# Analysis of porosity distribution of large-scale porous media and their reconstruction by Langevin equation

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Several methods have been developed in the past for analyzing the porosity and other types of well logs for large-scale porous media, such as oil reservoirs, as well as their permeability distributions. We developed a method for analyzing the porosity logs $\phi(h)$ (where $h$ is the depth) and similar data that are often nonstationary stochastic series. In this method one first generates a new stationary series based on the original data, and then analyzes the resulting series. It is shown that the series based on the successive increments of the log $y(h) = \phi(h + \delta h) - \phi(h)$ is a stationary and Markov process, characterized by a Markov length scale $h_M$. The coefficients of the Kramers-Moyal expansion for the conditional probability density function (PDF) $P(y,h|y_0,h_0)$ are then computed. The resulting PDFs satisfy a Fokker-Planck (FP) equation, which is equivalent to a Langevin equation for $y(h)$ that provides probabilistic predictions for the porosity logs. We also show that the Hurst exponent $H$ of the self-affine distributions, which have been used in the past to describe the porosity logs, is directly linked to the drift and diffusion coefficients that we compute for the FP equation. Also computed are the level-crossing probabilities that provide insight into identifying the high or low values of the porosity beyond the depth interval in which the data have been measured.

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## I. INTRODUCTION

Large-scale (LS) porous media are highly heterogeneous at several distinct length scales, ranging from the pore to the laboratory and field scales. Modeling the geology of LS porous media is critical to understanding a wide variety of important processes, such as increased production from oil, gas, and geothermal reservoirs, and monitoring the spread of industrial contaminants in groundwater flows. Predictive models for such processes require accurate characterization of the geology of LS porous media, and in particular the distributions of their properties. Although characterization of laboratory-scale porous media may be done with considerable detail [1,2], characterization of LS porous media is plagued by insufficient data. Complicating the problem is the fact that such data fluctuate widely over the length scales in which they have been measured, and they are often nonstationary.

Two important characteristics of LS porous media are the spatial distributions of their porosity and permeability. The porosity logs $\phi(h)$ (where $h$ is the depth at which the porosity $\phi$ has been estimated or measured) of LS porous media are measured or estimated routinely along wells [3]. The permeability distributions (or logs) can be obtained either by *in situ* nuclear magnetic resonance [4] or by coring and laboratory measurements [2,3]. The focus of the present paper is the analysis of the porosity logs, although the method that we describe in the present paper is equally applicable to the analysis of the permeability distributions and other types of logs, such as the temperature, $\gamma$-ray, and resistivity logs, and other important characteristics of LS porous media, as well as other stochastic series.

Large-scale porous media are typically anisotropic, with the anisotropy being due to stratification and the existence of a number of layers of contrasting properties. In addition, the porosity logs and other characteristics of LS porous media may contain extended correlations and have been modeled by self-affine fractal distributions. In particular, the first concrete evidence that the porosity logs in the directions perpendicular and parallel to the strata may follow, respectively, the statistics of a fractional Gaussian noise (FGN) and a fractional Brownian motion (FBM) [5] was provided by Hewett [6]. The semivariogram of a porosity log is defined by

$$\gamma(h) = \tfrac{1}{2}\langle[\phi(h) - \phi(h + h')]^2\rangle, \qquad (1)$$

where the averaging is over all values of $h'$. The semivariogram of a one-dimensional (1D) set of data that follow the statistics of an FGN is given by [7]

$$\gamma(h) = \gamma_0 s^{2H} - \frac{\gamma_1}{2s^2}[(h + s)^{2H} - 2r^{2H} + |h - s|^{2H}], \qquad (2)$$

where $s$ is a smoothing parameter, $\gamma_0$ and $\gamma_1$ are two constants, and $H$ is the Hurst exponent. The spectral density of a 1D FGN, the Fourier transform of its covariance, is then given by

$$S(\omega) = \frac{1}{\pi}\gamma_0 H\Gamma(2H)\sin(\pi H)\,\frac{1}{\omega^{2H-1}}. \qquad (3)$$

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For frequencies $\omega \gg 1/s$ the spectral density of a FGN becomes negative, which is unphysical. The corresponding spectral density of a 1D FBM is given by

$$S(\omega) = \frac{\gamma_0 H}{\Gamma(1-2H)\cos(\pi H)} \frac{1}{\omega^{2H+1}}. \tag{4}$$

Note that $H > 0.5$ ($<0.5$) indicates positive (negative) correlations in the data, with the extent of the correlations being the thickness of the zone in which the data are collected, while $H = 0.5$ implies that successive increments in the data are random and follow Brownian motion. Extensive studies [8–13] have provided evidence that the porosity logs of LSPM often follow the statistics of a FGN or FBM. Similarly, measurements on outcrop surfaces have provided evidence [14–17] that, at least in some cases, the spectral density of the permeabilities follows Eqs. (3) and (4). A more recent study [18] presented evidence that the seismic wave velocities and the elastic moduli of LS porous media may also be approximately described by a FBM.

