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The Logarithmic Conformal Field Theories

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Abstract

We study the correlation functions of logarithmic conformal field theories. First, assuming conformal invariance, we explicitly calculate two– and three– point functions. This calculation is done for the general case of more than one logarithmic field in a block, and more than one set of logarithmic fields. Then we show that one can regard the logarithmic field as a formal derivative of the ordinary field with respect to its conformal weight. This enables one to calculate any n– point function containing the logarithmic field in terms of ordinary n–point functions. At last, we calculate the operator product expansion (OPE) coefficients of a logarithmic conformal field theory, and show that these can be obtained from the corresponding coefficients of ordinary conformal theory by a simple derivation.

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1 Introduction

It has been shown by Gurarie [1], that conformal field theories (CFTs) whose correlation functions exhibit logarithmic behaviour, can be consistently defined and in the OPE of two given local fields which has at least two fields with the same conformal dimension, one may find some operators with a special property, known as logarithmic operators. As discussed in [1], these operators with the ordinary operators form the basis of the Jordan cell for the operators L_i. In some interesting physical theories, for example the WZNW model on the $GL(1,1)$ super-group [2], and edge excitation in fractional quantum Hall effect [3], one can naturally find logarithmic terms in correlators. Recently the role of logarithmic operators have been considered in study of some physical problems such as 2D-magnetohydrodynamic turbulence [4, 5, 6], 2D-turbulence [7, 8], $c_{p,1}$ models [9, 16], gravitationally dressed CFT's [10], and some critical disordered models [12, 13]. They play a role in the so called unifing $\mathcal{W}$ algebra [14] and in the description of normalizable zero modes for string backgrounds [11, 15].

The basic properties of logarithmic operators are that, they form a part of the basis of the Jordan cell for L_i's and in the correlator of such fields there is a logarithmic singularity [1, 12]. It has been shown that in rational minimal models such a situation, i.e. two fields with the same dimensions, doesn't occur [5]. The modular invariant partition functions for $c_{eff} = 1$ and the fusion rules of logarithmic conformal field theories (LCFT) are considered in [16, 17].

In this paper, we study the correlation functions of logarithmic conformal field theories (LCFT's). Assuming conformal invariance, we obtain all two- and three-point functions. This calculations have been already done for the case where the Jordanian cell is two dimensional [1, 12]. The key observation in this point is that, one can regard logarithmic fields *formally* as the derivative of ordinary fields with respect to their conformal weight and use this effectively to obtain logarithmic three- and more-point functions from ordinary ones. We think that many other results, if not all of them, for LCFT's can also be obtained by this technique from ordinary conformal field theories. We show that any n-point function for these theories can be obtained from their analouges in the ordinary conformal field theories. These results are then extended to the case of more than two fields in a Jordan cell, and more than one Jordan cell. At last we give the OPE coefficients of a LCFT with a two-dimensional Jordan cell, in terms of the OPE coefficients of the corresponding CFT, and then generalize it to the case of a more dimensional Jordan cell.

2 The Correlation Functions of a LCFT

In an ordinary conformal field theory, primary fields are the highest weights of the representations of the Virasoro algebra. The operator product expansion that defines a primary field $\Phi(w,\bar{w})$ is [18]

$$T(z)\Phi_i(w,\bar{w}) = \frac{\Delta_i}{(z-w)^2}\Phi_i(w,\bar{w}) + \frac{1}{(z-w)}\partial_w\Phi_i(w,\bar{w}) \tag{1}$$

$$T(\bar{z})\Phi_i(w,\bar{w}) = \frac{\bar{\Delta}_i}{(\bar{z}-\bar{w})}\Phi_i(w,\bar{w}) + \frac{1}{(\bar{z}-\bar{w})}\partial_{\bar{w}}\Phi_i(w,\bar{w}) \tag{2}$$

where $T(z) := T_{zz}(z)$ and $\bar{T}(\bar{z}) := T_{\bar{z}\bar{z}}(\bar{z})$. The primary fields are those which transform under $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$ as:

$$\Phi_i(z,\bar{z}) \to \Phi_i'(z,\bar{z}) = (\frac{\partial f^{-1}}{\partial z})^{\Delta_i}(\frac{\partial \bar{f}^{-1}}{\partial \bar{z}})^{\bar{\Delta}_i}\Phi_i(f^{-1}(z),\bar{f}^{-1}(\bar{z})) \tag{3}$$

