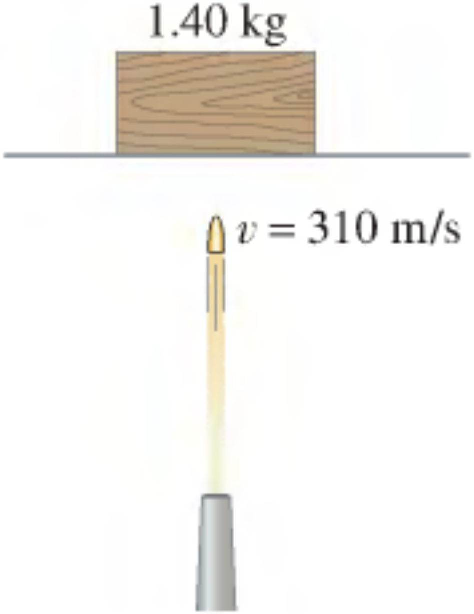


FIGURE 9–50
Problem 89.

Example:

99. Two balls, of masses $m_A = 45$ g and $m_B = 65$ g, are suspended as shown in Fig. 9–52. The lighter ball is pulled away to a 66° angle with the vertical and released. (*a*) What is the velocity of the lighter ball before impact? (*b*) What is the velocity of each ball after the elastic collision? (*c*) What will be the maximum height of each ball after the elastic collision?

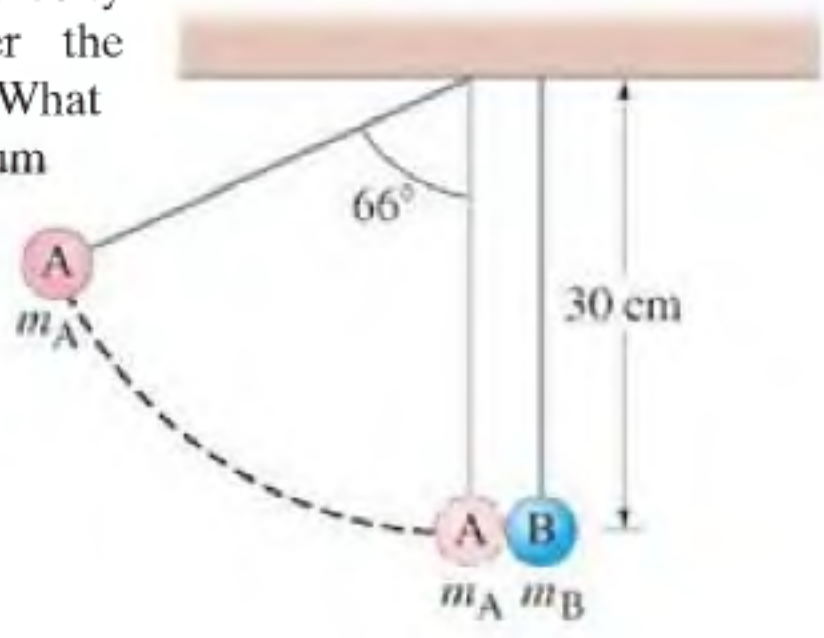
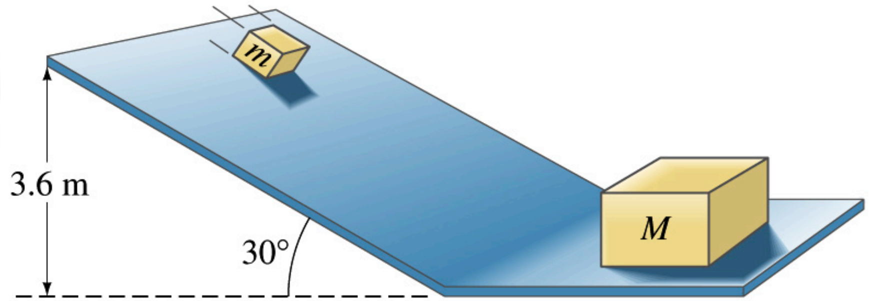


FIGURE 9–52 Problem 99.