The problem of analyzing the porosity and similar well logs is, however, still under study. The reason is threefold. (1) The porosity logs and other characteristics of LS porous media are usually nonstationary and represent widely fluctuating stochastic series. Accurate analysis of such series is fraught with difficulties. (2) Several studies have indicated that the FGN or FBM does not always describe the porosity logs and other types of data. For example, evidence was presented [13,19,20] that indicated, at least in some cases, that the porosities might follow a fractional Lévy motion, which represents a generalization of the FBM. Moreover, even in those cases in which the porosity logs can be represented by the FGN and FBM, lumping together all the information that the logs contain in a single exponent, the Hurst exponent $H$, is not wise. (3) Even if one could analyze the data accurately, it is not clear how one can gain information about those sectors of the LS porous media for which no data are available.

In this paper we utilize a new method for analyzing the porosity logs of LS porous media. The method has three features: (1) It generates a stationary process $y(h)$, given a nonstationary porosity log $\phi(h)$. (2) It analyzes the statistical properties of $y(h)$ and characterizes it in terms of two new quantities that have never been used in the analysis of the porosity logs. (3) It constructs stochastic continuum equations that not only *reconstruct* $y(h)$ [and, hence, $\phi(h)$] but also provide probabilistic predictions for the porosity logs over a certain length scale that we identify later. As we show later, a distinct advantage of the method is that, unlike the description of porosity logs by self-affine distributions that requires the logs to exhibit scaling properties, the approach described in the present paper needs no scaling feature in the data.

The plan of this paper is as follows. In the next section we briefly describe the data. Section III describes the construction of the stationary series $y(h)$, given a nonstationary porosity log. In Sec. IV we describe the new method of analyzing the stationary $y(h)$ in terms of Markov processes. The construction of the stochastic continuum equations for reconstruction of the series $y(h)$ [and, hence, $\phi(h)$] is described in Sec. V. The results are presented and discussed in Sec. VI. In Sec. VII we derive a relation between the Hurst exponent $H$ and some of the properties that we compute in this paper. The last section presents a summary of the results and their implications.

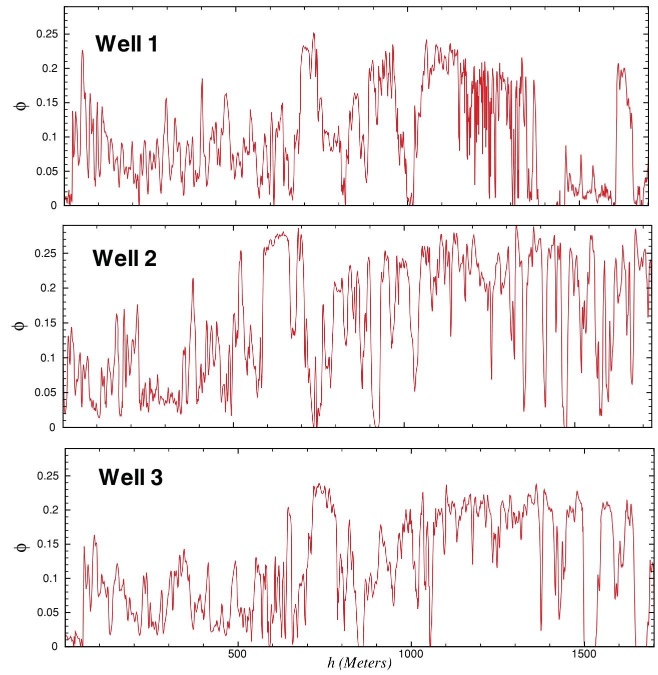


FIG. 1. (Color online) The three porosity logs analyzed.

## II. THE DATA

We analyze the fluctuations in three porosity logs, measured along three wells in an oil reservoir in central Iran. The reservoir is undersaturated with good quality ($H_2S$-free) oil and active wells, all producing from the Asmari formation, with an average production rate of about 50 000 bbl/day. Its recoverable oil is estimated to be more than 3 billion barrels. Figure 1 presents the three porosity logs.