One can write equation (1) in terms of the components of Laurent expansion of $T(z)$, L_n's,

$$[L_n,\Phi_i(z)] = z^{n+1}\partial_z\Phi_i + (n+1)z^n\Delta_i\Phi_i \tag{4}$$

One can regard Δ_i's as the diagonal elements of a diagonal matrix D,

$$[L_n,\Phi_i(z)] = z^{n+1}\partial_z\Phi_i + (n+1)z^n D_i^j\Phi_j \tag{5}$$

One can however, extend the above relation for any matrix D, which is not necessarily diagonal. This new representation of L_n also satisfies the Virasoro algebra for any arbitrary matrix D . Because we have not altered the first term in the right hand side of the equation (3), this is still a conformal transformation. By a suitable change of basis, one can make D diagonal or Jordanian. If it becomes diagonal, the field theory is nothing but the ordinary conformal field theory. The general case is that there are some Jordanian blocks in the matrix D. The latter is the case of a LCFT. Here, there arise some other fields which do not transform like ordinary primary fields, and are called quasi-primary fields [1]. For the simplest case, consider a two-dimensional Jordan cell. The fields Φ and Ψ satisfy

$$[L_n, \Phi(z)] = z^{n+1}\partial_z \Phi + (n+1) z^n \Delta \Phi \tag{6}$$

and

$$[L_n, \Psi(z)] = z^{n+1}\partial_z \Psi + (n+1) z^n \Delta \Psi + (n+1) z^n \Phi, \tag{7}$$

and they transform as below

$$\Phi(z) \to (\frac{\partial f^{-1}}{\partial z})^{\Delta} \Phi(f^{-1}(z)) \tag{8}$$

$$\Psi(z) \to (\frac{\partial f^{-1}}{\partial z})^{\Delta} [\Psi(f^{-1}(z)) + \log(\frac{\partial f^{-1}(z)}{\partial z}) \Phi(f^{-1}(z)] \tag{9}$$

Note that we have considered only the chiral fields. The logarithmic fields, however cannot be factorized to the left- and right-handed fields. For simplicity we derive the results for chiral fields. The corresponding results for full fields are simply obtained by changing

$$z^{\Delta} \to z^{\Delta} \bar{z}^{\bar{\Delta}} \tag{10}$$

and

$$\log z \to \log |z|^2 \tag{11}$$

Now compare the relations (6, 8) and (7, 9); one can assume the field Ψ as the derivation of the field Φ with respect to its conformal weight, Δ. This fact will be effectively used throughout this paper.

Now let us consider the action of Möbius generators $(L_0, L_\pm)$ on the correlation functions. Whenever the field Ψ is absent, the *form* of the correlators is the same as ordinary conformal field theory. By the term *form* we mean that some of the constants which cannot be determined in the ordinary conformal field theory may be fixed in the latter case. Now we want to compute correlators containing the field Ψ. At first we should compute the two-point functions. The two-point functions of the field Φ is as below

$$< \Phi(z)\Phi(w) >= \frac{c}{(z-w)^{2\Delta}} \tag{12}$$

In the ordinary conformal field theory the constant c cannot be determined only with assuming conformal invariance; to obtain it, one should know for example the stress-energy tensor, although for $c \neq 0$ one can set it equal to one by renormalizing the field. Assuming the conformal invariance of the two-point function $< \Psi(z)\Phi(w) >$, means that acting the set $\{L_0, L_{\pm 1}\}$ on the correlator yeilds zero. Action of L_{-1} ensures that the correlator depends only on the $z - w$. the relations for L_{+1} and L_0 are as below

$$[z^2\partial_z + w^2\partial_w + 2\Delta(z+w)] < \Psi(z)\Phi(w) > +2z < \Phi(z)\Phi(w) >= 0 \tag{13}$$

$$[z\partial_z + w\partial_w + 2\Delta] < \Psi(z)\Phi(w) > + < \Phi(z)\Phi(w) >= 0 \tag{14}$$

Consistency of these two equations for any z and w, fixes c to be zero. Then, solving the above equation for $< \Psi(z)\Phi(w) >$ leads to

$$< \Phi(z)\Phi(w) >= 0, \qquad < \Phi(z)\Psi(w) >= \frac{a}{(z-w)^{2\Delta}} \tag{15}$$

