General Physics I

Chapter 4

Sharif University of Technology
Mehr 1401 (2022-2023)

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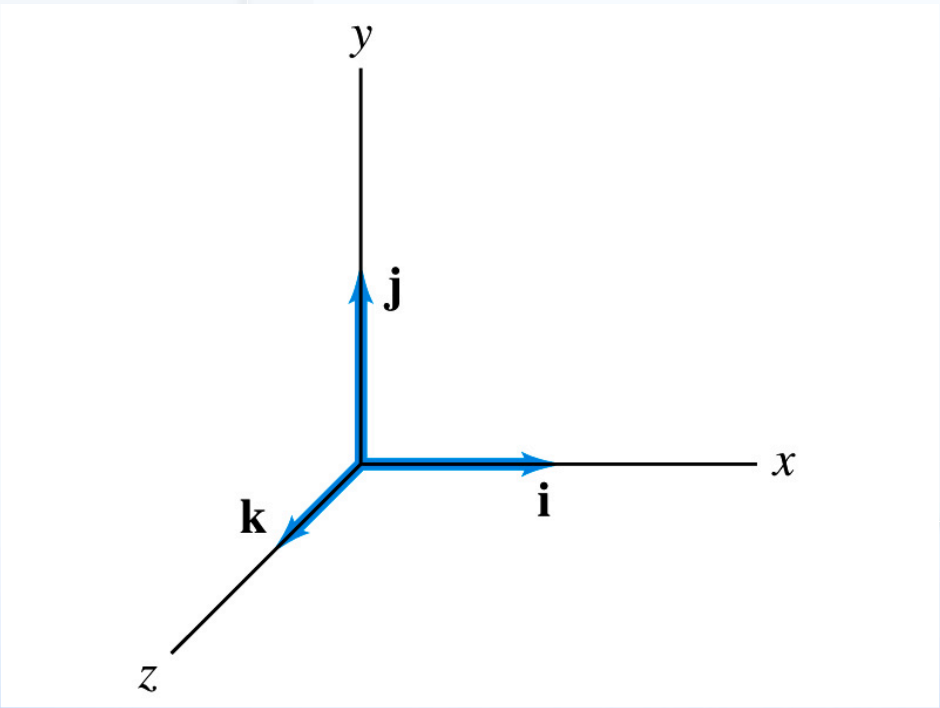
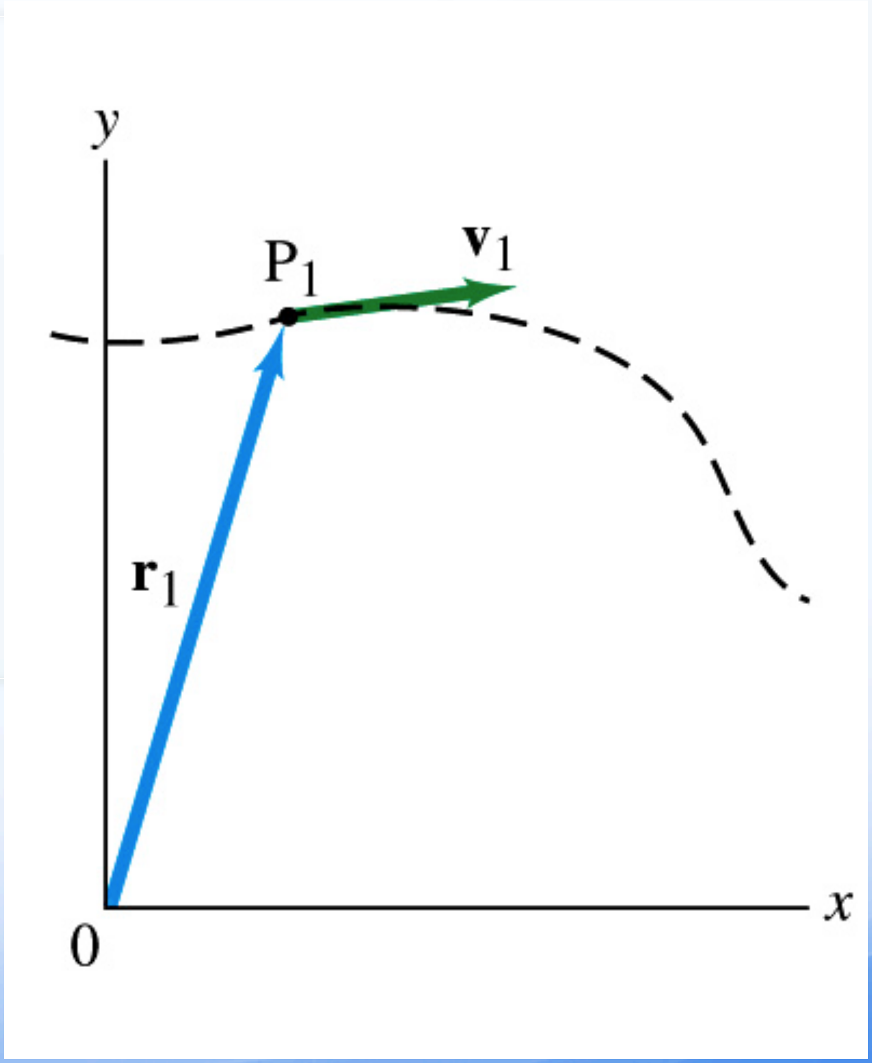
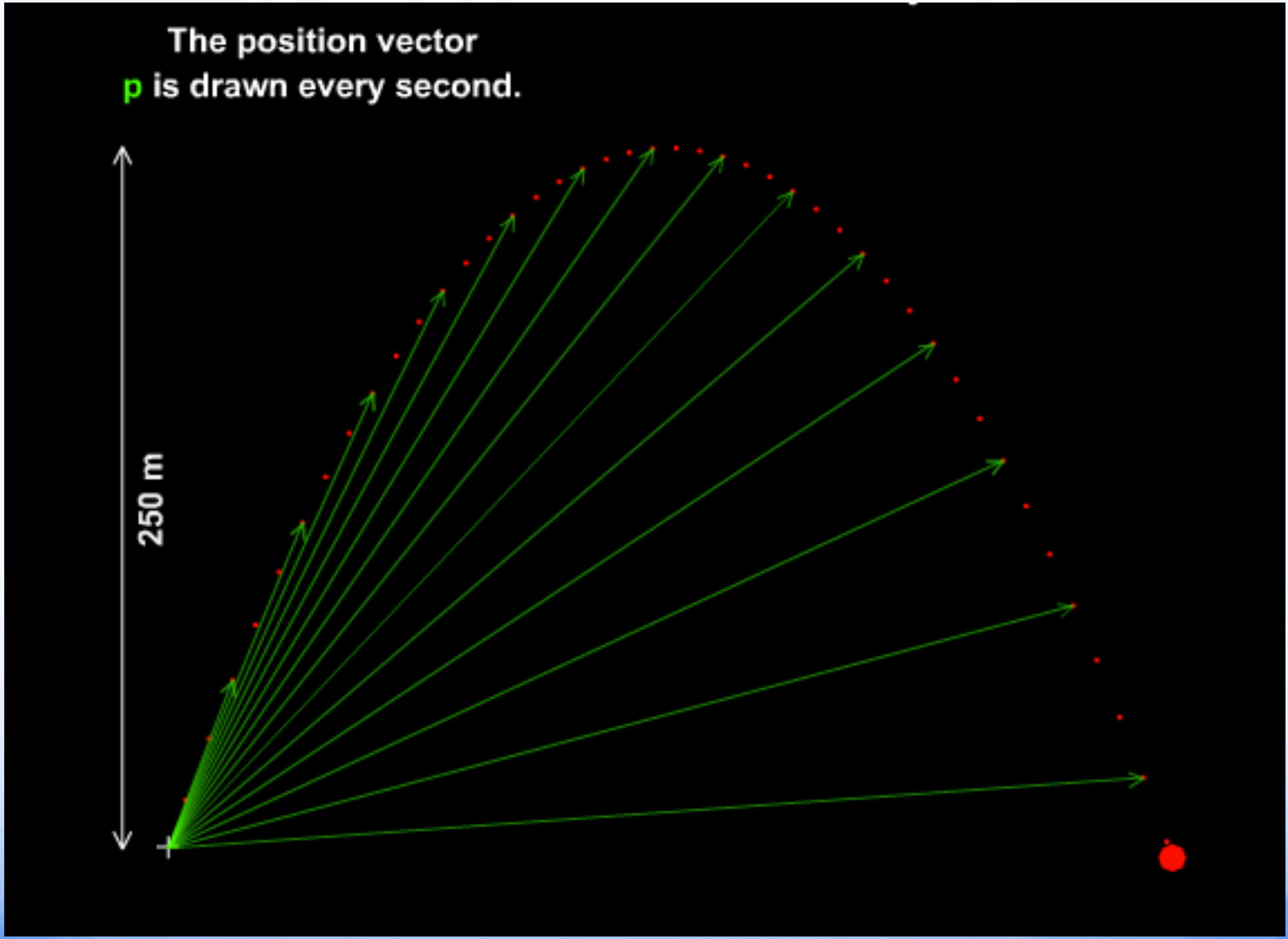
Chapter 4
Motion in Two and Three Dimensions

- ✓ 4.01 Draw two-dimensional and three-dimensional position vectors for a particle, indicating the components along the axes of a coordinate system.
- ✓ 4.02 On a coordinate system, determine the direction and magnitude of a particle's position vector from its components, and vice versa.
- ✓ 4.03 Apply the relationship between a particle's displacement vector and its initial and final position vectors.

4-2 Position and Displacement







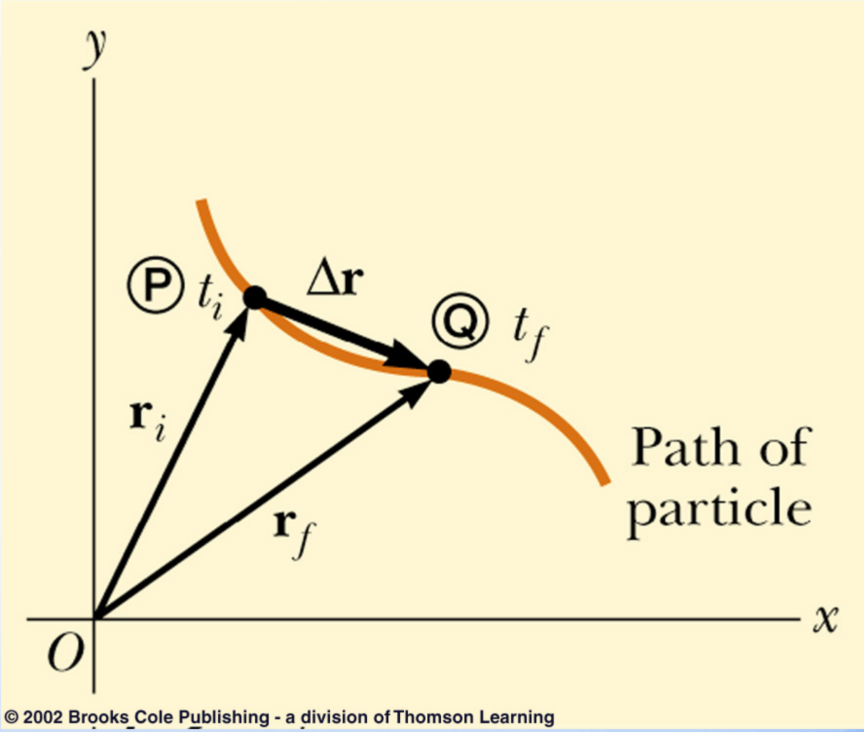
The position vector
p is drawn every second.
250 m

Motion in Two Dimensions

- Using + or – signs is not always sufficient to fully describe motion in more than one dimension
  - Vectors can be used to more fully describe motion
- Still interested in displacement, velocity, and acceleration

Displacement

- The position of an object is described by its position vector, **r**
- The **displacement** of the object is defined as the *change in its position*



$$\vec{\Delta r} = \vec{r_f} - \vec{r_i}$$

Velocity

- The average velocity is the ratio of the displacement to the time interval for the displacement

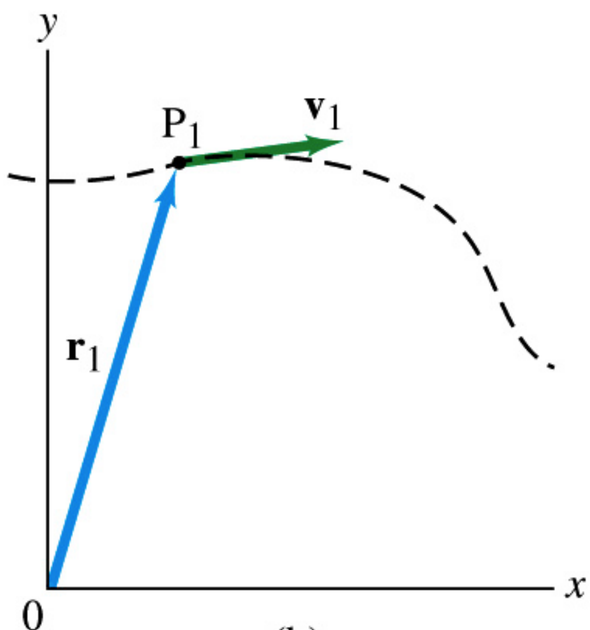
$$\bar{v} = \frac{\Delta \vec{r}}{\Delta t}$$

- The instantaneous velocity is the limit of the average velocity as Δt approaches zero
  - The direction of the instantaneous velocity is along a line that is tangent to the path of the particle and in the direction of motion

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

Velocity

$$\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}},$$



(b)

The result is the same in three dimensions: $\vec{v}$ is always tangent to the particle's path.

To write Eq. 4-10 in unit-vector form, we substitute for $\vec{r}$ from Eq. 4-1:

$$\vec{v} = \frac{d}{dt}(x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}) = \frac{dx}{dt}\hat{\mathrm{i}} + \frac{dy}{dt}\hat{\mathrm{j}} + \frac{dz}{dt}\hat{\mathrm{k}}.$$

This equation can be simplified somewhat by writing it as

$$\vec{v} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}}, \qquad (4\text{-}11)$$

where the scalar components of $\vec{v}$ are

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad \text{and} \quad v_z = \frac{dz}{dt}. \qquad (4\text{-}12)$$

Example

A rabbit runs across a parking lot on which a set of coordinate axes has, strangely enough, been drawn. The coordinates (meters) of the rabbit's position as functions of time t (seconds) are given by

$$x = -0.31t^2 + 7.2t + 28 \tag{4-5}$$

and

$$y = 0.22t^2 - 9.1t + 30. \tag{4-6}$$

(a) At $t = 15$ s, what is the rabbit's position vector $\vec{r}$ in unit-vector notation and in magnitude-angle notation?

Calculations: We can write

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}. \tag{4-7}$$

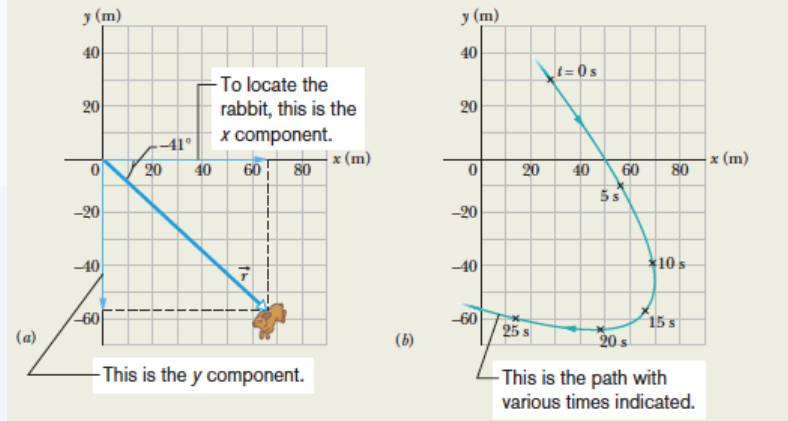
(We write $\vec{r}(t)$ rather than $\vec{r}$ because the components are functions of t, and thus $\vec{r}$ is also.)

At $t = 15$ s, the scalar components are

$$x = (-0.31)(15)^2 + (7.2)(15) + 28 = 66 \text{ m}$$

and $$y = (0.22)(15)^2 - (9.1)(15) + 30 = -57 \text{ m},$$

so $$\vec{r} = (66 \text{ m})\hat{i} - (57 \text{ m})\hat{j}, \qquad \text{(Answer)}$$



which is drawn in Fig. 4-2*a*. To get the magnitude and angle of $\vec{r}$, notice that the components form the legs of a right triangle and r is the hypotenuse. So, we use Eq. 3-6:

$$r = \sqrt{x^2 + y^2} = \sqrt{(66 \text{ m})^2 + (-57 \text{ m})^2}$$
$$= 87 \text{ m}, \qquad \text{(Answer)}$$

and $$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-57 \text{ m}}{66 \text{ m}}\right) = -41°. \qquad \text{(Answer)}$$

Check: Although $\theta = 139°$ has the same tangent as $-41°$, the components of position vector $\vec{r}$ indicate that the desired angle is $139° - 180° = -41°$.

(b) Graph the rabbit's path for $t = 0$ to $t = 25$ s.

Graphing: We have located the rabbit at one instant, but to see its path we need a graph. So we repeat part (a) for several values of t and then plot the results. Figure 4-2*b* shows the plots for six values of t and the path connecting them.

A rabbit runs across a parking lot on which a set of coordinate axes has, strangely enough, been drawn. The coordinates (meters) of the rabbit's position as functions of time t (seconds) are given by

$$x = -0.31t^2 + 7.2t + 28 \tag{4-5}$$

and

$$y = 0.22t^2 - 9.1t + 30. \tag{4-6}$$

(a) At $t = 15$ s, what is the rabbit's position vector $\vec{r}$ in unit-vector notation and in magnitude-angle notation?

For the rabbit in the preceding sample problem, find the velocity $\vec{v}$ at time $t = 15$ s.

KEY IDEA

We can find $\vec{v}$ by taking derivatives of the components of the rabbit's position vector.

Calculations: Applying the v_x part of Eq. 4-12 to Eq. 4-5, we find the x component of $\vec{v}$ to be

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(-0.31t^2 + 7.2t + 28) = -0.62t + 7.2. \tag{4-13}$$

At $t = 15$ s, this gives $v_x = -2.1$ m/s. Similarly, applying the v_y part of Eq. 4-12 to Eq. 4-6, we find

$$v_y = \frac{dy}{dt} = \frac{d}{dt}(0.22t^2 - 9.1t + 30) = 0.44t - 9.1. \tag{4-14}$$

At $t = 15$ s, this gives $v_y = -2.5$ m/s. Equation 4-11 then yields

$$\vec{v} = (-2.1 \text{ m/s})\hat{i} + (-2.5 \text{ m/s})\hat{j}, \qquad \text{(Answer)}$$

which is shown in Fig. 4-5, tangent to the rabbit's path and in the direction the rabbit is running at $t = 15$ s.

To get the magnitude and angle of $\vec{v}$, either we use a vector-capable calculator or we follow Eq. 3-6 to write

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(-2.1 \text{ m/s})^2 + (-2.5 \text{ m/s})^2} = 3.3 \text{ m/s} \qquad \text{(Answer)}$$

and

$$\theta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\left(\frac{-2.5 \text{ m/s}}{-2.1 \text{ m/s}}\right) = \tan^{-1} 1.19 = -130°. \qquad \text{(Answer)}$$

Check: Is the angle $-130°$ or $-130° + 180° = 50°$?

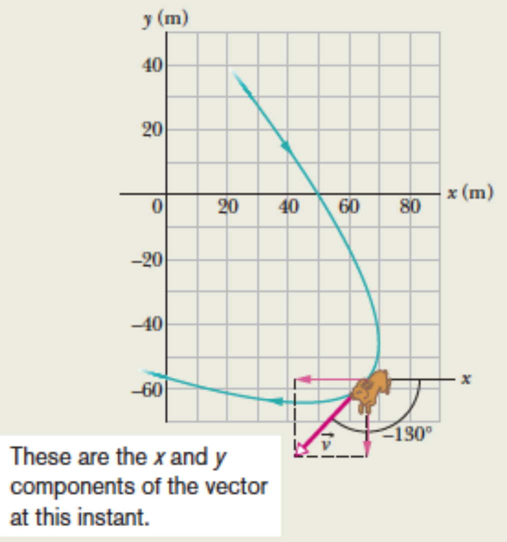


Figure 4-5 The rabbit's velocity $\vec{v}$ at $t = 15$ s.

4-4 Acceleration

- The average acceleration is defined as the rate at which the velocity changes

$$\bar{a} = \frac{\Delta \vec{v}}{\Delta t}$$

- The instantaneous acceleration is the limit of the average acceleration as Δt approaches zero

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t}$$

$$\vec{a} = \frac{d\vec{v}}{dt}. \tag{4-16}$$

If the velocity changes in *either* magnitude *or* direction (or both), the particle must have an acceleration.