## III. CONSTRUCTION OF THE STATIONARY SERIES

Given a porosity log $\phi(h)$, one may be able to construct a stationary process $y(h)$ by at least one of the two following methods. (1) Construct the successive increments $y(h) = \phi(h + \delta h) - \phi(h)$, where $\delta h$ is the distance between two neighboring points. The best-known example of such processes is the FBM with a power spectrum given by Eq. (4). It is well known that the successive increments of the FBM are stationary with their $S(\omega)$ given by Eq. (3). Moreover, when $H = 1/2$, the increments are uncorrelated, while for $H = -1/2$, $\phi(h)$ itself becomes white noise. (2) Let $Z(h) = \ln \phi(h)$. Then, one may construct a stationary series $x(h)$ by $x(h) = Z(h + \delta h) - Z(h) = \ln[\phi(h + \delta h)/\phi(h)]$, so that $x(h)$ represents the *log returns* [21].

It is not difficult to show that the porosity logs analyzed here are not stationary by showing, for example, that their variances, computed in a window, are not invariant if we increase the window's size or move it along the series. Hence, we constructed the algebraic increments $y(h)$ of the entire well logs. It was then straightforward to show, using three different methods, that the resulting series $y(h)$ are stationary.

(1) We computed the averages and variances of the three series $y(h)$ in moving windows of increasing sizes to check that they are essentially invariant. (2) We computed the spectral densities $S(\omega)$ of the three series $y(h)$. The result, $S(\omega) \propto \omega^{\beta}$ with $\beta \simeq 0$, indicated the absence of long-range correlations in $y(h)$. (3) We also analyzed the three series $y(h)$ using the detrended fluctuation analysis [22] and the rescaled-range method [23,24] to further check that the three series $y(h)$ are stationary. Both methods yielded $\beta \simeq 0$ and, thus, the three series $y(h)$ are, at least to a good degree of approximation, stationary.

Having established the stationarity of $y(h)$, we analyzed the new series based on the application of Markov processes and development of a Langevin equation for the series. The method is, however, general and applicable to a large class of stochastic processes that represent various properties of LS porous media.

## IV. MARKOV ANALYSIS

Although long-range correlations are absent in $y(h)$, short-range correlations may exist. To analyze the data, we check whether the three series $y(h)$ represent a Markov process (MP) [21,25,26]. If so, we estimate the Markov length scale $h_M$ of the three series $y(h)$, that is, the minimum length interval over which $y(h)$ may be represented by a MP. To characterize the statistical properties of $y(h)$, one must evaluate the joint probability density function (PDF) $P_n(y_1,h_1;\ldots;y_n,h_n)$ for an arbitrary $n$, the number of the data points. If, however, $y(h)$ is a MP, the $n$-point joint PDF $P_n$ is the product of the conditional probabilities $P(y_{i+1},h_{i+1}|y_i,h_i)$, for $i = 1,\ldots,n-1$. A necessary condition for the series $y(h)$ to be a MP is that the Chapman-Kolmogorov (CK) equation [27]

$$P(y_2,h_2|y_1,h_1) = \int dy_3\ P(y_2,h_2|y_3,h_3)P(y_3,h_3|y_1,h_1) \quad (5)$$

should hold for any $h_3$ in $h_1 < h_3 < h_2$. Note that the opposite is not necessarily true, namely, that if a stochastic process satisfies the CK equation, it is not necessarily a MP [28]. To check the validity of the CK equation for several values of $y_1$, we compare the directly evaluated $P(y_2,h_2|y_1,h_1)$ with those calculated according to the right side of Eq. (5).

To estimate $h_M$, we used the least-squares method. If $y(h)$ is a MP, one has

$$P(y_3,h_3|y_2,h_2;y_1,h_1) = P(y_3,h_3|y_2,h_2). \quad (6)$$

Thus, the PDF $P(y_3,h_3;y_2,h_2;y_1,h_1) = P(y_3,h_3|y_2,h_2;y_1,h_1)P(y_2,h_2;y_1,h_1)$ is compared with that obtained based on the MP. Using the statistical properties of Markov processes, we obtain

$$P_{\mathrm{M}}(y_3,h_3;y_2,h_2;y_1,h_1) = P(y_3,h_3|y_2,h_2)P(y_2,h_2;y_1,h_1). \quad (7)$$

It should be pointed out that the homogeneity of a stochastic process is not necessary for using Eqs. (6) and (7). To check whether $y(h)$ is a MP, one must compute the three-point joint PDF through Eq. (6) and compare the result with that obtained through Eq. (7). Thus, one first determines the quality of the fit by computing the least-squares fitting quantity $\chi^2$, defined by

$$\chi^2 = \int dy_3\, dy_2\, dy_1 [P(y_3,h_3;y_2,h_2;y_1,h_1) - P_{\mathrm{M}}(y_3,h_3;y_2,h_2;y_1,h_1)]^2 \big/ \left(\sigma_{3j}^2 + \sigma_{\mathrm{M}}^2\right), \quad (8)$$