Now assuming the conformal invariance of the two-point function $< \Psi(z)\Psi(w) >$, gives us a set of partial differential equation. Solving them, we obtain

$$< \Psi(z)\Psi(w) >= \frac{1}{(z-w)^{2\Delta}}[b - 2a\log(z-w)] \tag{16}$$

These correlations have been obtained in [1, 12], by assuming the consistency of some four point functions. In fact, in [1] the two point functions $\Psi\Phi$ and $\Psi\Psi$ were obtained using the four point function and the assumption that there exists a term with a logarithmic factor in the OPE of certain fields. In [12], using the same assumption, it is shown that the correlator $\Phi\Phi$ is zero, and some three point functions are calculated.

Now we extend the above results to the case where Jordanian block is $n+1$-dimensional. So there is $n+1$ fields with the same weight Δ.

$$[L_n, \Phi_i(z)] = z^{n+1}\partial_z\Phi_i + (n+1)z^n\Delta\Phi_i + (n+1)z^n\Phi_{i-1}, \tag{17}$$

where $\Phi_{-1} = 0$. All we use is the conformal invariance of the theory. From the above fields, only Φ_0 is primary. Acting L_{-1} on any two-point function of these fields, shows that

$$< \Phi_i(z)\Phi_j(w) >= f_{ij}(z-w). \tag{18}$$

Acting L_0 and L_{+1}, leads to

$$< [L_0, \Phi_i(z)\Phi_j(0)] >= (z\partial_z + 2\Delta) < \Phi_i(z)\Phi_j(0) > + < \Phi_{i-1}(z)\Phi_j(0) > + < \Phi_i(z)\Phi_{j-1}(0) >= 0 \tag{19}$$

$$< [L_{+1}, \Phi_i(z)\Phi_j(0)] >= (z^2\partial_z + 2z\Delta) < \Phi_i(z)\Phi_j(0) > +2z < \Phi_{i-1}(z)\Phi_j(0) >= 0. \tag{20}$$

Then it is easy to see that

$$< \Phi_{i-1}(z)\Phi_j(0) >=< \Phi_i(z)\Phi_{j-1}(0) > . \tag{21}$$

Using $\Phi_{-1} = 0$ and the above equation, gives us the following two-point functions.

$$< \Phi_i(z)\Phi_j(w) >= 0 \qquad \text{for} \qquad i+j < n \tag{22}$$

Now solving the Ward identities for $< \Phi_0(z)\Phi_n(w) >$ among with the relation (21), leads to

$$< \Phi_i(z)\Phi_{n-i}(w) >=< \Phi_0(z)\Phi_n(w) >= a_0(z-w)^{-2\Delta}. \tag{23}$$

The form of the correlation function $< \Phi_1(z)\Phi_n(w) >$ is as below

$$< \Phi_1(z)\Phi_n(w) >= (z-w)^{-2\Delta}[a_1 + b_1 \log(z-w)], \tag{24}$$

but the conformal invariance fixes b_1 to be equal to $-2a_0$. So

$$< \Phi_i(z)\Phi_{n+1-i}(w) >=< \Phi_1(z)\Phi_n(w) >= (z-w)^{-2\Delta}[a_1 - 2a_0 \log(z-w)] \quad \text{for} \quad i > 0 \tag{25}$$

Repeating this procedure for the two-point functions of the other fields Φ_i with Φ_n, and knowing that they are in the following form

$$< \Phi_i(z)\Phi_n(w) >= (z-w)^{-2\Delta} \sum_{j=0}^{i} a_{ij}(\log(z-w))^j, \tag{26}$$

gives

$$\sum_{j=1}^{i} j a_{ij}(\log(z-w))^{j-1} + 2\sum_{j=0}^{i-1} a_{i-1,j}(\log(z-w))^j = 0 \tag{27}$$

or

$$(j+1)a_{i,j+1} + 2a_{i-1,j} = 0$$

So

$$a_{i,j+1} = \frac{-2}{j+1}a_{i-1,j} = \cdots = \frac{(-2)^{j+1}}{(j+1)!}a_{i-j-1,0} =: \frac{(-2)^{j+1}}{(j+1)!}a_{i-j-1} \tag{28}$$

or

$$< \Phi_i(z)\Phi_n(w) >= (z-w)^{-2\Delta} \sum_{j=0}^{i} \frac{(-2)^j}{j!} a_{i-j}(\log(z-w))^j, \tag{29}$$

and also we have

$$< \Phi_i(z)\Phi_k(w) >=< \Phi_{i+k-n}(z)\Phi_n(w) > \qquad \text{for} \qquad i+k \geq n. \tag{30}$$

So for the case of n logarithmic field, we found all the two point functions. The interesting points are

i)some of the two-point functions become zero.

ii)some of the two-point functions are logarithmic, and the highest power of the logarithm, which occures in the $< \Phi_n \Phi_n >$, is n.