We can write Eq. 4-16 in unit-vector form by substituting Eq. 4-11 for $\vec{v}$ to obtain

$$\vec{a} = \frac{d}{dt}(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}) = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}.$$

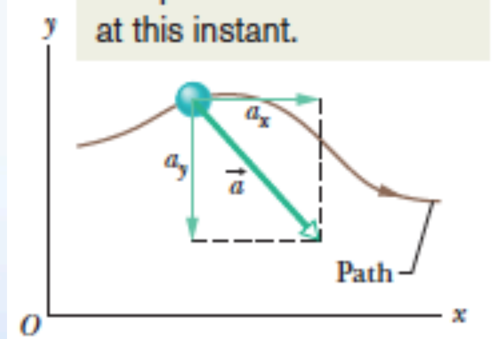
We can rewrite this as

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}, \tag{4-17}$$

where the scalar components of $\vec{a}$ are

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad \text{and} \quad a_z = \frac{dv_z}{dt}. \tag{4-18}$$

These are the x and y components of the vector at this instant.



Ways an Object Might Accelerate

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t}$$

- ✓ The magnitude of the velocity (the speed) can change

- ✓ The direction of the velocity can change

  Even though the magnitude is constant

- ✓ Both the magnitude and the direction can change

Example

For the rabbit in the preceding two sample problems, find the acceleration $\vec{a}$ at time $t = 15$ s.

KEY IDEA

We can find $\vec{a}$ by taking derivatives of the rabbit's velocity components.

Calculations: Applying the a_x part of Eq. 4-18 to Eq. 4-13, we find the x component of $\vec{a}$ to be

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(-0.62t + 7.2) = -0.62 \text{ m/s}^2.$$

Similarly, applying the a_y part of Eq. 4-18 to Eq. 4-14 yields the y component as

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}(0.44t - 9.1) = 0.44 \text{ m/s}^2.$$

We see that the acceleration does not vary with time (it is a constant) because the time variable t does not appear in the expression for either acceleration component. Equation 4-17 then yields

$$\vec{a} = (-0.62 \text{ m/s}^2)\hat{i} + (0.44 \text{ m/s}^2)\hat{j}, \quad \text{(Answer)}$$

which is superimposed on the rabbit's path in Fig. 4-7.

To get the magnitude and angle of $\vec{a}$, either we use a vector-capable calculator or we follow Eq. 3-6. For the magnitude we have

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-0.62 \text{ m/s}^2)^2 + (0.44 \text{ m/s}^2)^2}$$
$$= 0.76 \text{ m/s}^2. \quad \text{(Answer)}$$

For the angle we have

$$\theta = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\left(\frac{0.44 \text{ m/s}^2}{-0.62 \text{ m/s}^2}\right) = -35^\circ.$$

However, this angle, which is the one displayed on a calculator, indicates that $\vec{a}$ is directed to the right and downward in Fig. 4-7. Yet, we know from the components that $\vec{a}$ must be directed to the left and upward. To find the other angle that has the same tangent as -35° but is not displayed on a calculator, we add 180°:

$$-35^\circ + 180^\circ = 145^\circ. \quad \text{(Answer)}$$

This *is* consistent with the components of $\vec{a}$ because it gives a vector that is to the left and upward. Note that $\vec{a}$ has the same magnitude and direction throughout the rabbit's run because the acceleration is constant. That means that we could draw the very same vector at any other point along the rabbit's path (just shift the vector to put its tail at some other point on the path without changing the length or orientation).

This has been the second sample problem in which we needed to take the derivative of a vector that is written in unit-vector notation. One common error is to neglect the unit vectors themselves, with a result of only a set of numbers and symbols. Keep in mind that a derivative of a vector is always another vector.

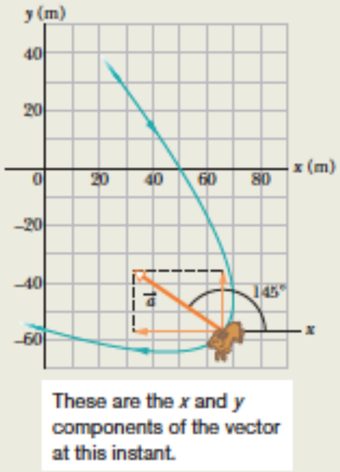
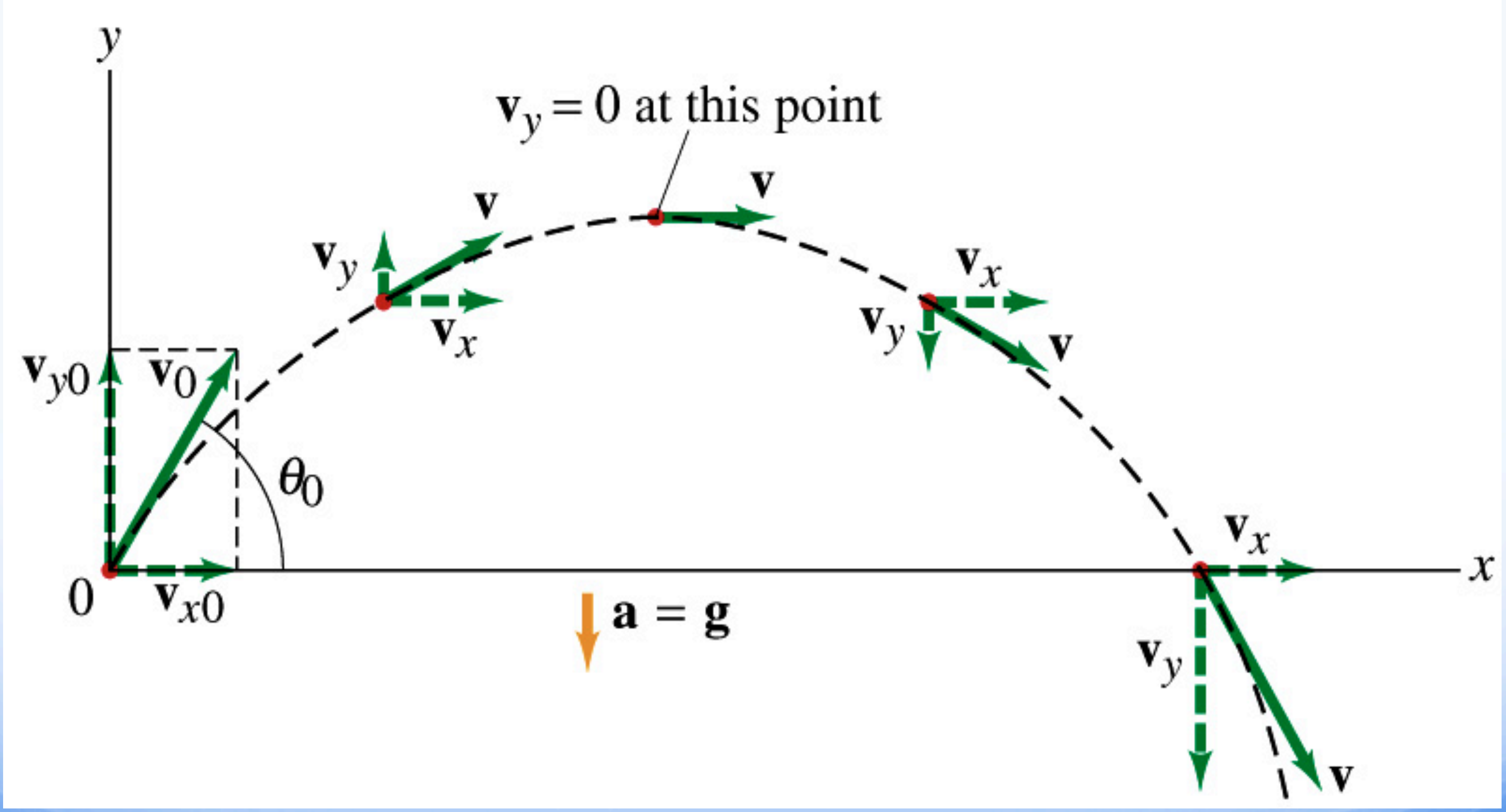


Figure 4-7 The acceleration $\vec{a}$ of the rabbit at $t = 15$ s. The rabbit happens to have this same acceleration at all points on its path.

4-4 Projectile Motion

- An object may move in both the x and y directions simultaneously (i.e. in two dimensions)
- The form of two dimensional motion we will deal with is called **projectile motion**
- We may:
  - ignore air friction
  - ignore the rotation of the earth
- With these assumptions, an object in projectile motion will follow a parabolic path



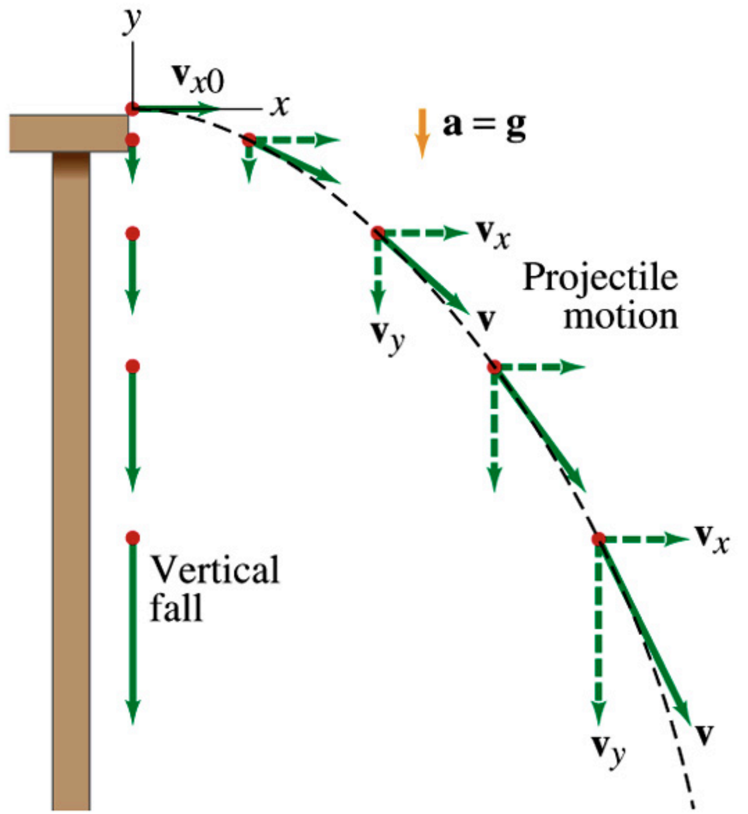
y
$\mathbf{v}_y = 0$ at this point
v
v
$\mathbf{v}_y$
$\mathbf{v}_x$
$\mathbf{v}_x$
$\mathbf{v}_y$
v
$\mathbf{v}_{y0}$
$\mathbf{v}_0$
θ_0
$\mathbf{v}_x$
x
0
$\mathbf{v}_{x0}$
$\mathbf{a} = \mathbf{g}$
$\mathbf{v}_y$
v

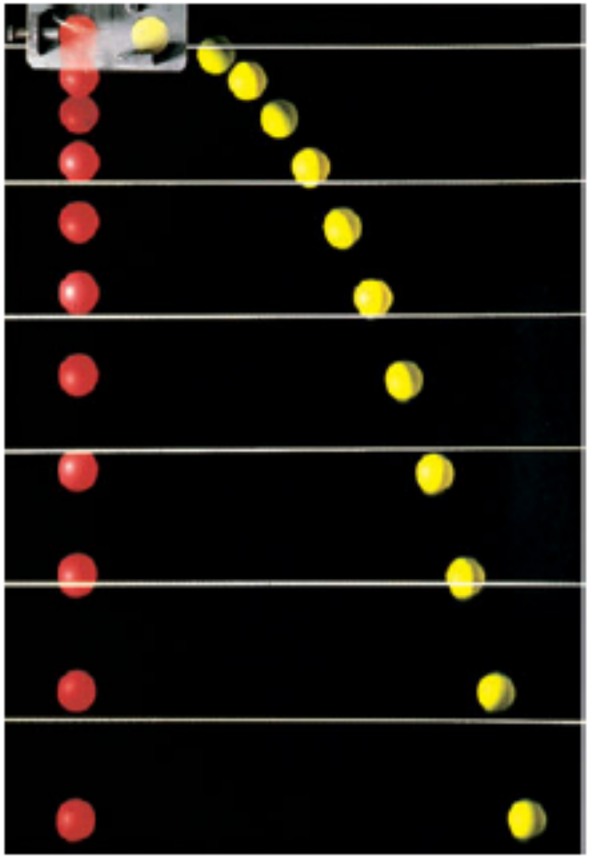
Notes on Projectile Motion:

- once released, only gravity pulls on the object, just like in up-and-down motion

- since gravity pulls on the object downwards:

  - ✓ vertical acceleration downwards
  - ✓ NO acceleration in horizontal direction





Richard Megna/Fundamental Photographs

Demonstration

- http://physics.wfu.edu/demolabs/demos/avimov/bychptr/chptr1_motion.html

FIGURE 3–27 Examples of projectile motion: a boy jumping, and glowing lava from the volcano Stromboli.

Rules of Projectile Motion

- Introduce coordinate frame: y is up
- The x- and y-components of motion can be treated independently
- Velocities (incl. initial velocity) can be broken down into its x- and y-components
- The x-direction is uniform motion

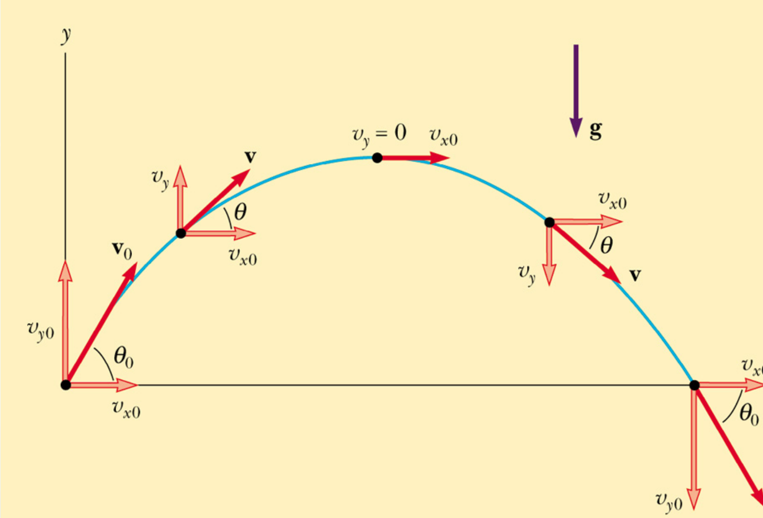
  $$a_x = 0$$

- The y-direction is free fall

  $$|a_y| = g$$

Some Details About the Rules

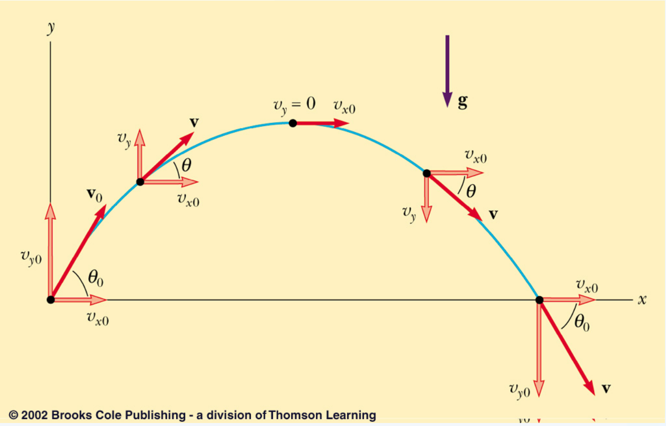
$$v_{xo} = v_o \cos\theta_o = v_x = \text{constant}$$



- x-direction
  - $a_x = 0$
  - $x = v_{xo}t$
    - This is the only operative equation in the x-direction since there is uniform velocity in that direction

More Details About the Rules

$$v_{yo} = v_o \sin\theta_o$$



- y-direction
  - take the positive direction as upward
  - then: free fall problem
    - only then: $a_y = -g$ (in general, $|a_y| = g$)
  - uniformly accelerated motion, so the motion equations all hold

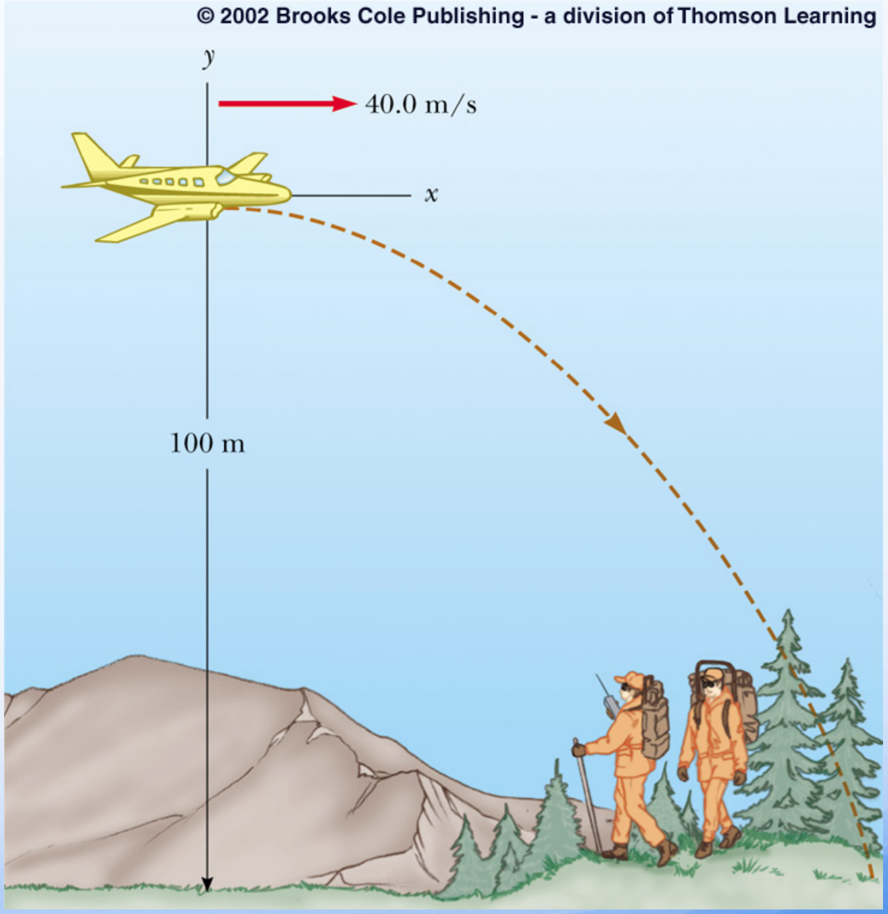
Velocity of the Projectile

- The velocity of the projectile at any point of its motion is the vector sum of its x and y components at that point

$$\boxed{v = \sqrt{v_x^2 + v_y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{v_y}{v_x}}$$

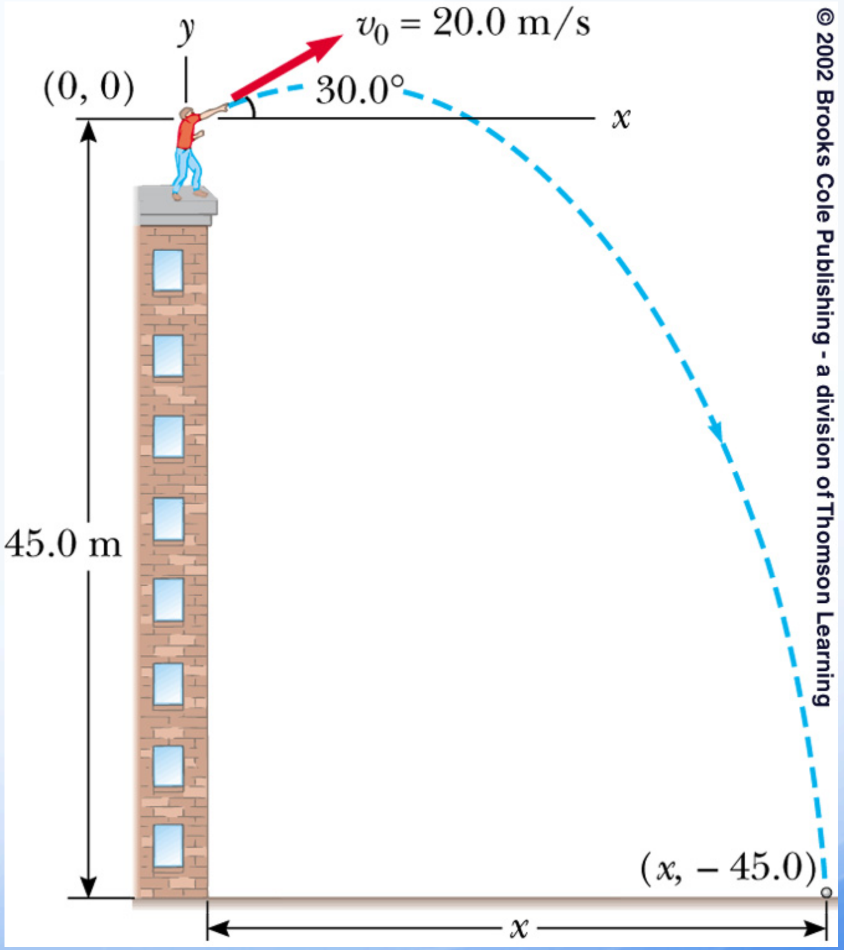
Examples of Projectile Motion:

- An object may be fired horizontally
- The initial velocity is all in the x-direction
  - $v_o = v_x$ and $v_y = 0$
- All the general rules of projectile motion apply



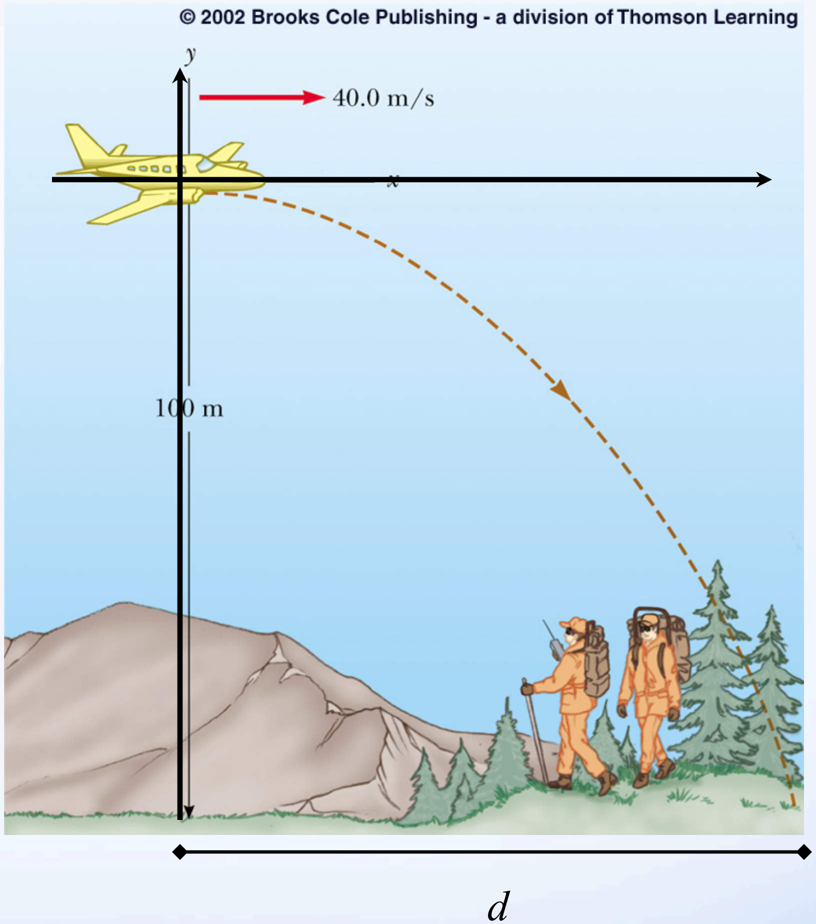
Non-Symmetrical Projectile Motion

- Follow the general rules for projectile motion
- Break the y-direction into parts
  - up and down
  - symmetrical back to initial height and then the rest of the height



Example problem:

An Alaskan rescue plane drops a package of emergency rations to a stranded party of explorers. The plane is traveling horizontally at 40.0 m/s at a height of 100 m above the ground.
Where does the package strike the ground relative to the point at which it was released?



Given:

velocity: v=40.0 m/s
height: h=100 m

Find:

Distance d=?

1. Introduce coordinate frame:
   Oy: y is directed up
   Ox: x is directed right

2. Note: $v_{ox} = v = +40$ m/s
   $v_{oy} = 0$ m/s

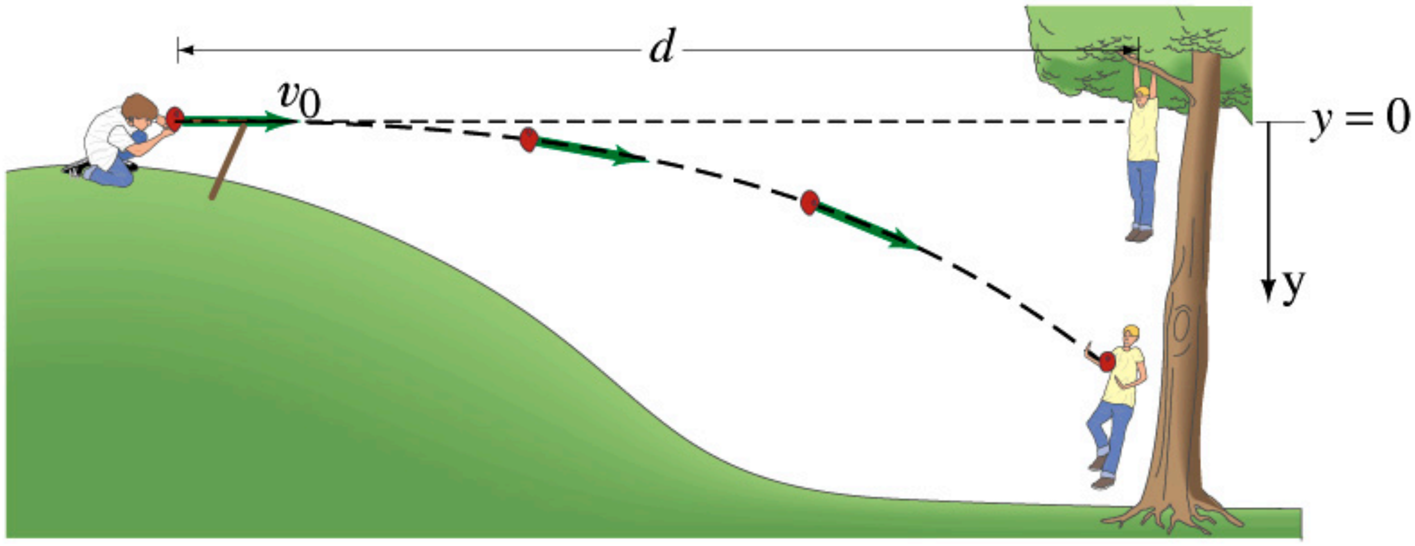
$$\underline{Oy}: y = \frac{1}{2} g t^2, so \quad t = \sqrt{\frac{2y}{g}}$$

$$or: \quad t = \sqrt{\frac{2(-100\,m)}{-9.8\,m/s^2}} = 4.51\,s$$

$$\underline{Ox}: x = v_{x0} t, so \quad x = (40\,m/\cancel{s})(4.51\cancel{s}) = \underline{180\,m}$$

✓

Example



Demonstration

- http://physics.wfu.edu/demolabs/demos/avimov/bychptr/chptr1_motion.html

Example

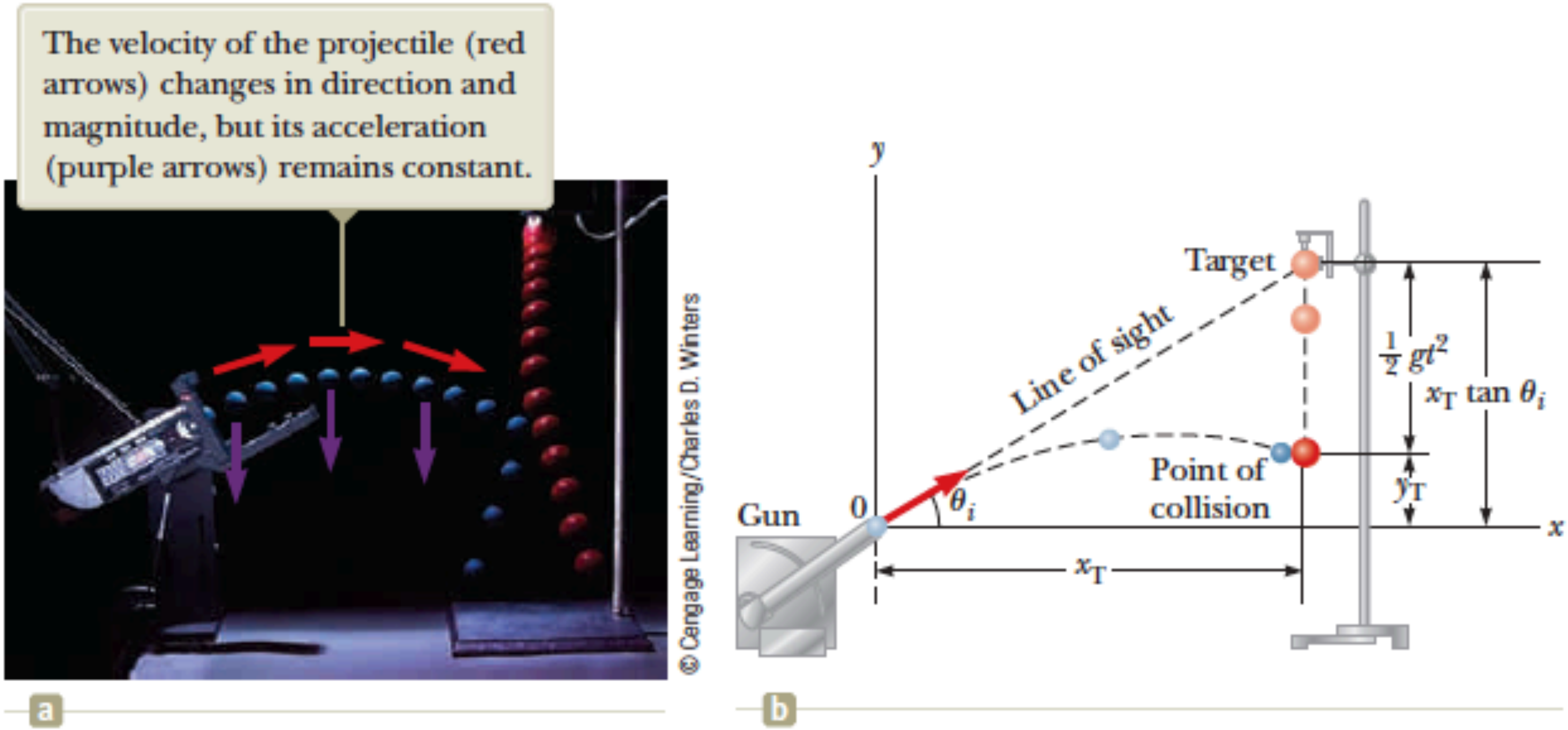
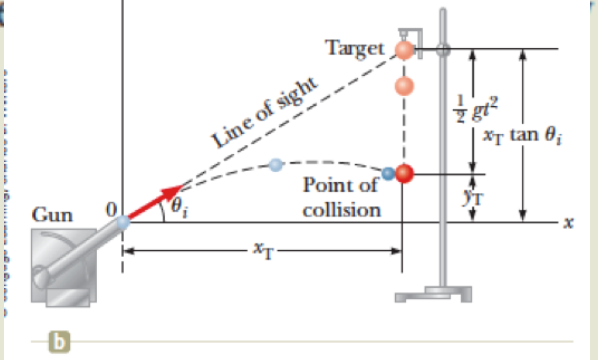


Figure 4.12 (Example 4.3) (a) Multiflash photograph of the projectile–target demonstration. If the gun is aimed directly at the target and is fired at the same instant the target begins to fall, the projectile will hit the target. (b) Schematic diagram of the projectile–target demonstration.

onstant acceleration in the y direction and a particl



$$(1)\ y_T = y_{iT} + (0)t - \tfrac{1}{2}gt^2 = x_T \tan\theta_i - \tfrac{1}{2}gt^2$$

$$(2)\ y_P = y_{iP} + v_{yiP}t - \tfrac{1}{2}gt^2 = 0 + (v_{iP}\sin\theta_i)t - \tfrac{1}{2}gt^2 = (v_{iP}\sin\theta_i)t - \tfrac{1}{2}gt^2$$

$$x_P = x_{iP} + v_{xiP}t = 0 + (v_{iP}\cos\theta_i)t = (v_{iP}\cos\theta_i)t$$

$$t = \frac{x_P}{v_{iP}\cos\theta_i}$$

$$(3)\ y_P = (v_{iP}\sin\theta_i)\left(\frac{x_P}{v_{iP}\cos\theta_i}\right) - \tfrac{1}{2}gt^2 = x_P\tan\theta_i - \tfrac{1}{2}gt^2$$

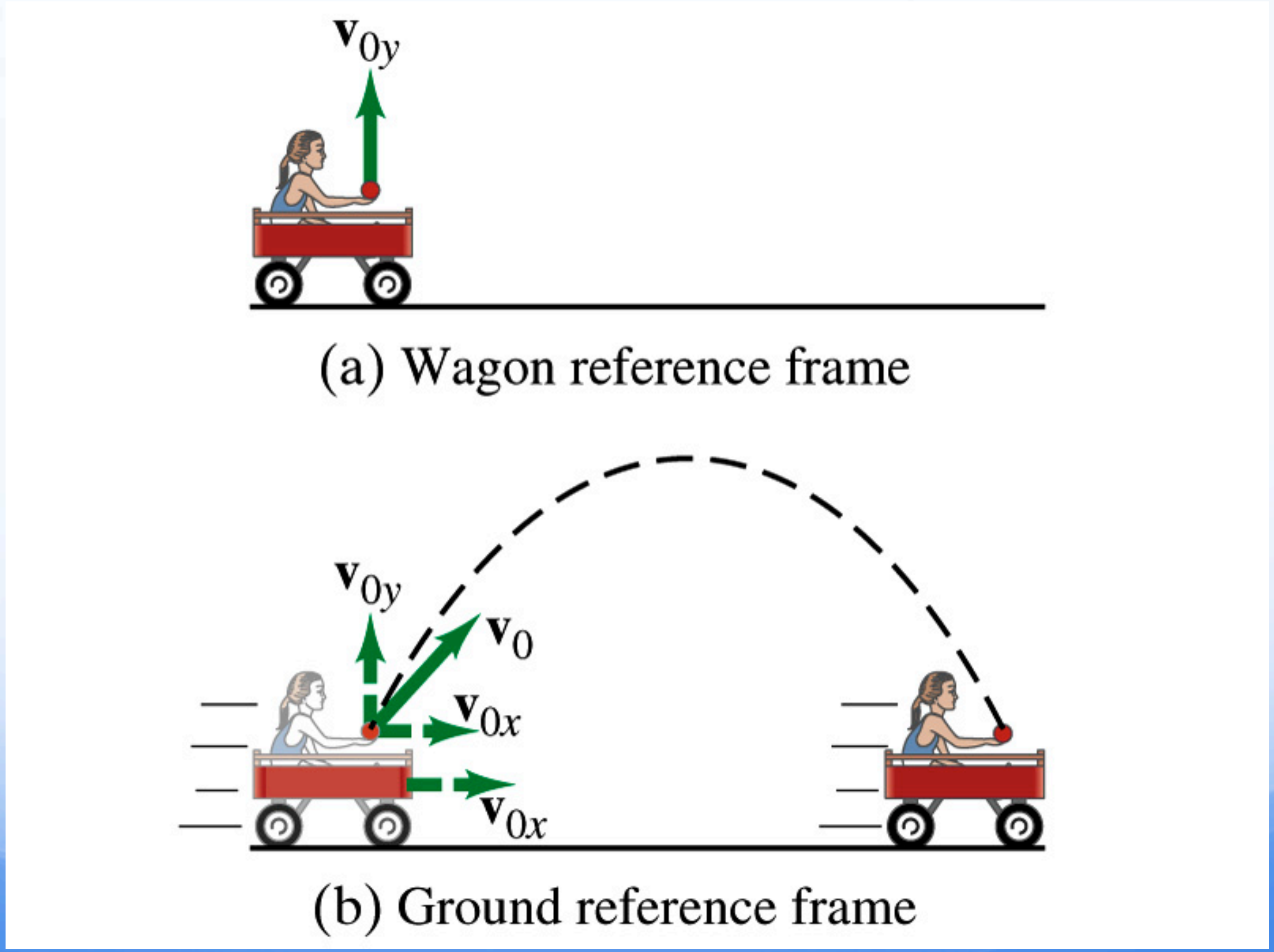
t when the x coordinates of the projectile and target are the same—that is, Equations (1) and (3) are the same and a collision results.

y when $v_{iP}\sin\theta_i \geq \sqrt{gd/2}$, where d is the initial elevation of the target above he projectile strikes the floor before reaching the target.

Example

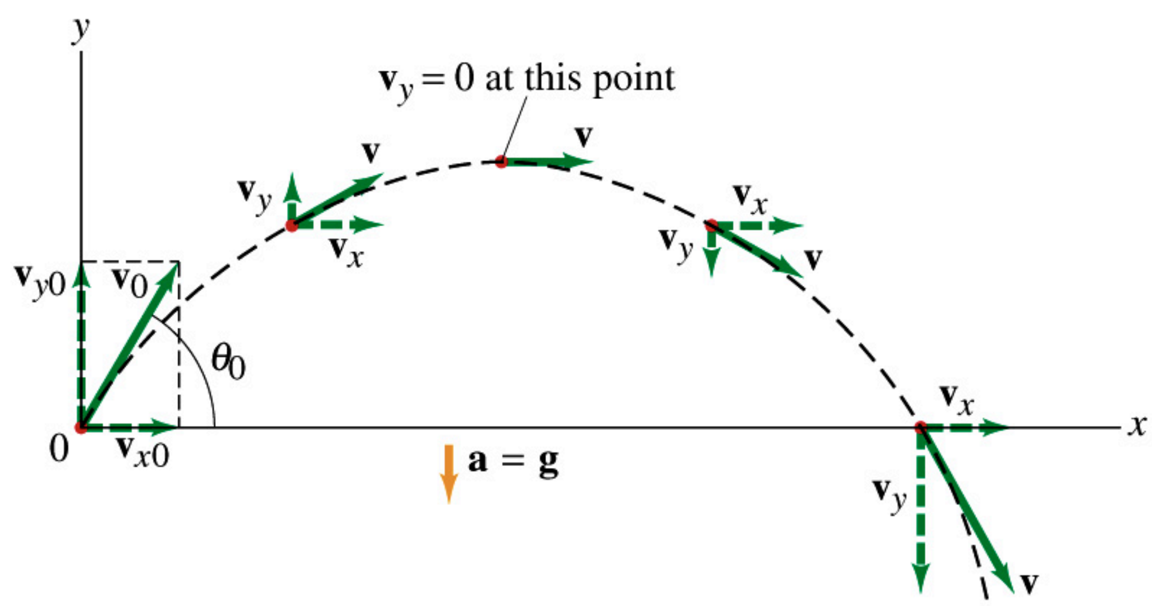
(a) Wagon reference frame

Example



$\mathbf{v}_{0y}$
(a) Wagon reference frame
$\mathbf{v}_{0y}$
$\mathbf{v}_0$
$\mathbf{v}_{0x}$
$\mathbf{v}_{0x}$
(b) Ground reference frame

Projectile Motion Analyzed



$$\vec{v}_0 = v_{0x}\hat{\mathrm{i}} + v_{0y}\hat{\mathrm{j}}$$

$$v_{0x} = v_0 \cos\theta_0 \quad \text{and} \quad v_{0y} = v_0 \sin\theta_0.$$

$$x - x_0 = v_{0x}t.$$

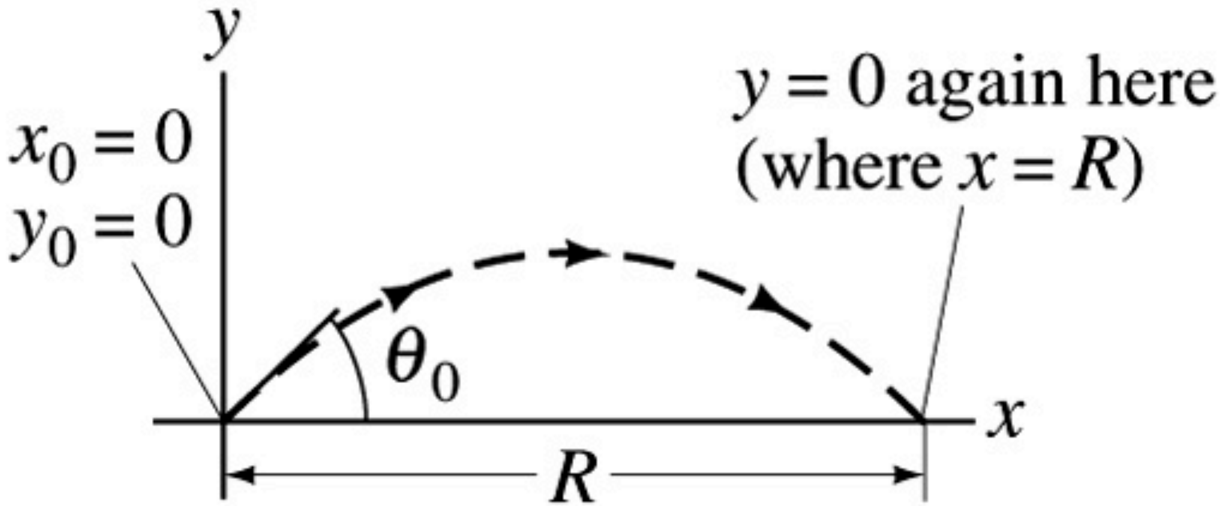
$$x - x_0 = (v_0 \cos \theta_0)t.$$

$$
\begin{aligned}
y - y_0 &= v_{0y}t - \tfrac{1}{2}gt^2 \\
&= (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2,
\end{aligned}
$$