where $\sigma_{3j}^2$ and $\sigma_{\mathrm{M}}^2$ are the variances of $P(y_3,h_3;y_2,h_2;y_1,h_1)$ and $P_{\mathrm{M}}(y_3,h_3;y_2,h_2;y_1,h_1)$, respectively. To estimate the Markov length scale $h_M$, a likelihood statistical analysis was utilized [29]. Because there is no *a priori* constraint, the probability of the set of three-point joint PDFs is given by [$P(x)$ must be normalized]

$$P(h_3 - h_1) = \Pi_{y_3,y_2,y_1} \frac{1}{\sqrt{2\pi\left(\sigma_{3j}^2 + \sigma_{\mathrm{M}}^2\right)}} \exp\left\{\frac{[P(y_3,h_3;y_2,h_2;y_1,h_1) - P_{\mathrm{M}}(y_3,h_3;y_2,h_2;y_1,h_1)]^2}{2\left(\sigma_{3j}^2 + \sigma_{\mathrm{M}}^2\right)}\right\}. \quad (9)$$

Evidently, when a set of the parameters $\chi_\nu^2 = \chi^2/N$ is minimum, with $N$ being the degree of freedom, the probability is maximum. This is shown in Fig. 2. The Markov length scale $h_M$ corresponds to that value of $(h_3 - h_1)_{\min}$ for which $\chi_\nu^2$ is minimum. This yields $h_M = (h_3 - h_1)_{\min} \simeq 0.25$ m for the three well logs. Therefore, the series $y(h) = \phi(h + \delta h) - \phi(h)$ is a Markov process. In the next section we construct a continuum equation that governs $y(h)$.

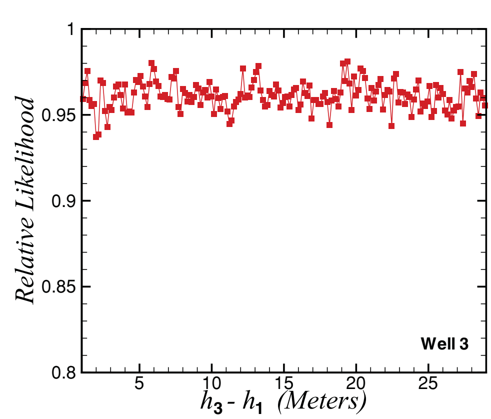


FIG. 2. (Color online) Relative likelihood function for $h_M$ and for well 3.

## V. KRAMERS-MOYAL EXPANSION AND THE LANGEVIN EQUATION

For a MP, knowledge of $P(y_2,h_2|y_1,h_1)$ is sufficient for generating the entire statistics of the series $y(h)$, encoded in the $n$-point PDF that satisfies a master equation that, in turn, is reformulated by a Kramers-Moyal (KM) expansion,

$$\frac{\partial}{\partial h}P(y,h|y_0,h_0)=\sum_k(-1)^k\frac{\partial^k}{\partial y^k}[D^{(k)}(y,h)P(y,h|y_0,h_0)]. \tag{10}$$

The KM coefficients $D^{(k)}(y,h)$ are given by

$$D^{(k)}(y,h)=\frac{1}{k!}\lim_{\Delta t\to 0}M^{(k)},$$
$$M^{(k)}=\frac{1}{\delta h}\int dy'(y'-y)^kP(y',h+\delta h|y,h). \tag{11}$$

In a sense, Eq. (11) identifies the method as a moment-generating function approach, as the coefficient $D^{(k)}$ is computed based on the moments of the conditional PDF $P(y,h|y_0,h_0)$.

For a general stochastic process, all the KM coefficients may be nonzero. However, provided that $D^{(4)}$ vanishes or is small compared to the first two coefficients [27], truncation of the KM expansion after the second term is meaningful in the statistical sense. For the porosity data, we found $D^{(4)}\simeq 10^{-2}D^{(2)}$, when $y(h)$ is measured in units of its maximum $y_{\max}$. Thus, we truncate the KM expansion after the second term, reducing it to a Fokker-Planck (FP) equation,

$$\frac{\partial}{\partial h}P(y,h|y_0,h_0)=\left[-\frac{\partial}{\partial y}D^{(1)}(y)+\frac{\partial^2}{\partial y^2}D^{(2)}(y)\right]\times P(y,h|y_0,h_0). \tag{12}$$

According to the Ito calculus [27], the FP equation is equivalent to a Langevin equation,

$$\frac{d}{dh}y(h)=D^{(1)}(y)+\sqrt{D^{(2)}(y)}\,f(h), \tag{13}$$

where $f(h)$ is a random "force" with zero mean and Gaussian statistics, $\delta$ correlated in $h$, that is, $\langle f(h)f(h')\rangle=2\delta(h-h')$. Furthermore, Eq. (13) enables one to reconstruct a series for $y(h)$ that is similar to the original one *in the statistical sense*.