The most general case is the case where there is more than one Jordanian block in the matrix D ,or in other words, there is more than one set of logarithmic operators. The dimension of these blocks may be equal or not equal. Using the same procedure, one can find that

$$< \Phi_i^I(z)\Phi_j^J(w) >= \begin{cases} (z-w)^{-2\Delta} \sum_{k=0}^{i+j-n} \frac{(-2)^k}{k!} a_{n-k}^{IJ} [\log(z-w)]^k, & i+j \geq n \\ \\ 0, & i+j < n \end{cases} \tag{31}$$

where I and J label the Jordan cells, $n = \max\{n_I, n_J\}$ and n_I and n_J are the dimensions of the corresponding Jordan cells. Also note that the conformal dimensions of the cells I and J must be equal, otherwise the two-point functions are trivially zero.

Now we want to consider the three-point functions of logarithmic fields. The simplest case is the case where, besides Φ, only one extra logarithmic field Ψ exists in the theory. The three-point functions of the fields Φ are the same as ordinary conformal field theory.

$$A(z_1, z_2, z_3) :=< \Phi(z_1)\Phi(z_2)\Phi(z_3) >= \frac{a}{(\xi_1\xi_2\xi_3)^\Delta} =: af(\xi_1, \xi_2, \xi_3), \tag{32}$$

where

$$\xi_i = \frac{1}{2} \sum_{j,k} \epsilon_{ijk}(z_j - z_k).$$

If one acts the set $\{L_0, L_{\pm 1}\}$ on the three-point function $< \Psi(z_1)\Phi(z_2)\Phi(z_3) >:= B(z_1, z_2, z_3)$, the result is an inhomogeneous partial differential equation for $B(z_1, z_2, z_3)$ where the inhomogeneous part is $A(z_1, z_2, z_3)$. So the form of $B(z_1, z_2, z_3)$ should be as below,

$$B(z_1, z_2, z_3) = [b + \sum b_i \log \xi_i] f(\xi_1, \xi_2, \xi_3). \tag{33}$$

Solving the above mentioned differential equations, we find the parameters b_i to be

$$b_1 = -b_2 = -b_3 = a. \tag{34}$$

The final result is

$$< \Psi(z_1)\Phi(z_2)\Phi(z_3) >= [b + a \log \frac{\xi_1}{\xi_2\xi_3}] f(\xi_1, \xi_2, \xi_3) \tag{35}$$

If there are two fields Ψ in the three-point function, one can write it in the following form

$$< \Psi(z_1)\Psi(z_2)\Phi(z_3) >= [c + \sum c_i \log \xi_i + \sum_{ij} c_{ij} \log \xi_i \log \xi_j] f(\xi_1, \xi_2, \xi_3). \tag{36}$$

Again the Ward identities can be used to determine the parameters c_i and c_{ij},

$$< \Psi(z_1)\Psi(z_2)\Phi(z_3) >= [c - 2b \log \xi_3 + a[(-\frac{\log \xi_1}{\log \xi_2})^2 + (\log \xi_3)^2] f(\xi_1, \xi_2, \xi_3). \tag{37}$$

Finally, for the correlator of three Ψ's we use

$$< \Psi(z_1)\Psi(z_2)\Psi(z_3) >= [d + d_1 D_1 + d_2 D_2 + d_2' D_1^2 + d_3 D_3 + d_3' D_1 D_2 + d_3' D_1^3] f(\xi_1, \xi_2, \xi_3) \tag{38}$$

where

$$D_1 := \log(\xi_1\xi_2\xi_3) \tag{39}$$

$$D_2 := \log \xi_1 \log \xi_2 + \log \xi_2 \log \xi_3 + \log \xi_1 \log \xi_3 \tag{40}$$

$$D_3 = \log \xi_1 \ \log \xi_2 \ \log \xi_3. \tag{41}$$

This is the most general symmetric up to third power logarithmic function of the relative positions. Using the Ward identities, this three-point function is calculated to be

$$< \Psi(z_1)\Psi(z_2)\Psi(z_3) >= [d - cD_1 + 4bD_2 - bD_1^2 + 8aD_3 - 4aD_1D_2 + aD_1^3] f(\xi_1, \xi_2, \xi_3) \tag{42}$$