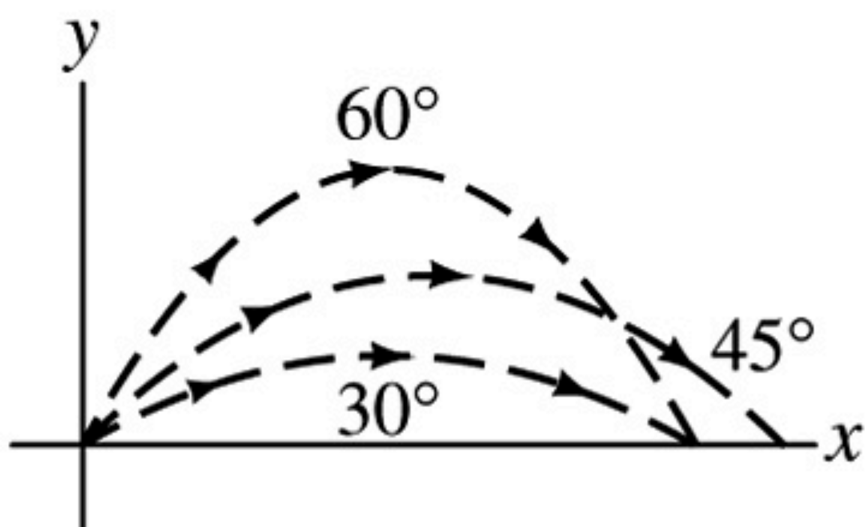
$$v_y = v_0 \sin \theta_0 - gt$$

$$v_y^2 = (v_0 \sin \theta_0)^2 - 2g(y - y_0).$$

$$y = (\tan \theta_0)x - \frac{gx^2}{2(v_0 \cos \theta_0)^2} \qquad \text{(trajectory).}$$



(a)



(b)

The Horizontal Range

The *horizontal range* R of the projectile is the *horizontal* distance the projectile has traveled when it returns to its initial height (the height at which it is launched). To find range R, let us put $x - x_0 = R$ in Eq. 4-21 and $y - y_0 = 0$ in Eq. 4-22, obtaining

$$R = (v_0 \cos \theta_0)t$$

and

$$0 = (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2.$$

Eliminating t between these two equations yields

$$R = \frac{2v_0^2}{g} \sin \theta_0 \cos \theta_0.$$

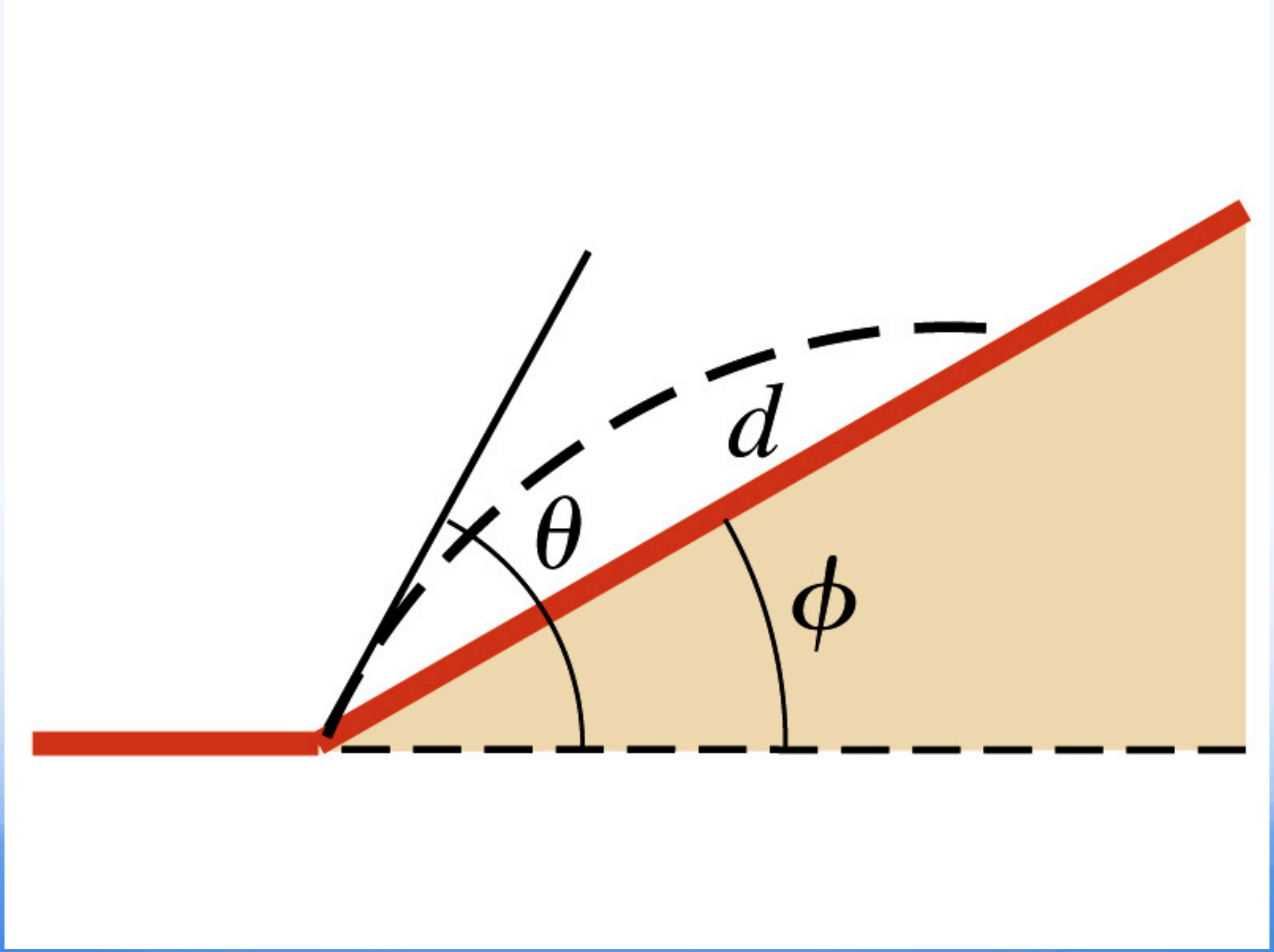
Using the identity $\sin 2\theta_0 = 2 \sin \theta_0 \cos \theta_0$ (see Appendix E), we obtain

$$R = \frac{v_0^2}{g} \sin 2\theta_0. \tag{4-26}$$

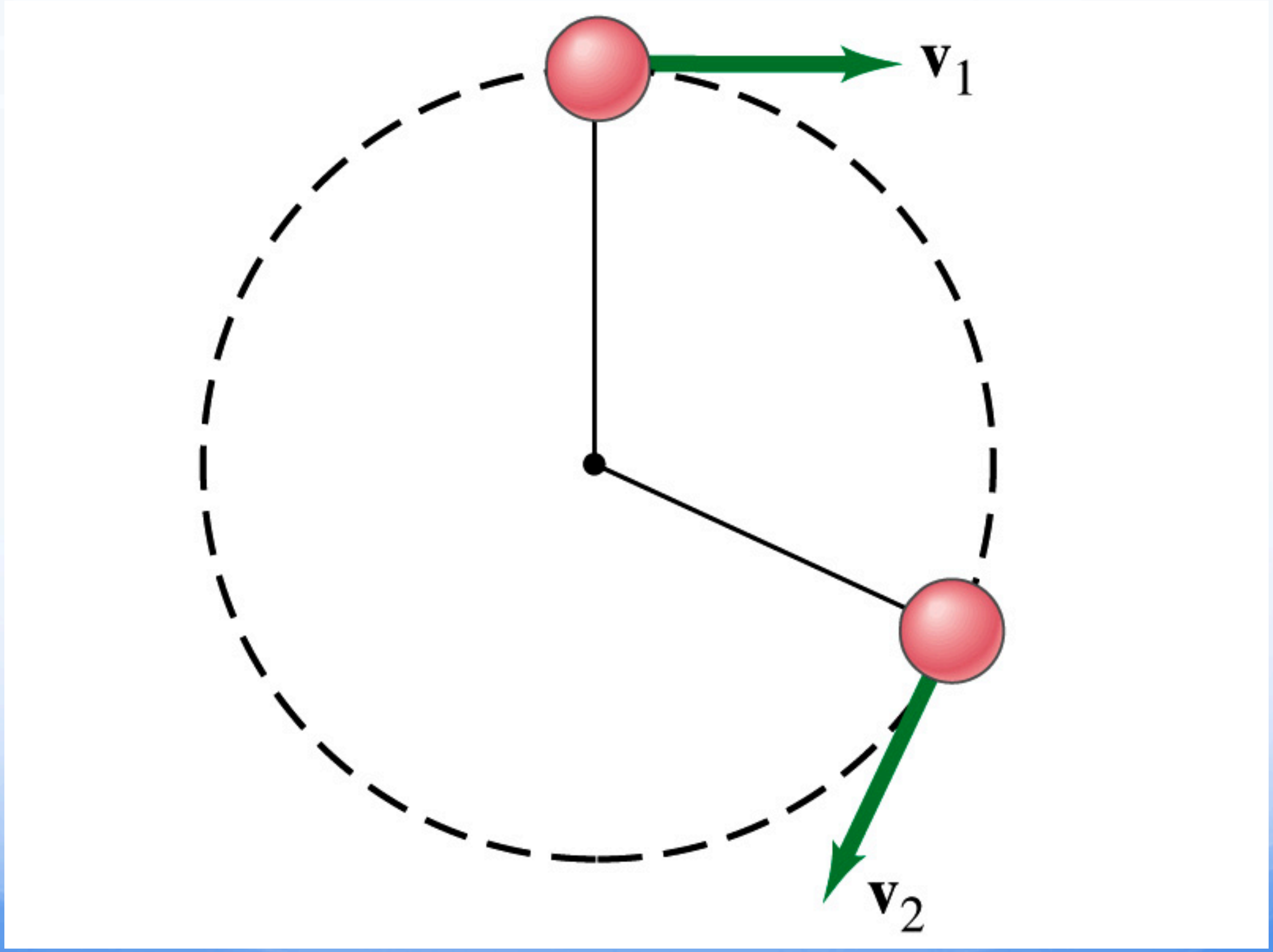
This equation does *not* give the horizontal distance traveled by a projectile when the final height is not the launch height. Note that R in Eq. 4-26 has its maximum value when $\sin 2\theta_0 = 1$, which corresponds to $2\theta_0 = 90°$ or $\theta_0 = 45°$.

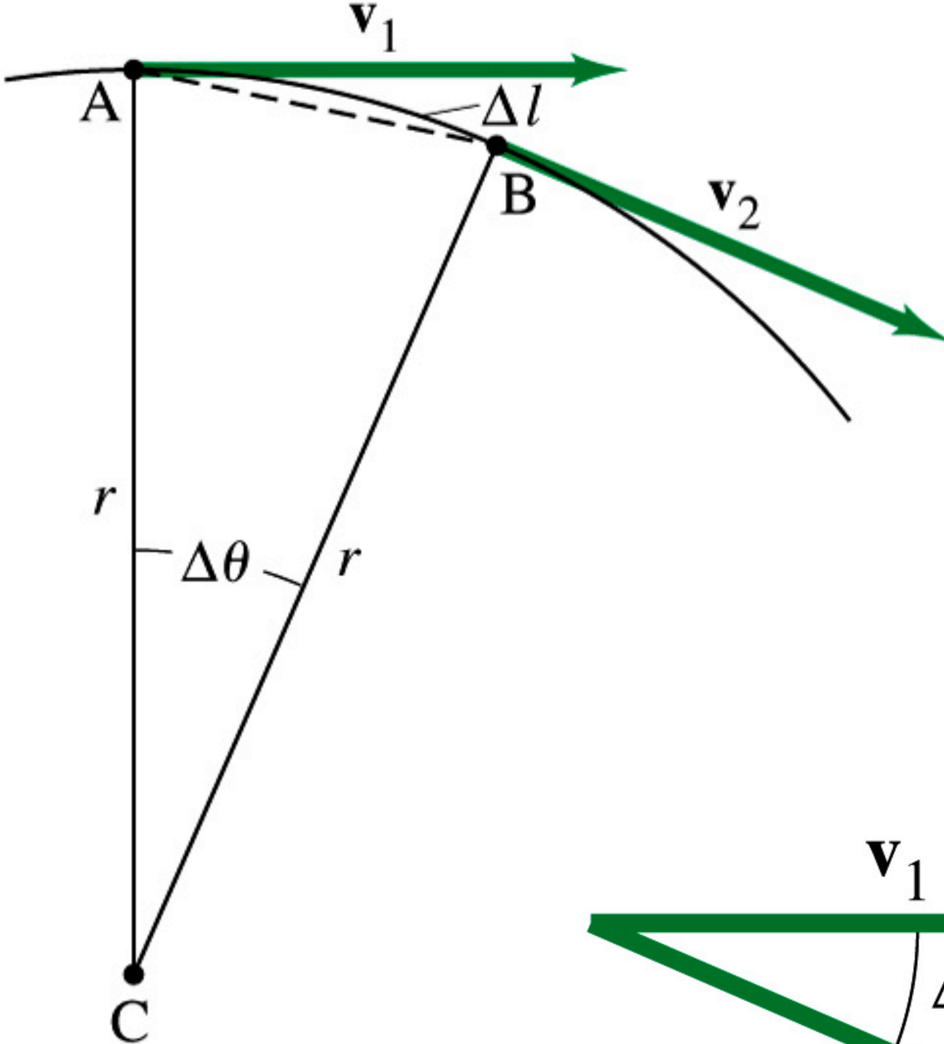
The horizontal range R is maximum for a launch angle of 45°.

Example



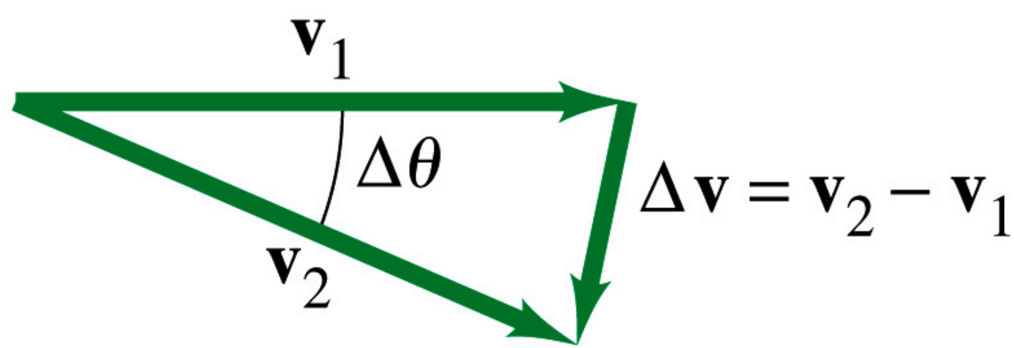
4-5 Uniform Circular Motion





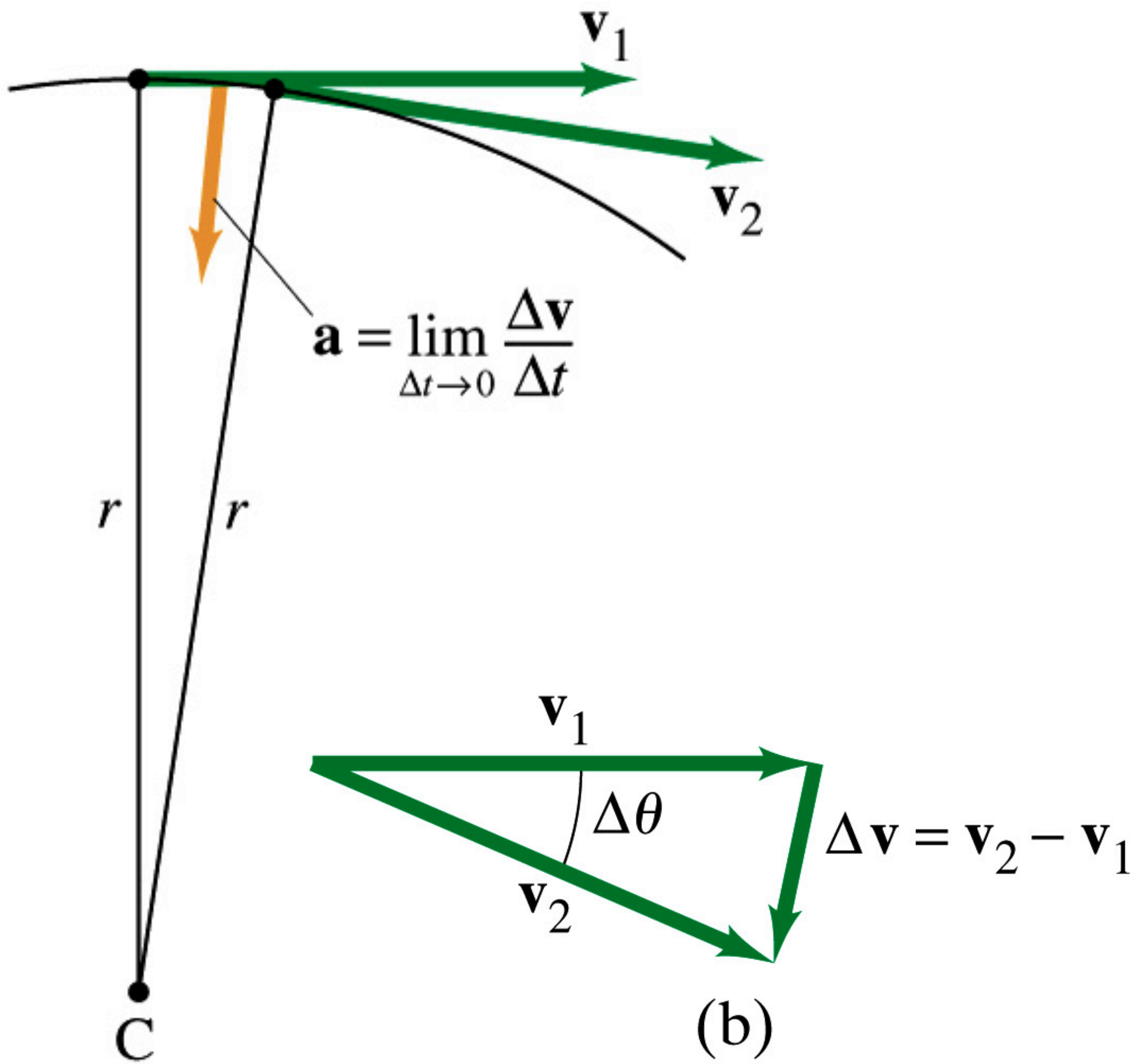
$\mathbf{v}_1$
A
Δl
B
$\mathbf{v}_2$
r
$\Delta\theta$
r
C

(a)

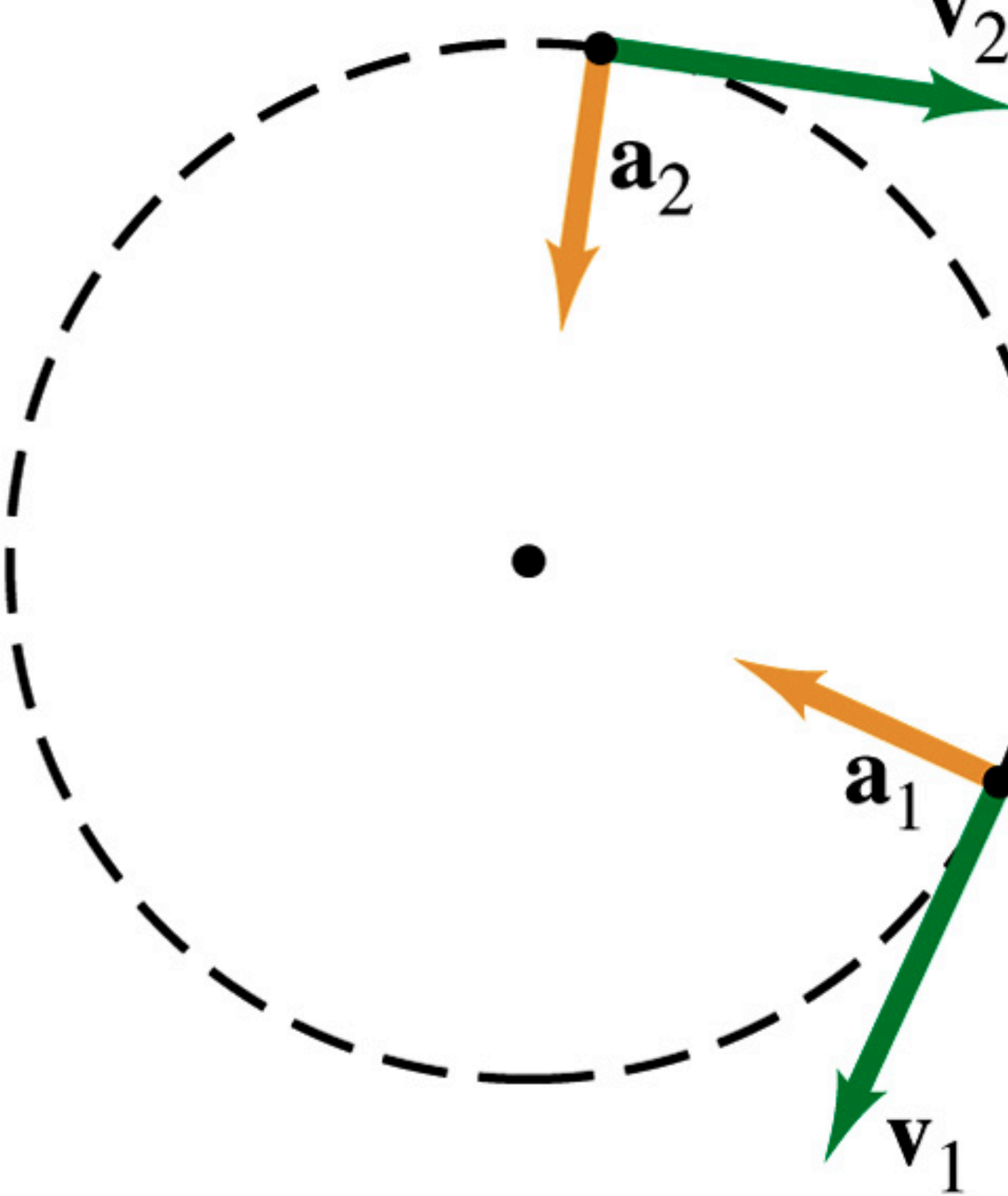


$\mathbf{v}_1$
$\Delta\theta$
$\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$
$\mathbf{v}_2$