## VI. RESULTS AND DISCUSSION

The porosity data described in Sec. II were used to construct the series $y(h)=\phi(h+\delta h)-\phi(h)$, and these were analyzed by the procedure described earlier. In addition, we also computed one additional important property of the data that may be useful in practice, particularly for modeling of the LS porous media. In what follows we describe and discuss the results.

### A. Reconstruction of the series via the Fokker-Planck and Langevin equations

The drift and diffusion coefficients $D^{(1)}$ and $D^{(2)}$ were estimated using the $y(h)$ series for the successive increments of the porosities. Since $y(h)$ is stationary, then $D^{(1)}(y,h)=D^{(1)}(y)$ and $D^{(2)}(y,h)=D^{(2)}(y)$. The analysis of the three series $y(h)$ then indicated that the two coefficients are well represented by the approximants

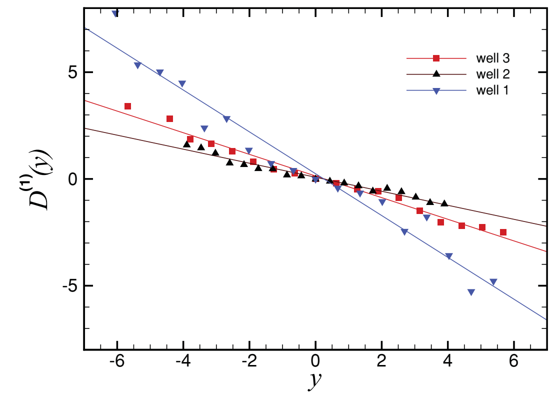


FIG. 3. (Color online) The drift coefficient $D^{(1)}$ for the three wells.

$$D^{(1)}(y)=\begin{cases}-1.090y\ \text{(well 1)},\\-0.266y\ \text{(well 2)},\\-0.549y\ \text{(well 3)}\end{cases} \tag{14}$$

and

$$D^{(2)}(y)=\begin{cases}0.003-0.003y+0.716y^2\ \text{(well 1)},\\0.149-1.280y+0.086y^2\ \text{(well 2)},\\0.079-0.202y+0.248y^2\ \text{(well 3)}.\end{cases} \tag{15}$$

The results are shown in Figs. 3 and 4.

Next, the precision of the reconstructed $y(h)$ is evaluated by computing the conditional PDF $P(y_2,h+\delta h|y_1,h)$, which is very sensitive to the numerical errors in $D^{(1)}(y)$ and $D^{(2)}(y)$. The solution of the FP equation for small $\delta h$, which also represents the left side of Eq. (12), is given by

$$P(y_2,h+\delta h|y_1,h)=\frac{1}{2\sqrt{\pi D^{(2)}(y_2)\delta h}}\times\exp\left\{\frac{-[y_2-y_1-D^{(1)}(y_2)\Delta h]^2}{4D^{(2)}(y_2)\delta h}\right\}. \tag{16}$$

Equation (16) enables one to predict the probability of an "observation" $y_2$ at depth $h+\delta h$, if we know $y_1$ at $h$. Figure 5 presents the directly computed conditional PDFs using the data

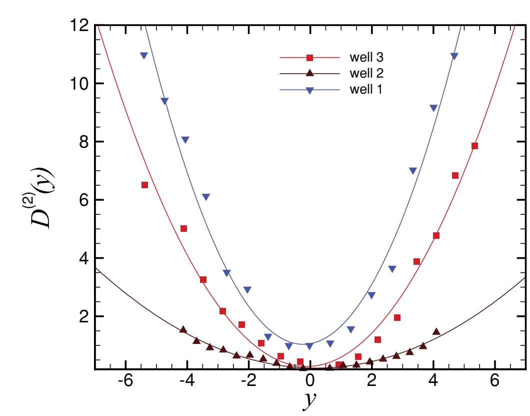


FIG. 4. (Color online) The diffusion coefficient $D^{(2)}$ for the three wells.