Now there is a simple way to obtain these correlators. Remember of the relation between $\Phi(z)$ and $\Psi(z)$

$$\Psi(z) = \frac{\partial}{\partial \Delta} \Phi(z). \tag{43}$$

The meaning of this relation will be clearer in section 3. Consider any three-point function which contains the field Ψ. This correlator is related to another correlator which has a Φ instead of Ψ according to

$$< \Psi(z_1)A(z_2)B(z_3) >= \frac{\partial}{\partial\Delta} < \Phi(z_1)A(z_2)B(z_3) >, \tag{44}$$

To be more exact, the left hand side satisfies the Ward identities if the correlator of the right hand side does so. But the three-point function for ordinary fields are known. So it is enough to differentiate it with respect to the weight Δ. Obviously, a logarithmic term appears in the result. In this way one can easily obtain the above three-point functions. In fact instead of solving certain partial differential equations, one can easily differentiate with respect to the conformal weight. This method can also be used when there are n logarithmic fields. To obtain the three- point function containing the field Φ_i, one should write the three-point function, which contains the field Φ_0, and then differentiate it i times with respect to Δ. Note that in the first three-point function, there may be more than one field with the same conformal weight Δ. Then one must treat the conformal weights to be independent variables, differentiate with respect to one of them, and finally put them equal to their appropriate value. Second, there are some constants, or unknown functions in the case of more than three-point functions, in any correlator. In differentiation with respect to a conformal weight, one must treat these *formally* as functions of the conformal weight as well. As an example consider

$$< \Phi(z_1)\Phi(z_2)\Phi(z_3) >= \frac{a}{(\xi_1)^{\Delta_2+\Delta_3-\Delta_1}(\xi_2)^{\Delta_3+\Delta_1-\Delta_2}(\xi_3)^{\Delta_2+\Delta_1-\Delta_3}}. \tag{45}$$

Differentiate with respect to Δ_1, and then put $\Delta_1 = \Delta_2 = \Delta_3$, and $\frac{\partial a}{\partial \Delta_1} = b$. This is (35).

This method can be used for any n-point function:

$$< \Phi_i(z_1)\cdots A(z_{n-1})B(z_n) >= \frac{\partial^i}{\partial\Delta^i} < \Phi_0(z_1)\cdots A(z_{n-1})B(z_n) >, \tag{46}$$

provided one treats the constants and functions of the correlator as functions of the conformal weight.

Another thing to be noted is that this technique does not work for the two point functions. The reason for this is that the two point function of two primary fields with different conformal weights is zero. So, the two point function is not a well-behaved differentiable function of the conformal weights.

3 OPE Coefficients of General LCFT

The most general expression for the operator product expansion of ordinary conformal fields is [18]:

$$\Phi_n(z)\Phi_m(0) = \sum_p z^{\Delta_p - \Delta_n - \Delta_m} \, C^p_{nm}\phi_p(z)\Phi_p(0) \tag{47}$$

where

$$\phi_p(z) = \sum_{\mathbf{k}} z^{\sum k_i} \, \beta^{p,\mathbf{k}}_{nm} L_{-k_1} \, \cdots L_{-k_n} \tag{48}$$

Here the coeficients $\beta^{p\mathbf{k}}_{nm}$ are completely determined in terms of conformal weights and the central charge of the theory. C^p_{nm}'s, however, are not determined just by conformal invariance. Now, concentrate on a specific value of p, and suppose that there are two conformal weights Δ_p, and $\Delta'_p := \Delta_p + \epsilon$, where ϵ is a small number. One can write