(b)



$\mathbf{v}_1$
$\mathbf{v}_2$
$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t}$
r
r
C
$\mathbf{v}_1$
$\Delta\theta$
$\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$
$\mathbf{v}_2$
(b)



$\mathbf{v}_2$
$\mathbf{a}_2$
$\mathbf{a}_1$
$\mathbf{v}_1$

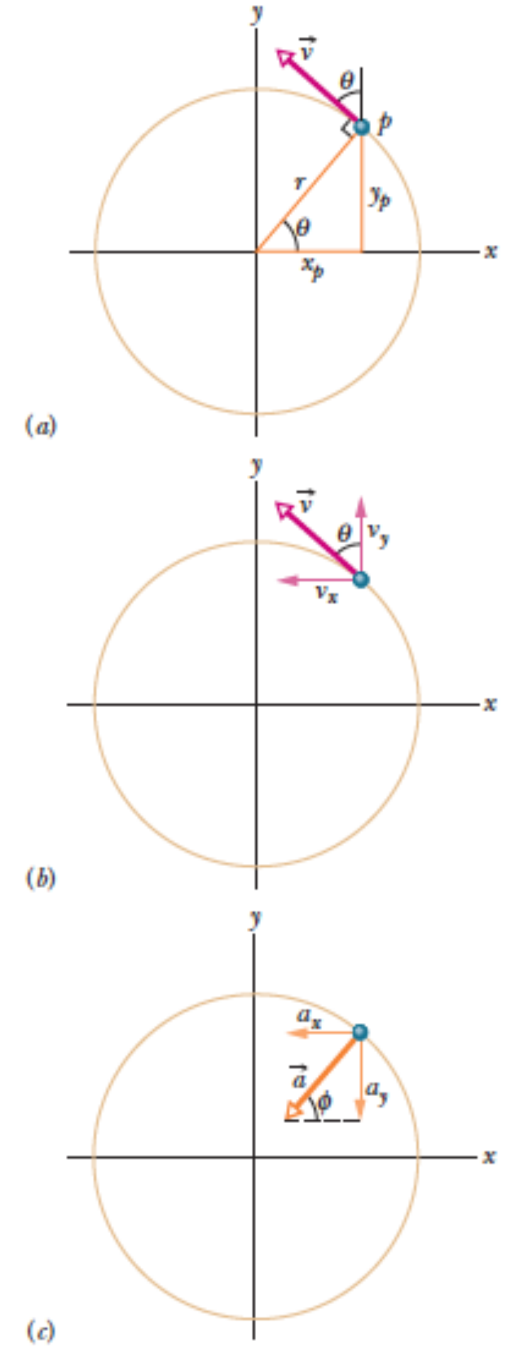
Uniform Circular Motion

$$\vec{v} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} = (-v \sin\theta)\hat{\mathrm{i}} + (v\cos\theta)\hat{\mathrm{j}}.$$

$$\vec{v} = \left(-\frac{vy_p}{r}\right)\hat{\mathrm{i}} + \left(\frac{vx_p}{r}\right)\hat{\mathrm{j}}.$$

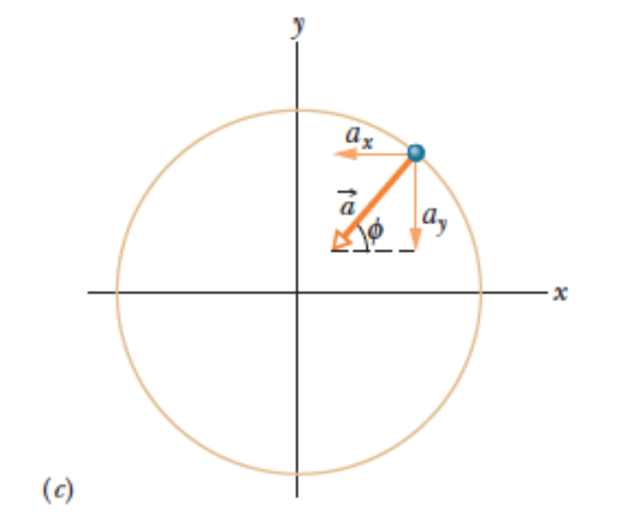
$$\vec{a} = \frac{d\vec{v}}{dt} = \left(-\frac{v}{r}\frac{dy_p}{dt}\right)\hat{\mathrm{i}} + \left(\frac{v}{r}\frac{dx_p}{dt}\right)\hat{\mathrm{j}}.$$

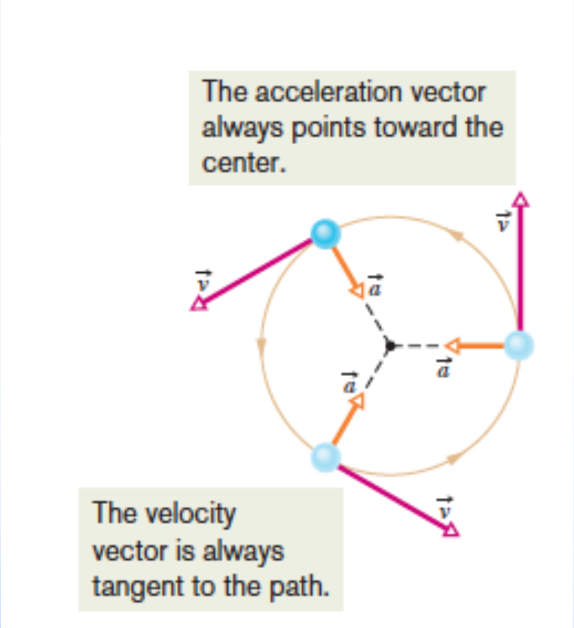
$$\vec{a} = \left(-\frac{v^2}{r}\cos\theta\right)\hat{\mathrm{i}} + \left(-\frac{v^2}{r}\sin\theta\right)\hat{\mathrm{j}}.$$



$$a = \sqrt{a_x^2 + a_y^2} = \frac{v^2}{r}\sqrt{(\cos\theta)^2 + (\sin\theta)^2} = \frac{v^2}{r}\sqrt{1} = \frac{v^2}{r},$$

$$\tan\phi = \frac{a_y}{a_x} = \frac{-(v^2/r)\sin\theta}{-(v^2/r)\cos\theta} = \tan\theta.$$





$$a = \frac{v^2}{r} \quad \text{(centripetal acceleration)},$$

$$T = \frac{2\pi r}{v} \quad \text{(period)}.$$

Example

Example 4.6 The Centripetal Acceleration of the Earth

What is the centripetal acceleration of the Earth as it moves in its orbit around the Sun?

SOLUTION

Conceptualize Think about a mental image of the Earth in a circular orbit around the Sun. We will model the Earth as a particle and approximate the Earth's orbit as circular (it's actually slightly elliptical, as we discuss in Chapter 13).

Categorize The Conceptualize step allows us to categorize this problem as one of a particle in uniform circular motion.

Analyze We do not know the orbital speed of the Earth to substitute into Equation 4.14. With the help of Equation 4.15, however, we can recast Equation 4.14 in terms of the period of the Earth's orbit, which we know is one year, and the radius of the Earth's orbit around the Sun, which is 1.496×10^{11} m.

Combine Equations 4.14 and 4.15:

$$a_c = \frac{v^2}{r} = \frac{\left(\frac{2\pi r}{T}\right)^2}{r} = \frac{4\pi^2 r}{T^2}$$

continued

4.6 *cont.*

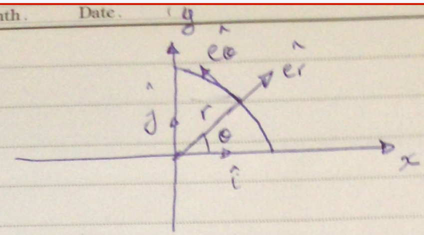
Substitute numerical values:

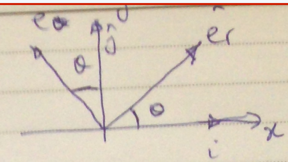
$$a_c = \frac{4\pi^2(1.496 \times 10^{11}\text{ m})}{(1\text{ yr})^2}\left(\frac{1\text{ yr}}{3.156 \times 10^7\text{ s}}\right)^2 = \boxed{5.93 \times 10^{-3}\text{ m/s}^2}$$

Finalize This acceleration is much smaller than the free-fall acceleration on the surface of the Earth. An important technique we learned here is replacing the speed v in Equation 4.14 in terms of the period T of the motion. In many problems, it is more likely that T is known rather than v.

Non-uniform circular motion

Hurricane Katrina





$\vec r = r\hat e_r \qquad \dot{\vec r} = \dot r\,\hat e_r + r\,\dot{\hat e}_r$

$\hat e_r = \cos\theta\,\hat i + \sin\theta\,\hat j$

$\hat e_\theta = -\sin\theta\,\hat i + \cos\theta\,\hat j$

$\dot{\hat e}_r = -\dot\theta\sin\theta\,\hat i + \dot\theta\cos\theta\,\hat j$

$= \dot\theta(-\sin\theta\,\hat i + \cos\theta\,\hat j)$

$\dot{\hat e}_r = \dot\theta\,\hat e_\theta$

Then

$\dot{\vec r} = \dot r\,\hat e_r + r\dot\theta\,\hat e_\theta$

$\ddot{\vec r} = \ddot r\,\hat e_r + \dot r\,\overset{\to\dot\theta\hat e_\theta}{\dot{\hat e}_r} + \dot r\dot\theta\,\hat e_\theta$

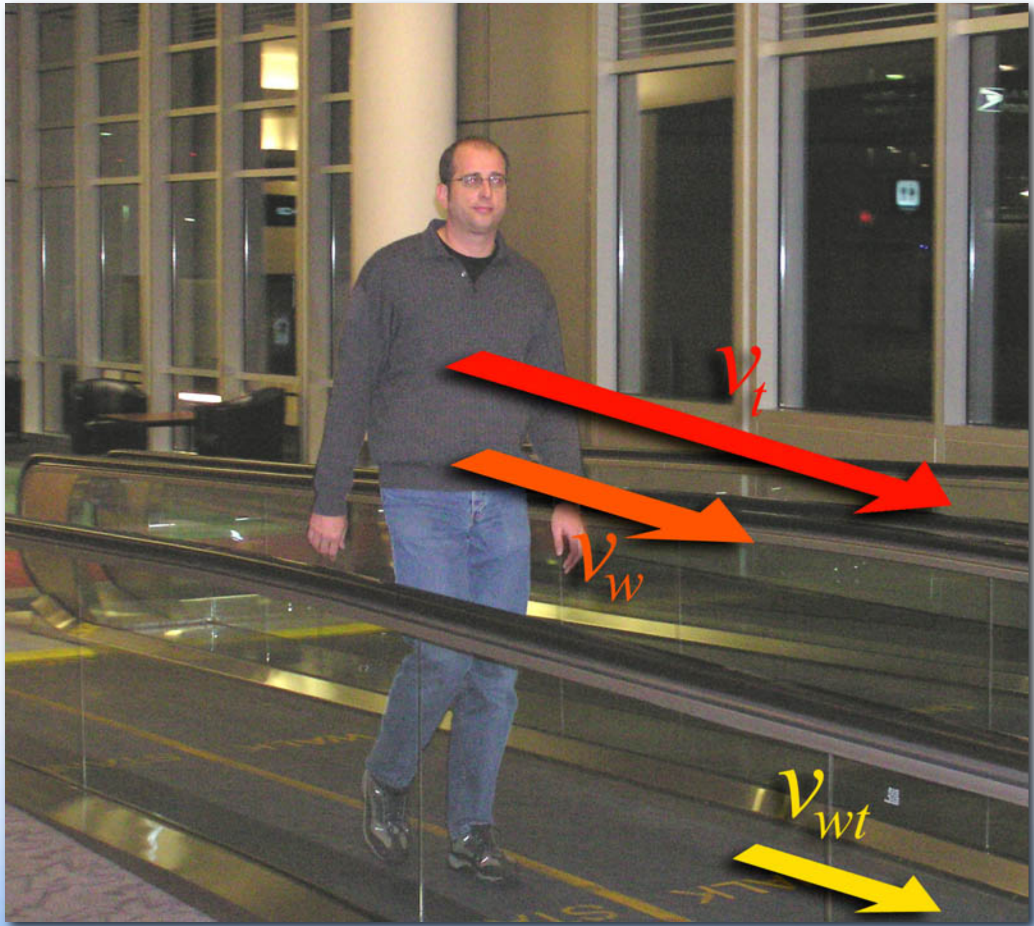
$+ r\ddot\theta\,\hat e_\theta + r\dot\theta\,\dot{\hat e}_\theta$

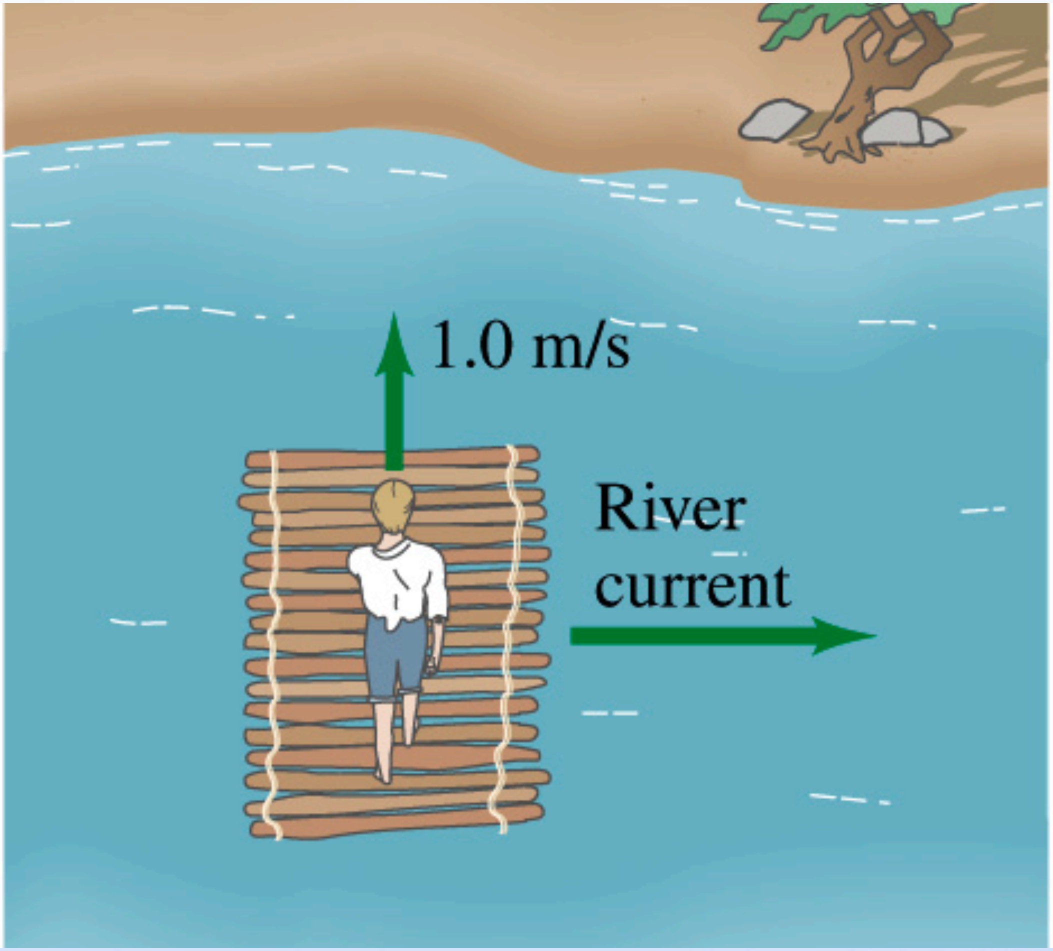
$\dot{\hat e}_\theta = -\dot\theta\cos\theta\,\hat i - \dot\theta\sin\theta\,\hat j = -\dot\theta\,\hat e_r$

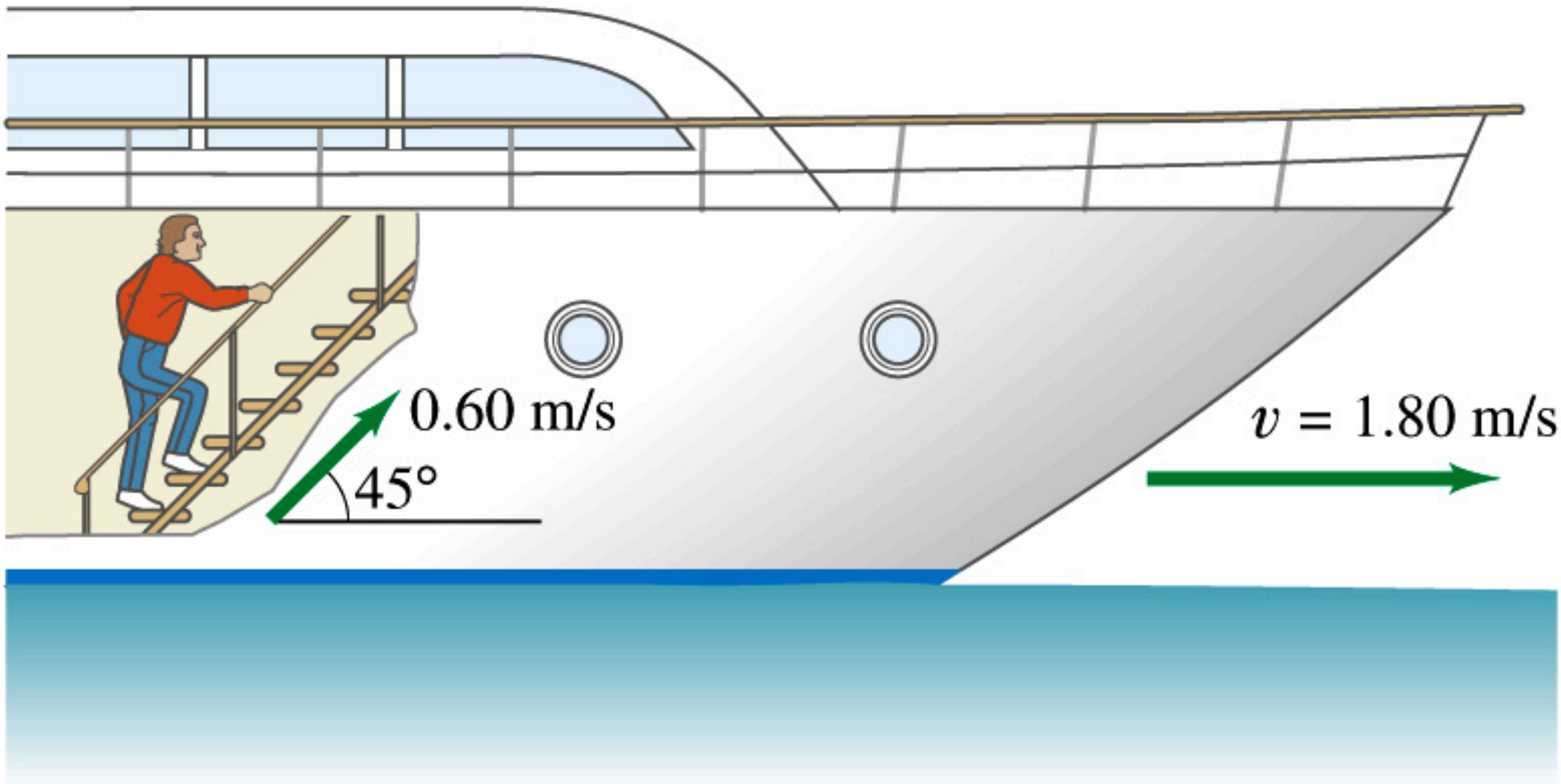
$$\boxed{\ddot{\vec r} = (\ddot r - r\dot\theta^2)\,\hat e_r + (r\ddot\theta + 2\dot r\dot\theta)\,\hat e_\theta}$$