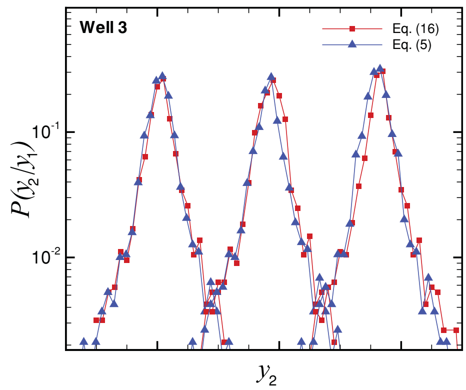


FIG. 5. (Color online) Comparison of the directly evaluated PDFs using the actual data and the PDFs obtained from Eq. (16). Values for $y_1$, from left to right, are $-0.1$, 0.0, and 0.1 [measured in units of $y_{\max}(t)$]. For better presentation, the PDFs have been shifted on the horizontal axis.

for one of the wells (with similar results being obtained for the other two wells and can be obtained by requesting them) and those using Eq. (16), for three values of $y_1$ with $\delta h = h_M = 0.25$ m. To further check the accuracy of the reconstructed $y(h)$ and the conditional PDFs, we used the Kolmogorov-Smirnov test to compare the cumulative distribution function for the original and reconstructed $y(h)$. We found that, for all values of the stochastic variable, the maximum difference between the two cumulative PDFs to be about 0.03. Although the method provides probabilistic predictions, the Langevin equation can produce many trajectories (realizations) of the stochastic process with the same statistical properties. In Fig. 6 the original and reconstructed $\phi(h)$ for the three porosity logs are shown. We show the trajectories that are very close to the original data. Note that the reconstructed series $y(h)$ are stationary, whereas the reconstructed logs $\phi(h)$ are not, as expected.

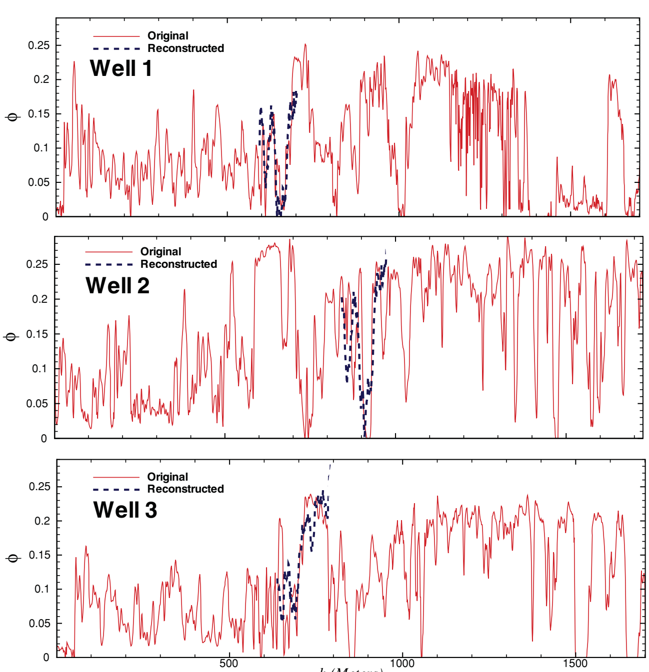


FIG. 6. (Color online) Comparison of the original data with the reconstructed ones for the three wells.

### B. The level-crossing probability

Another important quantity is the frequency of level crossings at a given level $\alpha$ [30,31] given by $\nu_\alpha^+ = P(y_i > \alpha, y_{i-1} < \alpha)$, where $\nu_\alpha^+$ is the number of positive-difference crossings of $y(h)$, $y(h) - \bar{y} = \alpha$, in the interval $l$. The quantity $l(\alpha) = 1/\nu_\alpha^+$ is the average depth interval that one should "wait" for in order to observe $y - \bar{y} = \alpha$ again, which is used to determine when (at what depth) a given value of the porosity is obtained again. This is a particularly important property for modeling LS porous media, because one would like to identify, for example, the high- or low-porosity sectors of porous media for which there are no data, using the available data for other sectors. The knowledge is then utilized in constructing stochastic models of LS porous media [1–3] by constraining the values of the porosity and permeability.

The frequency $\nu_\alpha^+$ is given by [21]

$$\begin{aligned}\nu_\alpha^+ &= \int_{-\infty}^{\alpha} dy_i \int_{\alpha}^{\infty} P(y_i, y_{i-1}) dy_{i-1} \\ &= \int_{-\infty}^{\alpha} dy_i \int_{\alpha}^{\infty} P(y_i|y_{i-1}) P(y_{i-1}) dy_{i-1},\end{aligned} \tag{17}$$

where

$$P(y_{i-1} = y) = \left[\frac{C}{D^{(2)}(y)}\right] \exp\left[\int_0^y dy'\ D^{(1)}(y')/D^{(2)}(y')\right], \tag{18}$$

and $P(y_i|y_{i-1})$ is given by Eq. (16) with $\delta h = h_M$, where $C$ is a normalization constant. In Fig. 7 we present the computed level-crossing frequency $\nu^+(\alpha)$ and $l(\alpha)$ over a depth interval for the actual data set representing well 3. Similar results are obtained for the other two wells and, hence, are not shown.