$$\begin{aligned}\Phi_n(z)\Phi_m(0) &= \cdots + z^{\Delta_p-\Delta_n-\Delta_m}\hat{C}^p_{nm}\phi_p(z)\Phi_p(0) + z^{\Delta_p-\Delta_n-\Delta_m+\epsilon}\hat{C}^{p'}_{nm}\phi_{p'}(z)\Phi_{p'}(0)\\ &= \cdots + (\hat{C}^p_{nm} + \hat{C'}^p_{nm})z^{\Delta_p-\Delta_n-\Delta_m}\phi_p(z)\Phi_p(0) + \epsilon\hat{C'}^p_{nm}\frac{\partial}{\partial\Delta_p}(z^{\Delta_p-\Delta_n-\Delta_m}\phi_p(z)\Phi_p(0))\end{aligned} \tag{49}$$

where

$$\hat{C'}^p_{nm} := \hat{C}^{p'}_{nm}, \tag{50}$$

and we have treated ϕ_p and Φ_p, formally, as functions of Δ_p.

Now let ϵ tend to zero. If the C's are kept finite, the second term vanishes and nothing new happens: this is just an ordinary conformal field theory. If, on the other hand, one keeps $\epsilon\hat{C}'$ and $\hat{C} + \hat{C}'$ finite. It turns out that

$$\Phi_n(z)\Phi_m(0) = \cdots + \bar{C}^p_{nm} z^{\Delta_p-\Delta_n-\Delta_m}\phi_p(z)\Phi_p(0) + C^p_{nm}\frac{\partial}{\partial\Delta_p}(z^{\Delta_p-\Delta_n-\Delta_m}\phi_p(z)\Phi_p(0)) \tag{51}$$

As C and $\bar{C}$ are arbitrary, we can define $\bar{C}$ as the formal derivative of C with respect to Δ_p. Then

$$\Phi_n(z)\Phi_m(0) = \cdots + \frac{\partial}{\partial\Delta_p}[C^p_{nm} z^{\Delta_p-\Delta_n-\Delta_m}\phi_p(z)\Phi_p(0)], \tag{52}$$

or

$$\Phi_n(z)\Phi_m(0) = \cdots + z^{\Delta_p-\Delta_n-\Delta_m}\{C^p_{nm}[\psi_p(z)\Phi_p(0) + \phi_p(z)\Psi_p(0) + \phi_p(z)\Phi_p(0)\log z] + {C'}^p_{nm}\phi_p(z)\Phi_p(0)\} \tag{53}$$

where we have defined

$$\psi_p(z) = \frac{\partial\phi_p(z)}{\partial\Delta_p}$$
$$\tag{54}$$
$$\Psi_p(z) = \frac{\partial\Phi_p(z)}{\partial\Delta_p}$$

Note that these derivations are formal. There are, of course, conformal field theories where the set of conformal weights is discrete, and it may seem that there, derivation with respect to the weight is meaningless. What is done, resembles very much to the case when one knows a function only in certain points. One cannot *obtain* the derivative of this function. One can, however, introduce other (unknown) quantities as the formal derivative of this function and use identities (such as Leibnitz's rule) concerning the derivation. There remains, of course, the unknown quantities introduced by derivation. That is the reason why new quantities (c-numbers such as $\bar{C}$ and operators such as ψ_p and Ψ_p) are introduced. The same is true also for derivating the correlators: one cannot *obtain* the derivative of constants appearing in the correlator. They simply introduce more constants in the theory. Also note that all we have used is conformal invariance, and no specific model has been taken into account.

Now, using the definitions

$$| \Delta_m >:= \Phi_m(0) | 0 >$$

$$| z, \Delta_p >:= \phi_p(z) | \Delta_p >$$

$$=: \sum_N z^N | N, \Delta_p > \tag{55}$$

$$| \Delta'_p >:= \frac{\partial}{\partial \Delta_p} | \Delta_p >$$

$$| z, \Delta'_p >:= \frac{\partial}{\Delta_p} | z, \Delta_p >$$

it is seen that

$$| z, \Delta'_p >= \phi_p(z) | \Delta'_p > + \psi_p(z) | \Delta_p >, \tag{56}$$

and

$$\Phi_n(z) | \Delta_m >= \sum_N z^{\Delta_p - \Delta_m - \Delta_n + N} [C^p_{nm}(| N, \Delta'_p > + \log z | N, \Delta_p >) + C'^p_{nm} | N, \Delta_p >] \tag{57}$$