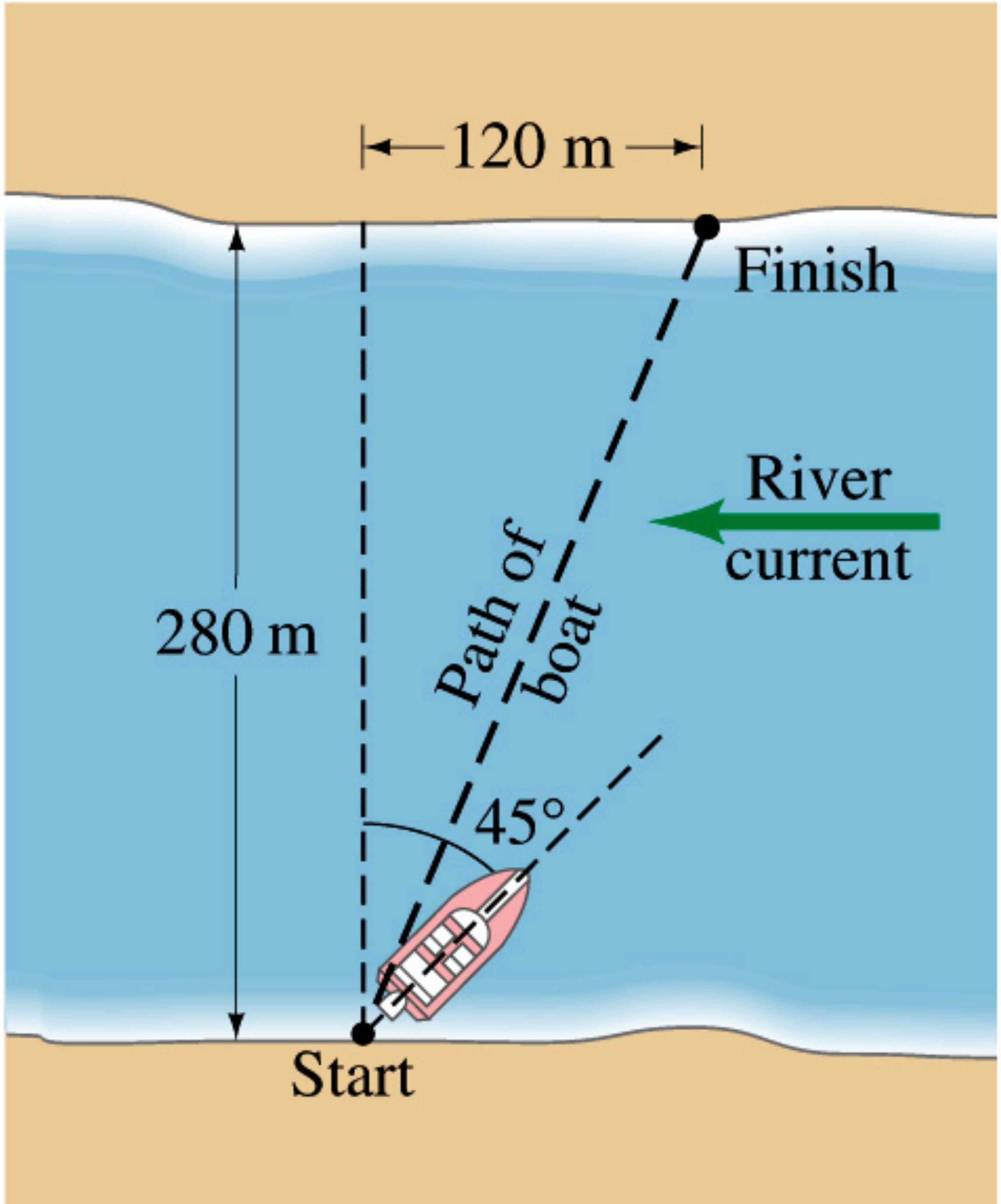
$\boxed{V_r = \dot r} \qquad \boxed{V_\theta = r\dot\theta = r\omega}$

4-8 (9) Relative Motion in One (two) Dimension(s)

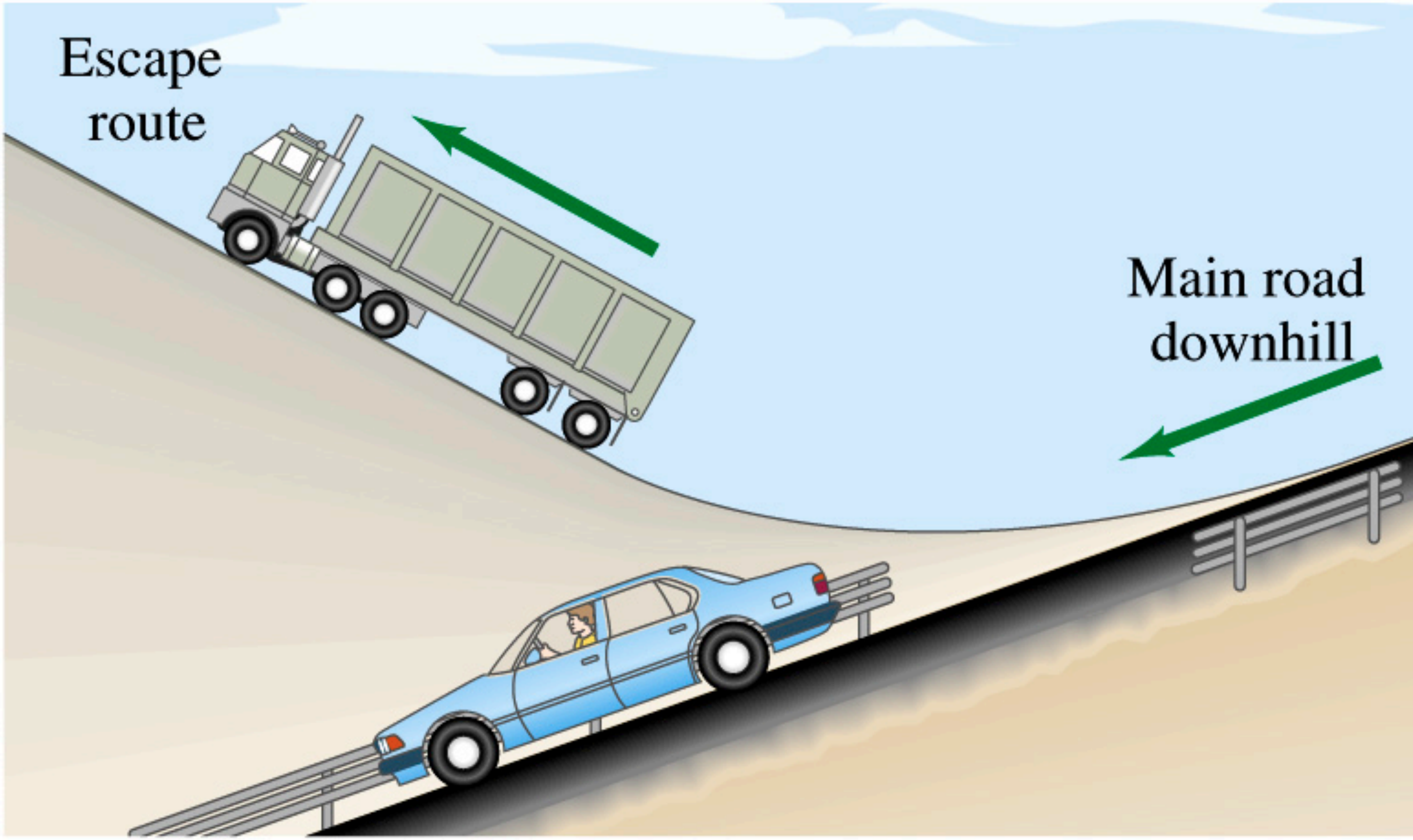



1.0 m/s
River
current

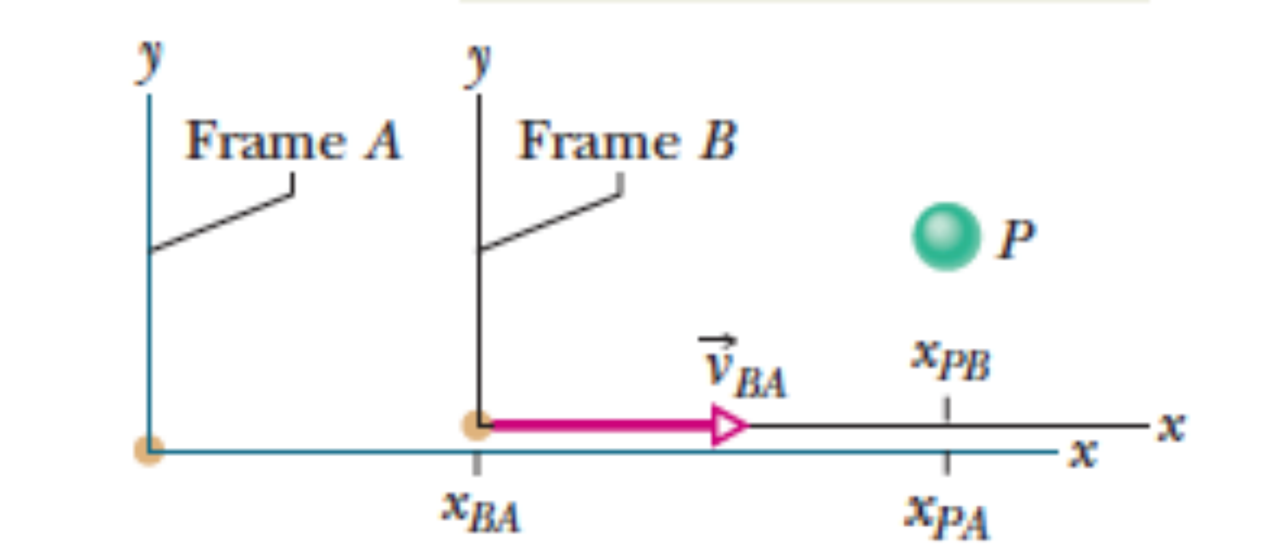

0.60 m/s
45°
v = 1.80 m/s



120 m
Finish
River
current
280 m
Path of
boat
45°
Start


Escape route
Main road downhill


θ



$$x_{PA} = x_{PB} + x_{BA}.$$

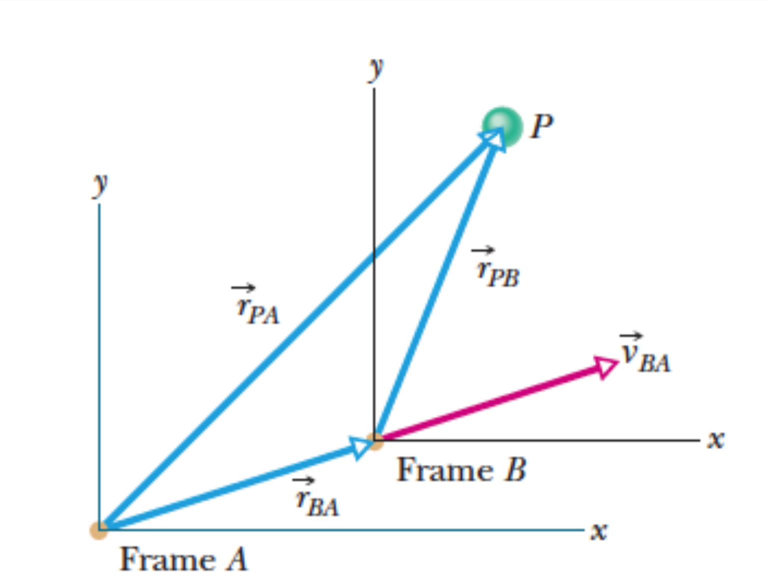
$$\frac{d}{dt}(x_{PA}) = \frac{d}{dt}(x_{PB}) + \frac{d}{dt}(x_{BA}).$$

$$\frac{d}{dt}(v_{PA}) = \frac{d}{dt}(v_{PB}) + \frac{d}{dt}(v_{BA}).$$

$$v_{PA} = v_{PB} + v_{BA}.$$

$$a_{PA} = a_{PB}.$$

Relative Motion in Two Dimensions

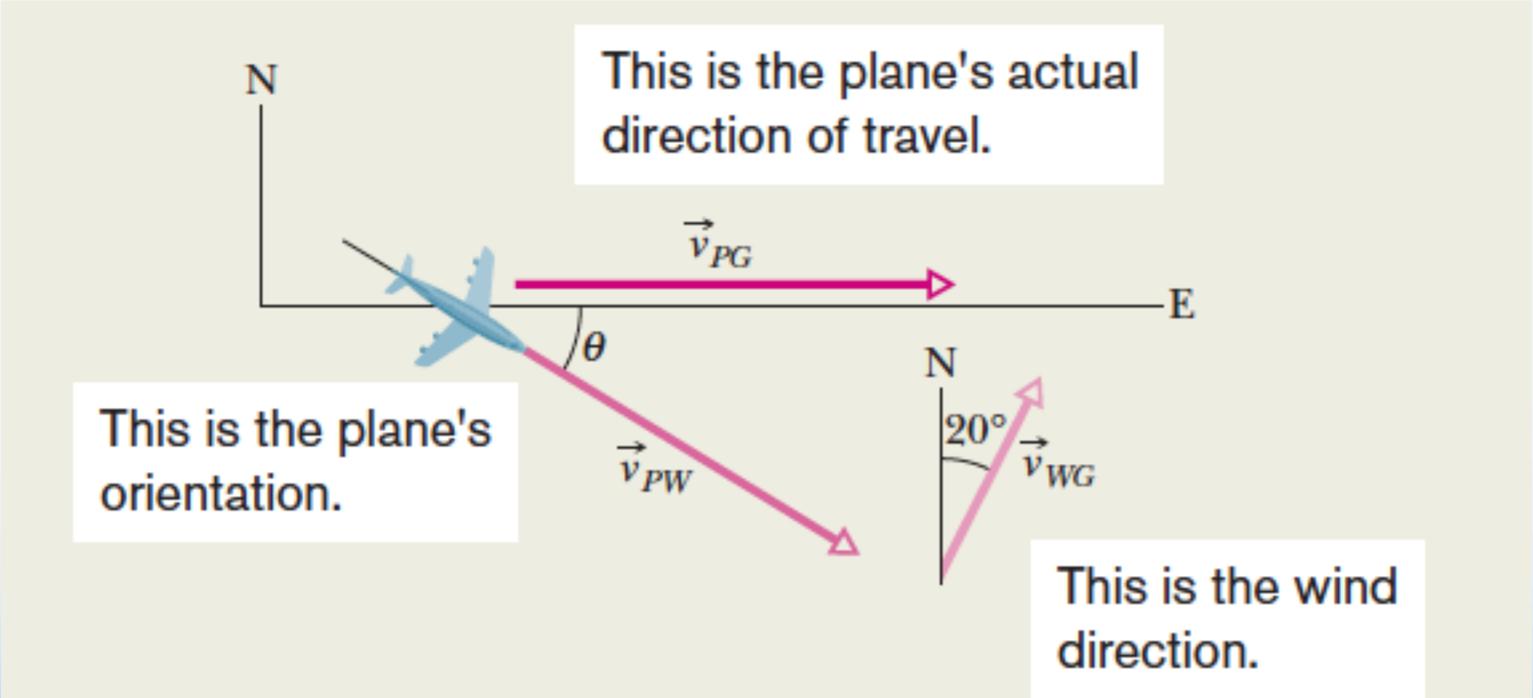


$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}.$$

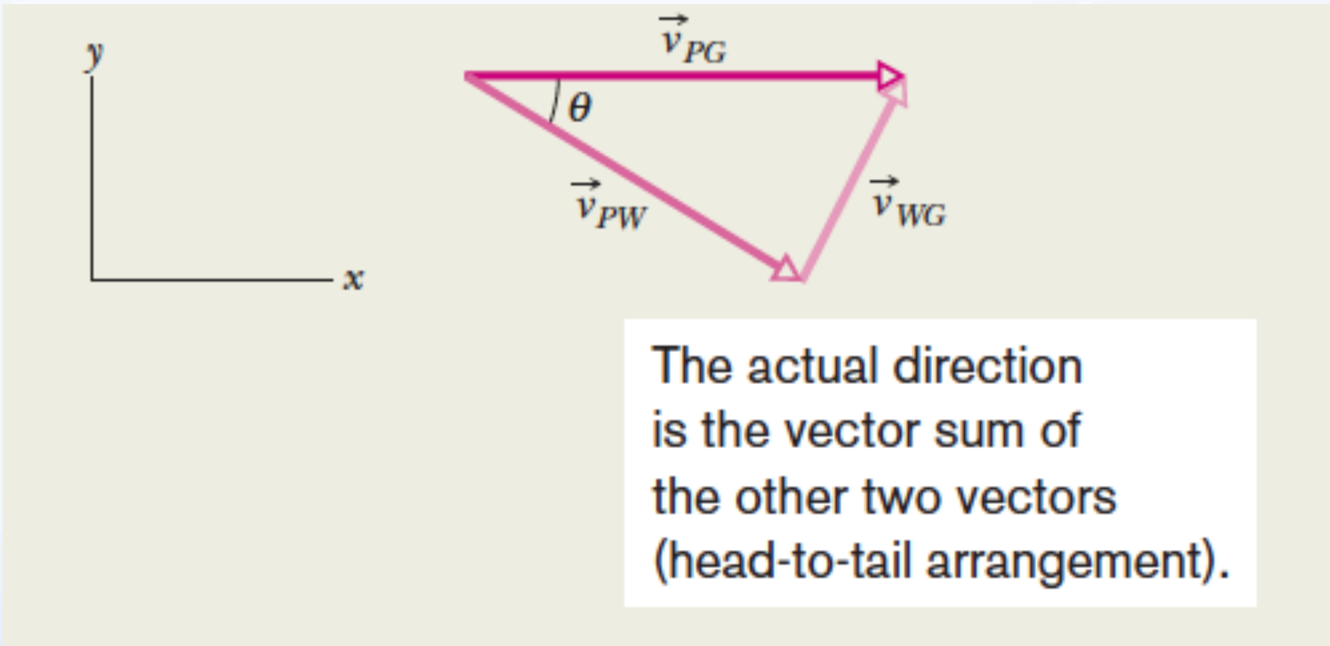
$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}.$$

$$\vec{a}_{PA} = \vec{a}_{PB}.$$

Example:


N
This is the plane's actual direction of travel.
$\vec{v}_{PG}$
E
θ
N
This is the plane's orientation.
$\vec{v}_{PW}$
20°
$\vec{v}_{WG}$
This is the wind direction.

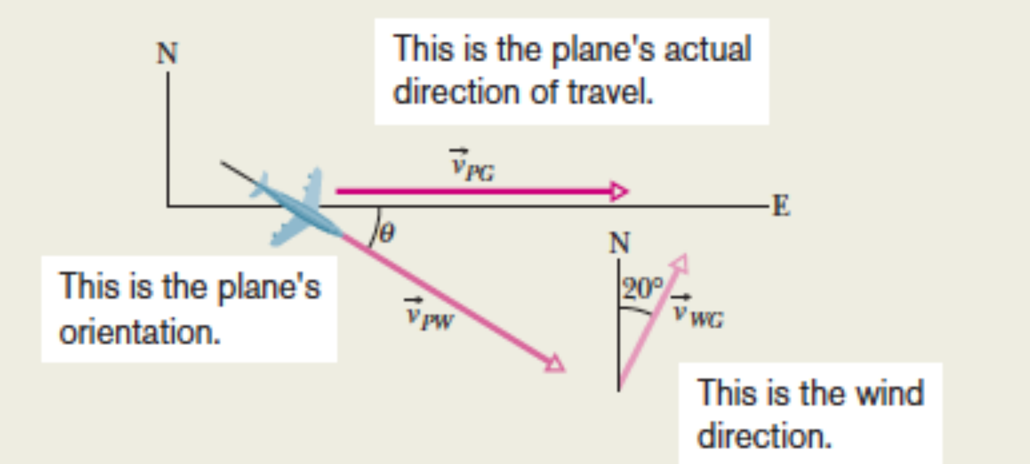
(a)


y
x
$\vec{v}_{PG}$
θ
$\vec{v}_{PW}$
$\vec{v}_{WG}$
The actual direction is the vector sum of the other two vectors (head-to-tail arrangement).

(b)

Example

In Fig. 4-20*a*, a plane moves due east while the pilot points the plane somewhat south of east, toward a steady wind that blows to the northeast. The plane has velocity $\vec{v}_{PW}$ relative to the wind, with an airspeed (speed relative to the wind) of 215 km/h, directed at angle θ south of east. The wind has velocity $\vec{v}_{WG}$ relative to the ground with speed 65.0 km/h, directed 20.0° east of north. What is the magnitude of the velocity $\vec{v}_{PG}$ of the plane relative to the ground, and what is θ?