## VII. RELATIONSHIP WITH SELF-AFFINE FRACTALS AND MULITFRACTAL ANALYSIS

As described in Sec. I, self-affine fractal distributions have been used for analyzing and describing the statistics of the porosity logs, permeability distributions, and other important properties of LS porous media [1–3]. Therefore, it is of both fundamental and practical interest to establish a relationship between the approach presented here and the fractal distributions. To do so, one must consider the PDF for $y(h,\Delta h) = \phi(h + \Delta h) - \phi(h)$ across a length scale $\Delta h$, in order to link the multifractal exponents that characterize the series to the drift and diffusion coefficients.

Thus, to establish the relationship between the two approaches, we begin with the FP equation for the probability density function of $y(h,\Delta h)$, that is, the truncated form of Eq.(10) [27,32,33], written for a scale $\Delta h$,

$$\begin{aligned}-\Delta h \frac{\partial}{\partial \Delta h} P(y,\Delta h) = \Bigg[ &- \frac{\partial}{\partial y} D^{(1)}(y,\Delta h) \\ &+ \frac{\partial^2}{\partial y^2} D^{(2)}(y,\Delta h) \Bigg] P(y,\Delta h).\end{aligned} \tag{19}$$

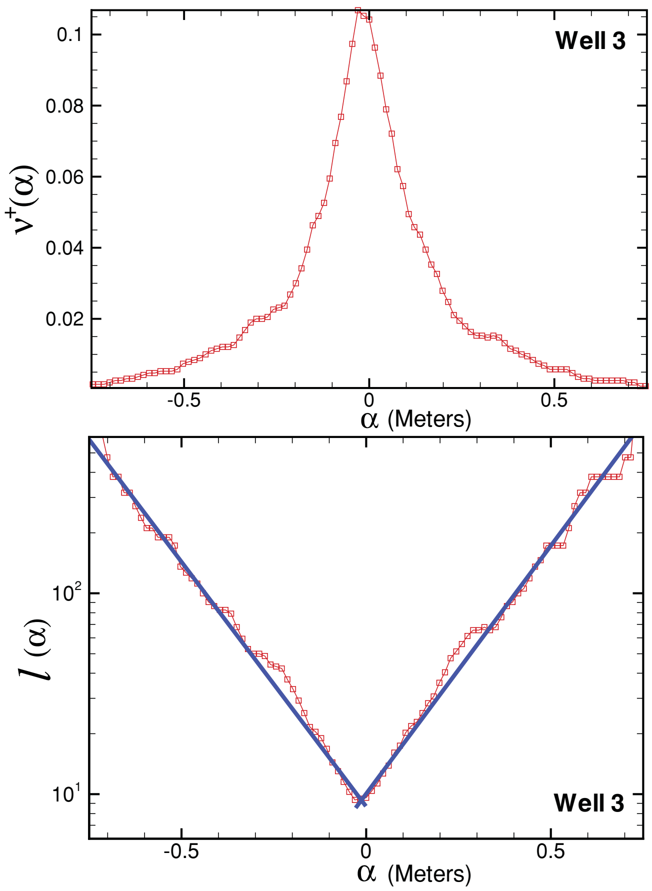


FIG. 7. (Color online) Frequency of crossing $\nu^+$ (top) and the average depth interval $l$ for observing $y(h) = \alpha$ again.

The drift and diffusion coefficients are still determined by Eq. (11), written for the scale $\Delta h$ and using $y(h,\Delta h)$. According to Eqs. (14) and (15) the drift and diffusion coefficients of the increment series are given by the general forms (at least to a very good degree of approximation)

$$D^{(1)}(y,\Delta h) \simeq D_0^{(1)} - D_1^{(1)} y, \tag{20}$$

$$D^{(2)}(y,\Delta h) \simeq D_0^{(2)} - D_1^{(2)} y + D_2^{(2)} y^2. \tag{21}$$

Thus, using Eq. (19), we derive an equation for the structure function $S_q(\Delta h) \equiv \langle |y|^q \rangle = \langle |\phi(h+\Delta h) - \phi(h)|^q \rangle$, which is as follows:

$$\Delta h \frac{\partial}{\partial \Delta h} \langle |y|^q \rangle = q \langle |y|^{q-1} D^{(1)}(y,\Delta h) \rangle + q(q-1) \times \langle |y|^{q-2} D^{(2)}(y,\Delta h) \rangle. \tag{22}$$

If we now substitute Eqs. (20) and (21) into Eq. (22), we find that

$$-\Delta h \frac{\partial}{\partial \Delta h} \langle |y|^q \rangle = \left[ q D_1^{(1)} - D_2^{(2)} q(q-1) \right] \langle |y|^q \rangle + O(\langle |y|^{q-1} \rangle). \tag{23}$$