Example:

100. A block of mass $m = 2.20\,\text{kg}$ slides down a $30.0°$ incline which is 3.60 m high. At the bottom, it strikes a block of mass $M = 7.00\,\text{kg}$ which is at rest on a horizontal surface, Fig. 9–53. (Assume a smooth transition at the bottom of the incline.) If the collision is elastic, and friction can be ignored, determine (*a*) the speeds of the two blocks after the collision, and (*b*) how far back up the incline the smaller mass will go.



Example:

107. In a physics lab, a cube slides down a frictionless incline as shown in Fig. 9–57 and elastically strikes another cube at the bottom that is only one-half its mass. If the incline is 35 cm high and the table is 95 cm off the floor, where does each cube land? [*Hint*: Both leave the incline moving horizontally.]

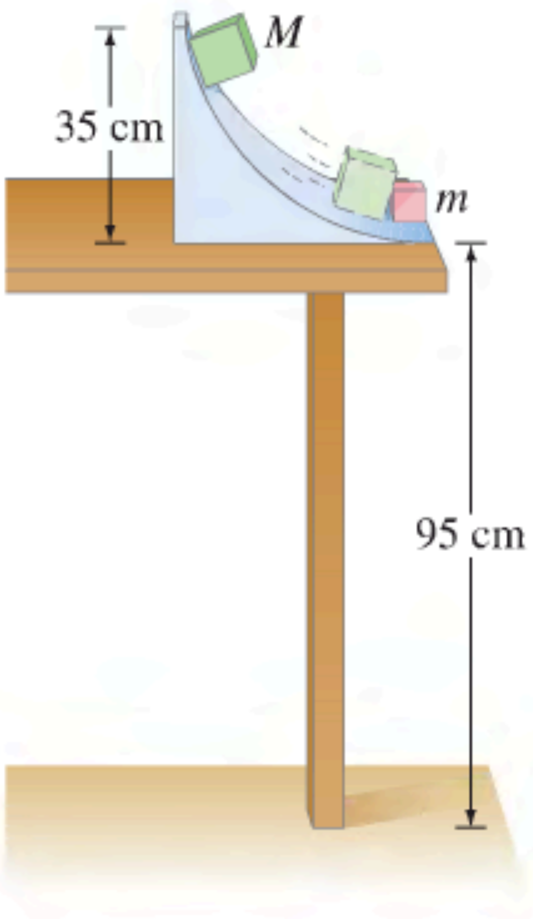


FIGURE 9–57
Problem 107.

Example:

13. (II) A child in a boat throws a 5.70-kg package out horizontally with a speed of 10.0 m/s, Fig. 9–37. Calculate the velocity of the boat immediately after, assuming it was initially at rest. The mass of the child is 24.0 kg and that of the boat is 35.0 kg.

FIGURE 9–37 Problem 13.

Example:

25. (II) A tennis ball of mass $m = 0.060\,\text{kg}$ and speed $v = 25\,\text{m/s}$ strikes a wall at a 45° angle and rebounds with the same speed at 45° (Fig. 9–38). What is the impulse (magnitude and direction) given to the ball?

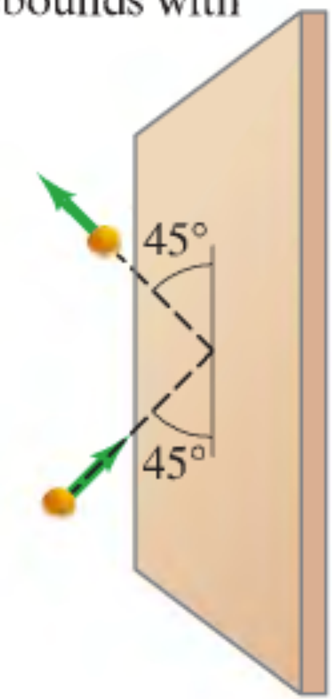


FIGURE 9–38
Problem 25.

Take-Home:

$V_{fM}, V_{fm}?$