Acting on both side of this relation by L_j, one gets

$$\sum_N z^{\Delta_p - \Delta_m - \Delta_n + N} [C^p_{nm} L_j(| N, \Delta'_p > + \log z L_j | N, \Delta_p >) + C'^p_{nm} L_j | N, \Delta_p >] = \sum_N z^{\Delta_p - \Delta_m - \Delta_n + N + j} \times$$

$$\times \{C^p_{nm}[(\Delta_p - \Delta_m + j\Delta_n + N)(| N, \Delta'_p > + \log z | N, \Delta_p >) + | N, \Delta_p >] + C'^p_{nm}(\Delta_p - \Delta_m + j\Delta_n + N) | N, \Delta_p >\} \tag{58}$$

using the independency of z^k and $z^k \log z$, it is seen that

$$\begin{cases} L_j | N + j, \Delta_p >= (\Delta_p - \Delta_m + j\Delta_n + N) | N, \Delta_p > \\ L_j | N + j, \Delta'_p >= (\Delta_p - \Delta_m + j\Delta_n + N) | N, \Delta'_p > + | N, \Delta_p > \end{cases}, \quad j > 0 \tag{59}$$

The last relation is obviously the derivative of the first, with respect to Δ_p.

Similarly, the action of L_0 yields

$$L_0(\mid N, \Delta_p >= (\Delta_p + N) \mid N, \Delta_p >$$
$$L_0 \mid N, \Delta'_p >= (\Delta_p + N) \mid N, \Delta'_p > + \mid N, \Delta - p > . \tag{60}$$

In [4], a method was proposed to obtain $\mid N, \Delta_p >$ in terms of $\mid \Delta_p >$:

$$\mid N, \Delta_p >= \sum_{\mathbf{k} \mid \sum k_i = N} \beta^{p\mathbf{k}}_{nm} L_{-k_1} \cdots L_{-k_n} \mid \Delta_p > \tag{61}$$

where β's satisfy a linear equation of the form

$$M\beta = \gamma \tag{62}$$

Using the definition of $\mid N, \Delta'_p >$, it is obvious that

$$\mid N, \Delta'_p >= \sum_{\mathbf{k} \mid \sum k_i = N} \beta'^{p,\mathbf{k}}_{nm} L_{-k_1} \cdots L_{-k_n} \mid \Delta_p > + \sum_{\mathbf{k} \mid \sum k_i = N} \beta^{p\mathbf{k}}_{nm} L_{-k_1} \cdots L_{-k_n} \mid \Delta_p > \tag{63}$$

where β''s are the derivative of β's with respect to Δ_p. But taking the derivative of (62) yields

$$M'\beta + M\beta' = \gamma' \tag{64}$$

Combining this with (62), we get

$$\begin{pmatrix} M & 0 \\ M' & M \end{pmatrix} \begin{pmatrix} \beta \\ \beta' \end{pmatrix} = \begin{pmatrix} \gamma \\ \gamma' \end{pmatrix} \tag{65}$$

which, among with (63), it is precisely what was obtained in [19].

Now, the general form of (52) can be written as

$$\Phi_n(z)\Phi_m(0) = \sum_p \left(\frac{\partial}{\partial \Delta p} \right)^{q_p} \left[C^p_{nm} z^{\Delta_p - \Delta_n - \Delta_m} \phi_p(z) \Phi_p(0) \right] , \tag{66}$$

where q_p is the dimension of the pth. Jordanian block.

In a manner similar to that of the previous discussion, one can define

$$\begin{cases} \Phi_p^{(n)} := \left(\frac{\partial}{\partial\Delta p}\right)^n \Phi_p \\ \phi_p^{(n)} := \left(\frac{\partial}{\partial\Delta p}\right)^n \phi_p & , 0 \leq n < q_p \\ \mid N, D_p^{(n)} > := \left(\frac{\partial}{\partial\Delta p}\right)^n \mid N, \Delta_p > \end{cases} \tag{67}$$

and the equation corresponding to (65) becomes

$$\sum_{j=0}^{q_p-1} M_{ij}\beta_j = \gamma_i, \tag{68}$$

where

$$\gamma_i := \left(\frac{\partial}{\partial\Delta p}\right)^i \gamma, \tag{69}$$

and

$$M_{ij} := \frac{i!}{j!(i-j)!}\left(\frac{\partial}{\partial\Delta p}\right)^{i-j} M. \tag{70}$$

M's up to the third level are given in [4].

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