Equation (23) implies that if the structure function follows the following power law:

$$S_q(\Delta h) \sim (\Delta h)^{\xi(q)}, \tag{24}$$

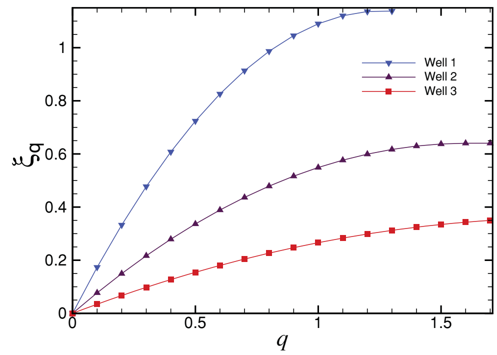


FIG. 8. (Color online) The scaling exponents $\xi(q)$ for the porosity logs of the three wells.

then

$$\xi(q) = q D_1^{(1)} - q(q-1) D_2^{(2)}, \tag{25}$$

which establishes a direct link between the scaling exponents in a multifractal formulation and the drift and diffusion coefficients that we compute.

We observe that $D_1^{(1)} = \xi(q=1)$, implying that the drift coefficient is linked through a simple relationship to the exponent for the $q=1$ moment of the structure function. Moreover, since the Hurst exponent $H$ defined by Eq. (2) is given by $2H = \xi(q=2)$, Eq. (25) yields $H = D_1^{(1)} - D_2^{(2)}$. That is, the Hurst exponent in the fractal or multifractal formulation is linked through a simple relationship to the drift and diffusion coefficients in the Markov analysis. That the Hurst exponent $H$ in a fractal or multifractal formulation is related to the drift and diffusion coefficients in the present approach is not perhaps totally surprising because, as pointed out earlier, the method that we describe in this paper is one of a moment-generating function approach, which is closely related to the structure function used in multifractal analysis. Figure 8 presents the results for the scaling exponents $\xi(q)$.

The coefficients $D^{(1)}(y,\Delta h)$ and $D^{(2)}(y,\Delta h)$ for the three well logs remain to be determined. As already pointed out, the two coefficients are not the same as those given by Eqs. (14) and (15) that are for the *successive increments* of the porosities. The drift and diffusion coefficients in Eqs. (23) and (25) are for fluctuations of $y(h)$ across scales $\Delta h$. The procedure for computing $D^{(1)}(y,\Delta h)$ and $D^{(2)}(y,\Delta h)$ is the same as before. We obtain

$$D^{(1)}(y,\Delta h) \simeq \begin{cases} -0.61y & \text{(well 1)}, \\ -0.32y & \text{(well 2)}, \\ -0.43y & \text{(well 3)} \end{cases} \tag{26}$$

and

$$D^{(2)}(y,\Delta h) \simeq \begin{cases} 0.47y^2 & \text{(well 1)}, \\ 0.13y^2 & \text{(well 2)}, \\ 0.24y^2 & \text{(well 3)}, \end{cases} \tag{27}$$

which, as expected, are similar to but not identical with those given by Eqs. (14) and (15). Then, using Eqs. (25)–(27) we obtain $H = 0.14$, 0.19, and 0.19 for wells 1, 2, and 3, respectively, which are the same as those computed based on the spectral density of the data.

## VIII. SUMMARY

The porosity logs and other important properties of large-scale porous media often represent nonstationary series that are very difficult to analyze. In this paper we analyzed the porosity logs of one such porous media by a method that is completely different from what has been used in the past [1–3]. The method is based on (1) constructing a stationary series based on the successive porosity increments, (2) checking whether the new series follows the properties of a Markov process, and (3), if so, analyzing the series based on the Markov processes and the Kramers-Moyal expansion. In many cases, such as the present study, the expansion terminates after the second-order term, hence yielding a Fokker-Planck equation, which is equivalent to a Langevin equation. The Langevin equation enables us to reconstruct the series and make probabilistic predictions for its "future" beyond the depth interval in which the logs have been measured.

Also computed was the level-crossing probability, which is an important quantity for constructing models of large-scale porous media, because it enables one to identify the areas with potentially high or low values of the porosity (and permeability, if such data are available).

Finally, in view of the past attempts to use the statistics of self-affine fractal distributions to describe the porosity logs, we established a direct link between such methods and the Markov approach described and utilized in this paper. In particular, the Hurst exponent $H$ is directly related to the drift and diffusion coefficients of Fokker-Planck (and, hence, the Langevin) equation. The advantage of the present method over a fractal or multifractal description of the data is that, while the latter method requires that the data exhibit scaling, the former, as was described in this paper, does not.

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