(*a*)

Calculations: First we construct a sentence that relates the three vectors shown in Fig. 4-20*b*:

velocity of plane relative to ground (PG) = velocity of plane relative to wind (PW) + velocity of wind relative to ground. (WG)

This relation is written in vector notation as

$$\vec{v}_{PG} = \vec{v}_{PW} + \vec{v}_{WG}. \tag{4-46}$$

We need to resolve the vectors into components on the coordinate system of Fig. 4-20*b* and then solve Eq. 4-46 axis by axis. For the y components, we find

$$v_{PG,y} = v_{PW,y} + v_{WG,y}$$

or $\quad 0 = -(215 \text{ km/h}) \sin\theta + (65.0 \text{ km/h})(\cos 20.0°).$

Solving for θ gives us

$$\theta = \sin^{-1}\frac{(65.0 \text{ km/h})(\cos 20.0°)}{215 \text{ km/h}} = 16.5°. \qquad \text{(Answer)}$$

Similarly, for the x components we find

$$v_{PG,x} = v_{PW,x} + v_{WG,x}.$$

Here, because $\vec{v}_{PG}$ is parallel to the x axis, the component $v_{PG,x}$ is equal to the magnitude v_{PG}. Substituting this notation and the value $\theta = 16.5°$, we find

$$v_{PG} = (215 \text{ km/h})(\cos 16.5°) + (65.0 \text{ km/h})(\sin 20.0°)$$
$$= 228 \text{ km/h}. \qquad \text{(Answer)}$$

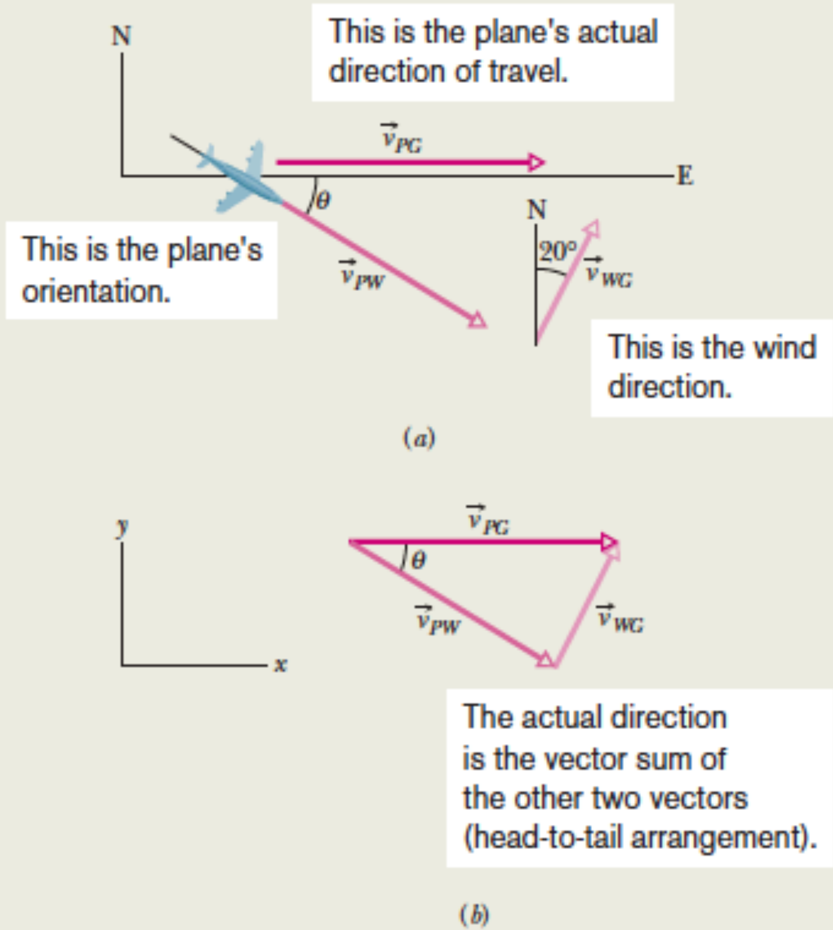


Figure 4-20 A plane flying in a wind.

Example

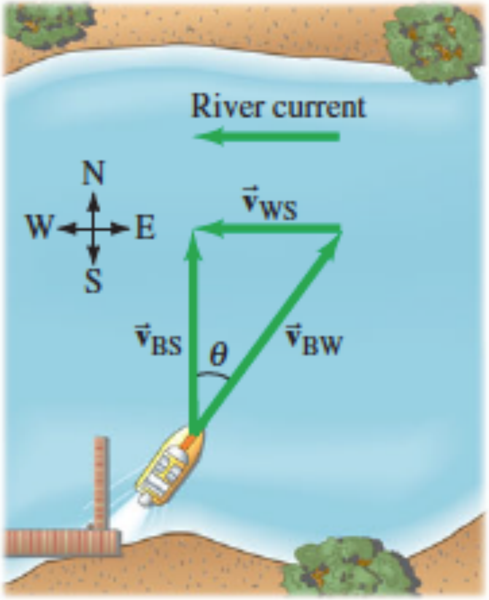


FIGURE 3–29 Example 3–10.

EXAMPLE 3–10 **Heading upstream.** A boat's speed in still water is $v_{BW} = 1.85$ m/s. If the boat is to travel north directly across a river whose westward current has speed $v_{WS} = 1.20$ m/s, at what upstream angle must the boat head? (See Fig. 3–29.)

APPROACH If the boat heads straight across the river, the current will drag the boat downstream (westward). To overcome the river's current, the boat must have an upstream (eastward) component of velocity as well as a cross-stream (northward) component. Figure 3–29 has been drawn with $\vec{v}_{BS}$, the velocity of the **B**oat relative to the **S**hore, pointing directly across the river because this is where the boat is supposed to go. (Note that $\vec{v}_{BS} = \vec{v}_{BW} + \vec{v}_{WS}$.)

SOLUTION Vector $\vec{v}_{BW}$ points upstream at angle θ as shown. From the diagram,

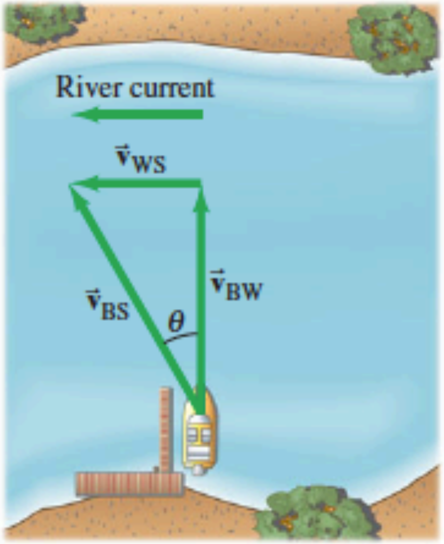
$$\sin\theta = \frac{v_{WS}}{v_{BW}} = \frac{1.20\ \text{m/s}}{1.85\ \text{m/s}} = 0.6486.$$

Thus $\theta = 40.4°$, so the boat must head upstream at a 40.4° angle.

Example

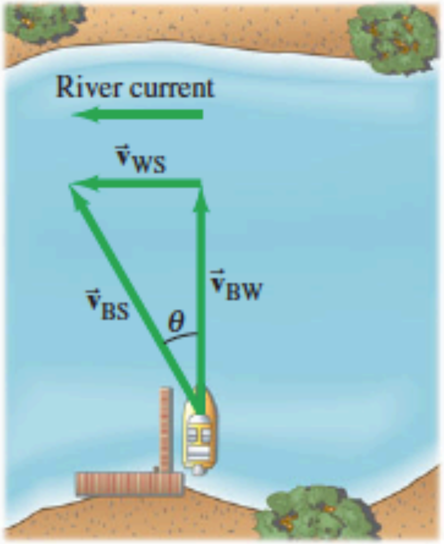
EXAMPLE 3–11 **Heading across the river.** The same boat ($v_{BW} = 1.85\ \mathrm{m/s}$) now heads directly across the river whose current is still $1.20\ \mathrm{m/s}$. (*a*) What is the velocity (magnitude and direction) of the boat relative to the shore? (*b*) If the river is 110 m wide, how long will it take to cross and how far downstream will the boat be then?

FIGURE 3–30 Example 3–11. A boat heading directly across a river whose current moves at $1.20\ \mathrm{m/s}$.



Example

FIGURE 3–30 Example 3–11. A boat heading directly across a river whose current moves at 1.20 m/s.



EXAMPLE 3–11 **Heading across the river.** The same boat ($v_{BW} = 1.85$ m/s) now heads directly across the river whose current is still 1.20 m/s. (*a*) What is the velocity (magnitude and direction) of the boat relative to the shore? (*b*) If the river is 110 m wide, how long will it take to cross and how far downstream will the boat be then?

APPROACH The boat now heads directly across the river and is pulled downstream by the current, as shown in Fig. 3–30. The boat's velocity with respect to the shore, $\vec{v}_{BS}$, is the sum of its velocity with respect to the water, $\vec{v}_{BW}$, plus the velocity of the water with respect to the shore, $\vec{v}_{WS}$: just as before,

$$\vec{v}_{BS} = \vec{v}_{BW} + \vec{v}_{WS}.$$

SOLUTION (*a*) Since $\vec{v}_{BW}$ is perpendicular to $\vec{v}_{WS}$, we can get v_{BS} using the theorem of Pythagoras:

$$v_{BS} = \sqrt{v_{BW}^2 + v_{WS}^2} = \sqrt{(1.85\text{ m/s})^2 + (1.20\text{ m/s})^2} = 2.21\text{ m/s}.$$

We can obtain the angle (note how θ is defined in Fig. 3–30) from:

$$\tan\theta = v_{WS}/v_{BW} = (1.20\text{ m/s})/(1.85\text{ m/s}) = 0.6486.$$

A calculator with a key INV TAN or ARC TAN or TAN^{-1} gives $\theta = \tan^{-1}(0.6486) = 33.0°$. Note that this angle is not equal to the angle calculated in Example 3–10.

(*b*) The travel time for the boat is determined by the time it takes to cross the river. Given the river's width $D = 110$ m, we can use the velocity component in the direction of D, $v_{BW} = D/t$. Solving for t, we get $t = 110\text{ m}/1.85\text{ m/s} = 59.5$ s. The boat will have been carried downstream, in this time, a distance

$$d = v_{WS}t = (1.20\text{ m/s})(59.5\text{ s}) = 71.4\text{ m} \approx 71\text{ m}.$$

NOTE There is no acceleration in this Example, so the motion involves only constant velocities (of the boat or of the river).

Example

•••20 GO In Fig. 4-32, particle A moves along the line $y = 30$ m with a constant velocity $\vec{v}$ of magnitude 3.0 m/s and parallel to the x axis. At the instant particle A passes the y axis, particle B leaves the origin with a zero initial speed and a constant acceleration $\vec{a}$ of magnitude 0.40 m/s^2. What angle θ between $\vec{a}$ and the positive direction of the y axis would result in a collision?

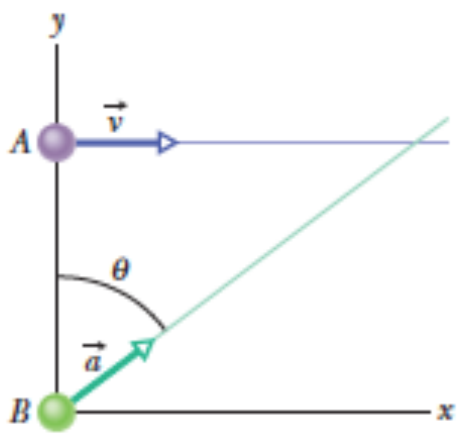


Figure 4-32 Problem 20.

Example

15. **S** A firefighter, a distance d from a burning building, directs a stream of water from a fire hose at angle θ_i above the horizontal as shown in Figure P4.15. If the initial speed of the stream is v_i, at what height h does the water strike the building?

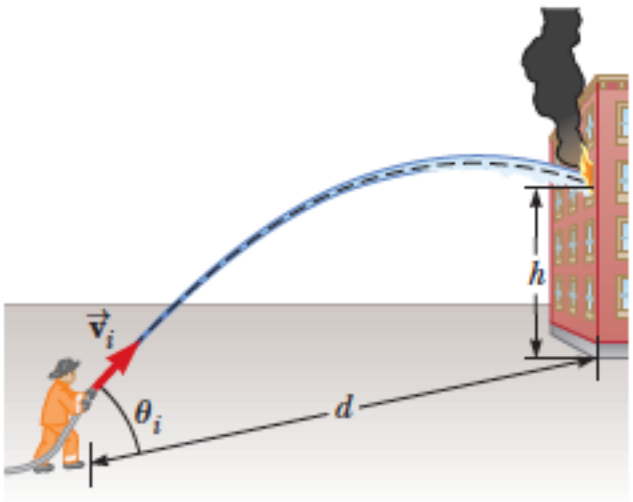


Figure P4.15

Example

29. **Review.** The 20-g centrifuge at NASA's Ames Research Center in Mountain View, California, is a horizontal, cylindrical tube 58 ft long and is represented in Figure P4.29. Assume an astronaut in training sits in a seat at one end, facing the axis of rotation 29.0 ft away. Determine the rotation rate, in revolutions per second, required to give the astronaut a centripetal acceleration of $20.0g$.

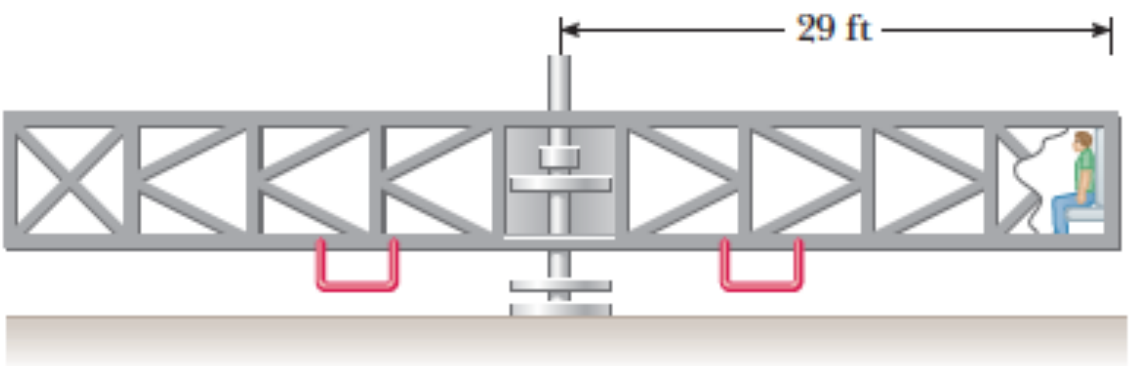


Figure P4.29

Example

42. **Q|C** A farm truck moves due east with a constant velocity of 9.50 m/s on a limitless, horizontal stretch of road. A boy riding on the back of the truck throws a can of soda upward (Fig. P4.42) and catches the projectile at the same location on the truck bed, but 16.0 m farther down the road. (a) In the frame of reference of the truck, at what angle to the vertical does the boy throw the can? (b) What is the initial speed of the can relative to the truck? (c) What is the shape of the can's trajectory as seen by the boy? An observer on the ground watches the boy throw the can and catch it. In this observer's frame of reference, (d) describe the shape of the can's path and (e) determine the initial velocity of the can.

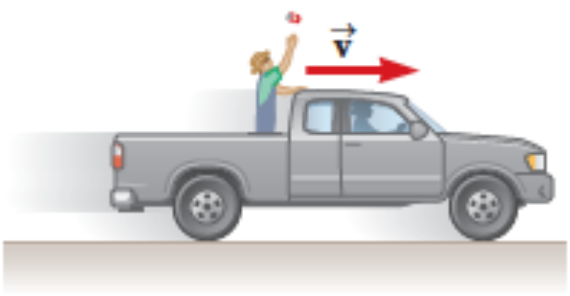


Figure P4.42

Example

52. **S** As some molten metal splashes, one droplet flies off to the east with initial velocity v_i at angle θ_i above the horizontal, and another droplet flies off to the west with the same speed at the same angle above the horizontal as shown in Figure P4.52. In terms of v_i and θ_i, find the distance between the two droplets as a function of time.

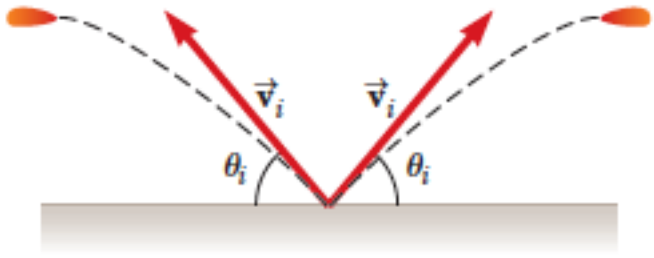


Figure P4.52

Example

68. **S** A person standing at the top of a hemispherical rock of radius R kicks a ball (initially at rest on the top of the rock) to give it horizontal velocity $\vec{\mathbf{v}}_i$ as shown in Figure P4.68.

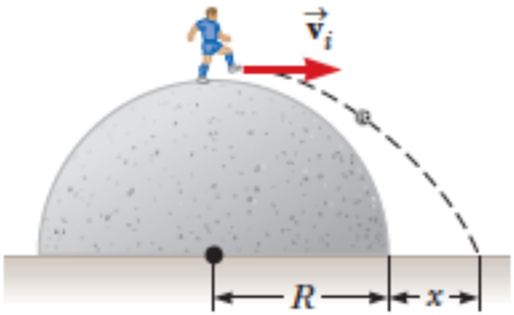


Figure P4.68

(a) What must be its minimum initial speed if the ball is never to hit the rock after it is kicked? (b) With this initial speed, how far from the base of the rock does the ball hit the ground?

Example

71. An enemy ship is on the east side of a mountain island as shown in Figure P4.71. The enemy ship has maneuvered to within 2 500 m of the 1 800-m-high mountain peak and can shoot projectiles with an initial speed of 250 m/s. If the western shoreline is horizontally 300 m from the peak, what are the distances from the western shore at which a ship can be safe from the bombardment of the enemy ship?

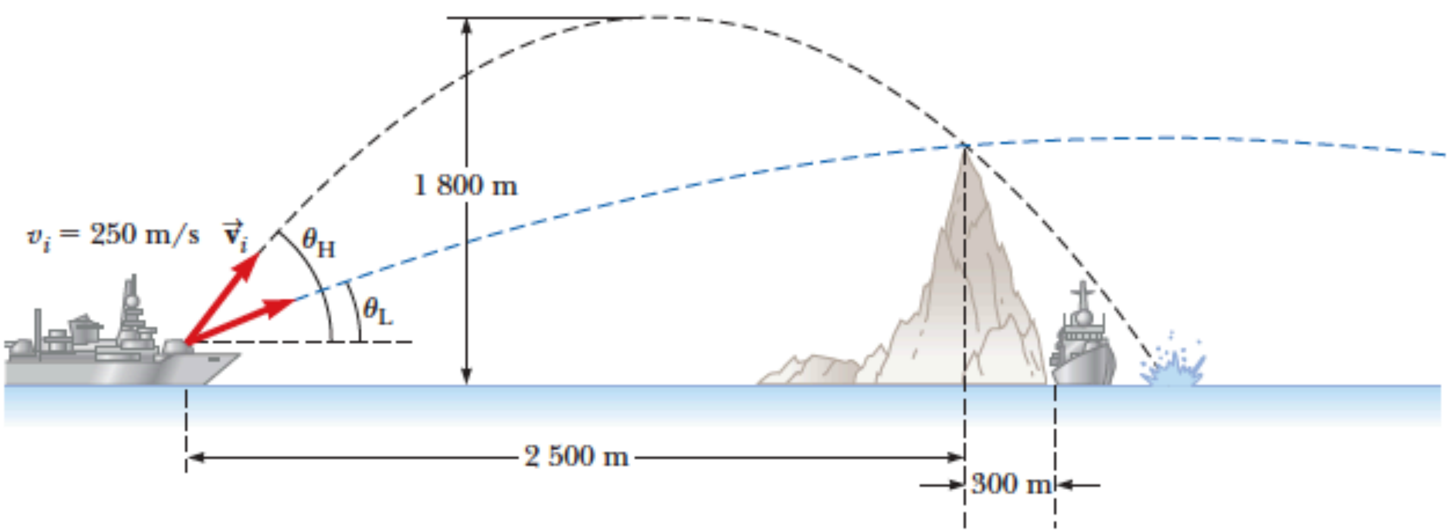
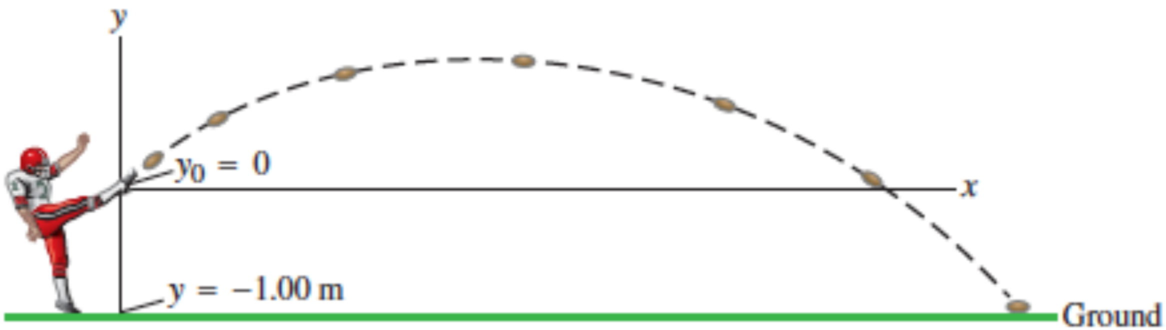


Figure P4.71

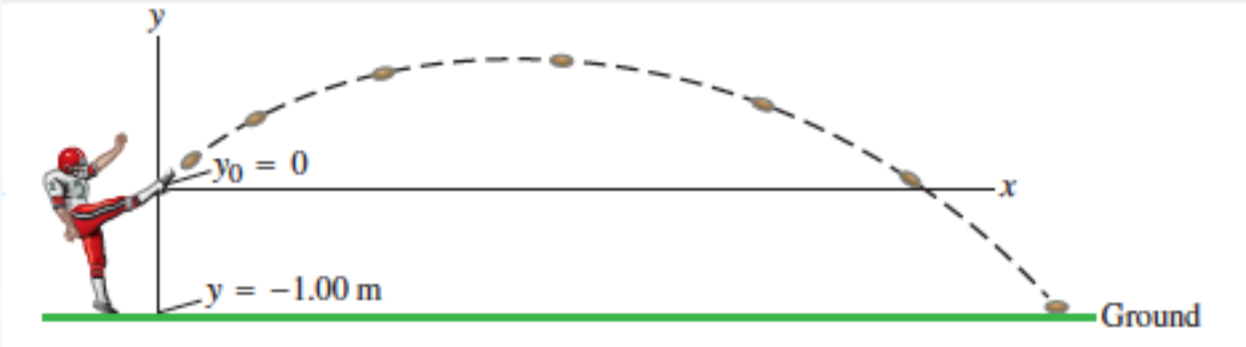
Example

FIGURE 3–26 Example 3–9: the football leaves the punter's foot at $y = 0$, and reaches the ground where $y = -1.00$ m.



PHYSICS APPLIED
Sports

EXAMPLE 3–9 **A punt.** Suppose the football in Example 3–6 was punted, and left the punter's foot at a height of 1.00 m above the ground. How far did the football travel before hitting the ground? Set $x_0 = 0$, $y_0 = 0$.



EXAMPLE 3–9 **A punt.** Suppose the football in Example 3–6 was punted, and left the punter's foot at a height of 1.00 m above the ground. How far did the football travel before hitting the ground? Set $x_0 = 0,\ y_0 = 0$.

APPROACH The only difference here from Example 3–6 is that the football hits the ground *below* its starting point of $y_0 = 0$. That is, the ball hits the ground at $y = -1.00$ m. See Fig. 3–26. Thus we cannot use the range formula which is valid only if y (final) $= y_0$. As in Example 3–6, $v_0 = 20.0$ m/s, $\theta_0 = 37.0°$.

SOLUTION With $y = -1.00$ m and $v_{y0} = 12.0$ m/s (see Example 3–6), we use the y version of Eq. 2–11b with $a_y = -g$,

$$y = y_0 + v_{y0}t - \tfrac{1}{2}gt^2,$$

and obtain

$$-1.00\,\text{m} = 0 + (12.0\,\text{m/s})t - (4.90\,\text{m/s}^2)t^2.$$

We rearrange this equation into standard form $(ax^2 + bx + c = 0)$ so we can use the quadratic formula:

$$(4.90\,\text{m/s}^2)t^2 - (12.0\,\text{m/s})t - (1.00\,\text{m}) = 0.$$

The quadratic formula (Appendix A–4) gives

$$t = \frac{12.0\,\text{m/s} \pm \sqrt{(-12.0\,\text{m/s})^2 - 4(4.90\,\text{m/s}^2)(-1.00\,\text{m})}}{2(4.90\,\text{m/s}^2)}$$
$$= 2.53\,\text{s} \quad \text{or} \quad -0.081\,\text{s}.$$

The second solution would correspond to a time prior to the kick, so it doesn't apply. With $t = 2.53$ s for the time at which the ball touches the ground, the horizontal distance the ball traveled is (using $v_{x0} = 16.0$ m/s from Example 3–6):

$$x = v_{x0}t = (16.0\,\text{m/s})(2.53\,\text{s}) = 40.5\,\text{m}.$$

Our assumption in Example 3–6 that the ball leaves the foot at ground level would result in an underestimate of about 1.3 m in the distance our punt traveled.

Example

23. (II) A fire hose held near the ground shoots water at a speed of 6.5 m/s. At what angle(s) should the nozzle point in order that the water land 2.5 m away (Fig. 3–36)? Why are there two different angles?
Sketch the two trajectories.

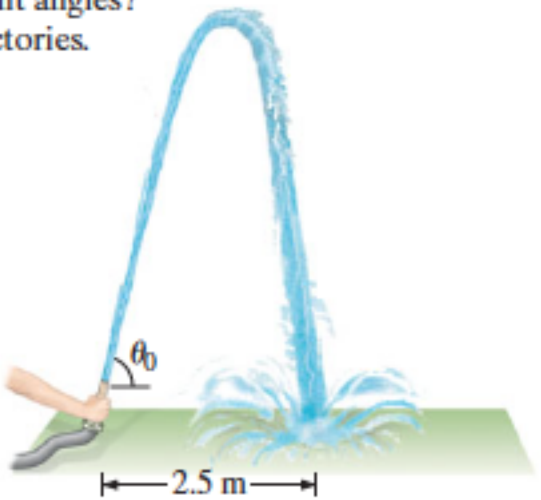


FIGURE 3–36
Problem 23.

Example

43. (II) A person in the passenger basket of a hot-air balloon throws a ball horizontally outward from the basket with speed 10.0 m/s (Fig. 3–44). What initial velocity (magnitude and direction) does the ball have relative to a person standing on the ground (*a*) if the hot-air balloon is rising at 3.0 m/s relative to the ground during this throw, (*b*) if the hot-air balloon is descending at 3.0 m/s relative to the ground?

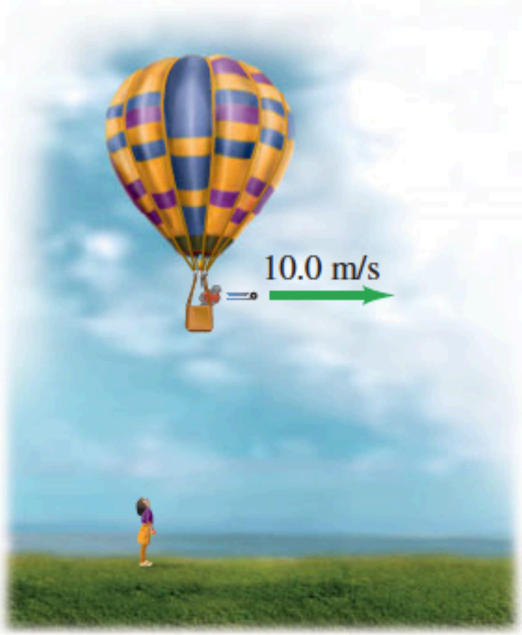


FIGURE 3–44
Problem 43.

Example

58. (*a*) A long jumper leaves the ground at 45° above the horizontal and lands 8.0 m away. What is her "takeoff" speed v_0? (*b*) Now she is out on a hike and comes to the left bank of a river. There is no bridge and the right bank is 10.0 m away horizontally and 2.5 m vertically below. If she long jumps from the edge of the left bank at 45° with the speed calculated in (*a*), how long, or short, of the opposite bank will she land (Fig. 3–50)?

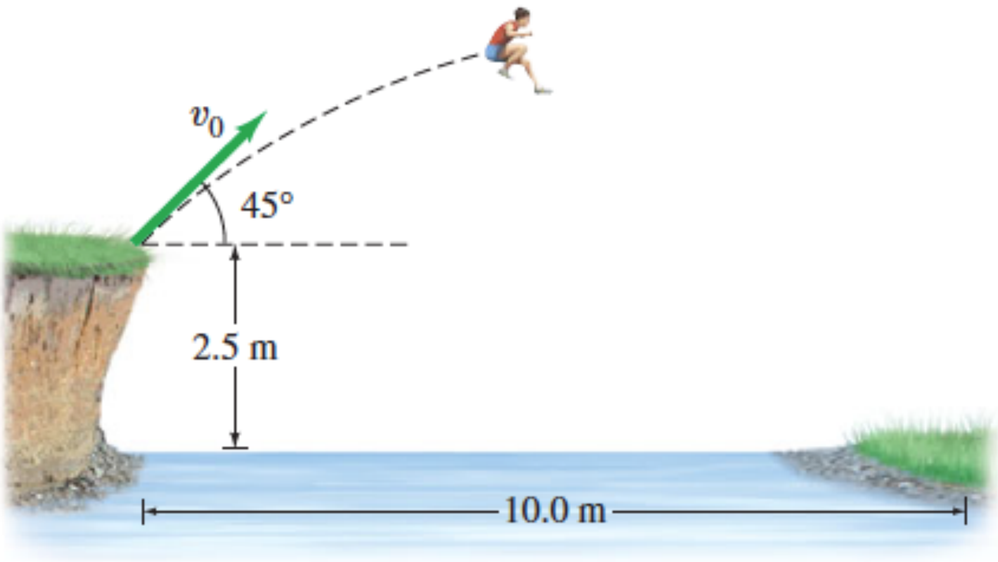


FIGURE 3–50 Problem 58.

Example

56. Romeo is throwing pebbles gently up to Juliet's window, and he wants the pebbles to hit the window with only a horizontal component of velocity. He is standing at the edge of a rose garden 8.0 m below her window and 8.5 m from the base of the wall (Fig. 3–49). How fast are the pebbles going when they hit her window?

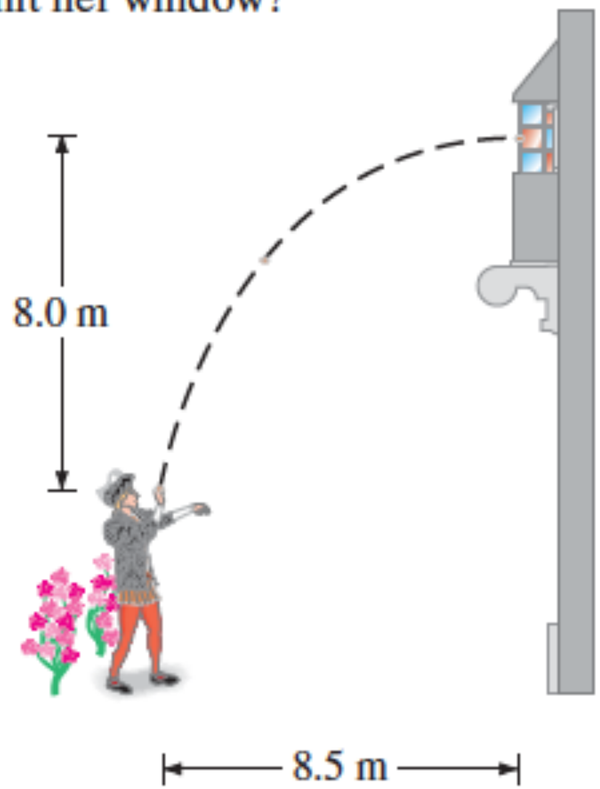


FIGURE 3–49
Problem 56.

Example

67. Spymaster Chris, flying a constant 208 km/h horizontally in a low-flying helicopter, wants to drop secret documents into her contact's open car which is traveling 156 km/h on a level highway 78.0 m below. At what angle (with the horizontal) should the car be in her sights when the packet is released (Fig. 3–55)?

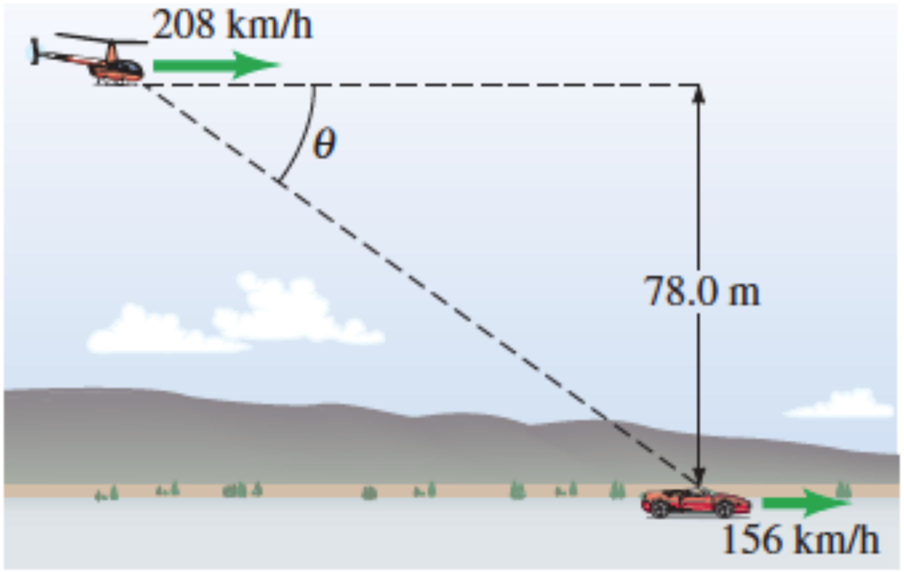


FIGURE 3–55 Problem 67.

Example

66. At serve, a tennis player aims to hit the ball horizontally. What minimum speed is required for the ball to clear the 0.90-m-high net about 15.0 m from the server if the ball is "launched" from a height of 2.50 m? Where will the ball land if it just clears the net (and will it be "good" in the sense that it lands within 7.0 m of the net)? How long will it be in the air? See Fig. 3–54.

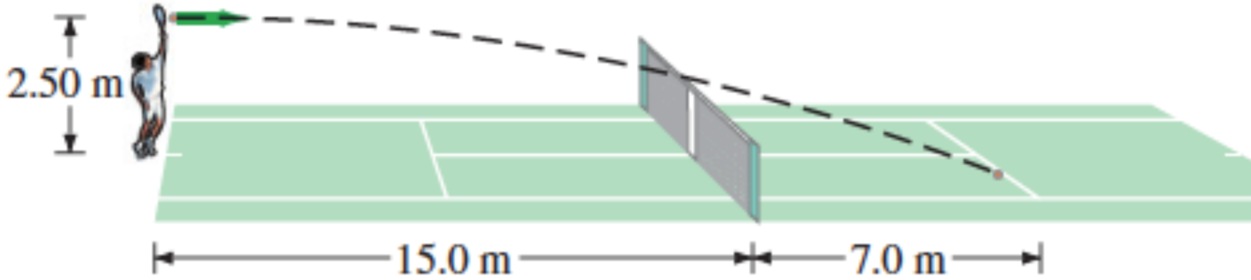


FIGURE 3–54 Problem 66.

Example

72. A rock is kicked horizontally at 15 m/s from a hill with a 45° slope (Fig. 3–58). How long does it take for the rock to hit the ground?

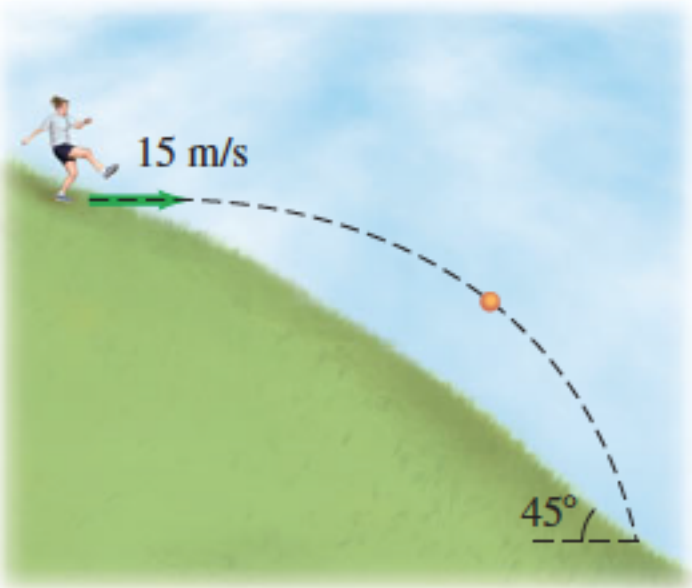


FIGURE 3–58 Problem 72.

Summary

Review & Summary

Position Vector The location of a particle relative to the origin of a coordinate system is given by a *position vector* $\vec{r}$, which in unit-vector notation is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}. \tag{4-1}$$

Here $x\hat{i}$, $y\hat{j}$, and $z\hat{k}$ are the vector components of position vector $\vec{r}$, and x, y, and z are its scalar components (as well as the coordinates of the particle). A position vector is described either by a magnitude and one or two angles for orientation, or by its vector or scalar components.

Displacement If a particle moves so that its position vector changes from $\vec{r}_1$ to $\vec{r}_2$, the particle's *displacement* $\Delta\vec{r}$ is

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1. \tag{4-2}$$

The displacement can also be written as

$$\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \tag{4-3}$$

$$= \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}. \tag{4-4}$$

Average Velocity and Instantaneous Velocity If a particle undergoes a displacement $\Delta\vec{r}$ in time interval Δt, its *average velocity* $\vec{v}_{avg}$ for that time interval is

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}. \tag{4-8}$$

As Δt in Eq. 4-8 is shrunk to 0, $\vec{v}_{avg}$ reaches a limit called either the *velocity* or the *instantaneous velocity* $\vec{v}$:

$$\vec{v} = \frac{d\vec{r}}{dt}, \tag{4-10}$$

which can be rewritten in unit-vector notation as

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}, \tag{4-11}$$

where $v_x = dx/dt$, $v_y = dy/dt$, and $v_z = dz/dt$. The instantaneous velocity $\vec{v}$ of a particle is always directed along the tangent to the particle's path at the particle's position.

Average Acceleration and Instantaneous Acceleration If a particle's velocity changes from $\vec{v}_1$ to $\vec{v}_2$ in time interval Δt, its *average acceleration* during Δt is

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t}. \tag{4-15}$$

As Δt in Eq. 4-15 is shrunk to 0, $\vec{a}_{avg}$ reaches a limiting value called either the *acceleration* or the *instantaneous acceleration* $\vec{a}$:

$$\vec{a} = \frac{d\vec{v}}{dt}. \tag{4-16}$$

In unit-vector notation,

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}, \tag{4-17}$$

where $a_x = dv_x/dt$, $a_y = dv_y/dt$, and $a_z = dv_z/dt$.

Projectile Motion *Projectile motion* is the motion of a particle that is launched with an initial velocity $\vec{v}_0$. During its flight, the particle's horizontal acceleration is zero and its vertical acceleration is the free-fall acceleration $-g$. (Upward is taken to be a positive direction.) If $\vec{v}_0$ is expressed as a magnitude (the speed v_0) and an angle θ_0 (measured from the horizontal), the particle's equations of motion along the horizontal x axis and vertical y axis are

$$x - x_0 = (v_0 \cos \theta_0)t, \tag{4-21}$$

$$y - y_0 = (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2, \tag{4-22}$$

$$v_y = v_0 \sin \theta_0 - gt, \tag{4-23}$$

$$v_y^2 = (v_0 \sin \theta_0)^2 - 2g(y - y_0). \tag{4-24}$$

The **trajectory** (path) of a particle in projectile motion is parabolic and is given by

$$y = (\tan \theta_0)x - \frac{gx^2}{2(v_0 \cos \theta_0)^2}, \tag{4-25}$$

if x_0 and y_0 of Eqs. 4-21 to 4-24 are zero. The particle's **horizontal range** R, which is the horizontal distance from the launch point to the point at which the particle returns to the launch height, is

$$R = \frac{v_0^2}{g} \sin 2\theta_0. \tag{4-26}$$

Uniform Circular Motion If a particle travels along a circle or circular arc of radius r at constant speed v, it is said to be in *uniform circular motion* and has an acceleration $\vec{a}$ of constant magnitude

$$a = \frac{v^2}{r}. \tag{4-34}$$

The direction of $\vec{a}$ is toward the center of the circle or circular arc, and $\vec{a}$ is said to be *centripetal*. The time for the particle to complete a circle is

$$T = \frac{2\pi r}{v}. \tag{4-35}$$

T is called the *period of revolution*, or simply the *period*, of the motion.

Relative Motion When two frames of reference A and B are moving relative to each other at constant velocity, the velocity of a particle P as measured by an observer in frame A usually differs from that measured from frame B. The two measured velocities are related by

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}, \tag{4-44}$$

where $\vec{v}_{BA}$ is the velocity of B with respect to A. Both observers measure the same acceleration for the particle:

$$\vec{a}_{PA} = \vec{a}_{PB}. \tag{4-45}$$