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Anderson localization and propagation of electromagnetic waves through disordered media

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We have used the dynamic method to calculate the frequency dependence of the localization length in a disordered medium, using the amplitude change and the redshift of the spectral density of the propagating incident pulse. The frequency dependence of the localization length in an effectively one-dimensional disordered medium is computed in terms of the strength of the disorder. The results obtained with the dynamic method are confirmed by computing the same results using the transfer-matrix method.

1. Introduction

Heterogeneous materials and media, both natural and man-made, are ubiquitous [1], and are of tremendous importance and interest. Examples include porous media, composite materials, and biological systems. Due to their significance, static and dynamical properties of heterogeneous materials have been studied for decades, using experimental, theoretical, and computer simulation methods. Among the dynamical phenomena that occur in disordered materials that have been studied are transport of charge, stress, and waves. It is the last subject that is of interest to us in this paper.

Propagation of waves in heterogeneous materials has been studied for several decades [2]. Many rigorous results have been derived. For example, it has been shown [3,4] rigorously that in one-dimensional (1D) disordered media with diagonal disorder and short-range correlations, even infinitesimally small disorder is sufficient to localize the wavefunction, irrespective of the energy. In that case, the envelope of the wavefunction $\psi(r)$ decays exponentially at large distances r from the domain's center, $\psi(r) \sim \exp[-r/\xi(E)]$, with $\xi(E)$ being the localization length at energy level E.

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The known exact results are limited mostly to low-dimensional media. Thus, the main tool for studying wave propagation in heterogeneous media has been numerical simulations. Among the various numerical techniques used in the simulations to estimate the effective properties of disordered media, such as the localization length, is the transfer-matrix (TM) method [5,6]. In the TM method one fixes the frequency of the wave, or the energy of the incident particles, and calculates the Lyapunov exponent, the inverse of ξ. In the limit of weak disorder, however, the convergence of the TM method requires intensive computations.

The localization concept has been extended to the propagation of classical waves, e.g. ultrasound [7] and electromagnetic [8–11] waves. Such studies have been followed by observation of weak localization and studying the shape of the coherent back-scattering cone [12–16]. The experimental evidence for localization (of Anderson type) of light in optical systems was provided by Wiersma et al. [8] in semiconducting powders that are highly scattering. They reported a diffusive regime for large particle sizes, and a localized regime, i.e. one in which the transmission coefficients decays exponentially with the sample thickness, for particles with small diameters. The transverse localization of light was also observed experimentally in a 2D disordered photonic lattice [17]. The intensity distribution of the Gaussian beam of light in the transverse directions was studied by propagation in the longitudinal direction. The average effective width of the light beam, w_{eff}, was found to increase with the propagation distance Z, and to follow a power-law relation, $w_{\mathrm{eff}} = Z^{\nu}$. By increasing the disorder strength the transport mechanism changed from the ballistic ($\nu = 1$) type to the diffusive ($\nu = 0.5$), and eventually the localized ($\nu = 0$) regime.

Propagation of a light pulse, known as the dynamic method, is another way of investigating Anderson localization. Various experiments have reported the existence of a non-exponential long-time tail of the transmitted intensity of the pulse in strongly disordered media [18–20]. The same has also been investigated numerically [21,22]. The self-consistent theory of Anderson localization has also been applied to the study of the dynamics of localized waves in random media [23,24].

In this paper we use the dynamic method in order to compute the localization length ξ of electromagnetic fields in a disordered medium, using the amplitude change and the redshift of the intensity of the spectrum density (SD) of an incident pulse propagating in the medium. We consider the transmission of a wave pulse that contains a wide range of frequencies, which allows us to compute $\xi(\omega)$. The wave disperses during its propagation and loses energy by multiple scattering caused by the fluctuations of the background dielectric constant ϵ that varies spatially. By calculating the SD of the transmitted pulse at several distances from the source, we determine the exponential decay of the SD as a function of the distance, particularly for high frequencies. This would enable us to determine the frequency dependence of the localization length. The spectrum of the high frequencies decreases faster than that of the low frequencies, and leads to a change in the maximum value of the spectrum that indicates a redshift of the SD due to the propagation in random media. We also utilize the TM method to compute the localization length over a range of frequencies, and compare the results with those obtained with the dynamic method.

The rest of the paper is organized as follows. In Section 2 we describe the model that we use, which is based on Maxwell's equations and discuss its details.

In Section 3 we present and discuss our numerical results. A summary of the paper is presented in Section 4.

2. Model and methods

To study the propagation of electromagnetic waves in a disordered medium, we use the dynamic Maxwell equations,

$$\begin{aligned} \nabla \times \mathbf{H} &= \partial \mathbf{D}/\partial t \\ \nabla \times \mathbf{E} &= -\partial \mathbf{B}/\partial t \end{aligned} \tag{1}$$

where $\mathbf{D} = \epsilon \mathbf{E}$, $\mathbf{H} = \mu^{-1}\mathbf{B}$, and $\mathbf{E}$ is the electric field, while $\mathbf{H}$ represents the magnetic field. The permittivity $\epsilon(\mathbf{x})$ is assumed to be a random variable and is given by $\epsilon(\mathbf{x}) = \bar{\epsilon} + \eta(\mathbf{x}) = k\epsilon_0 + \eta(\mathbf{x})$, where $\bar{\epsilon} = \langle \epsilon(\mathbf{x}) \rangle$ is its mean value, and $\eta(\mathbf{x})$ is a Gaussian white noise with variance W, i.e. $\langle \eta(\mathbf{x}_1)\eta(\mathbf{x}_2) \rangle = W^2 \delta(\mathbf{x}_1 - \mathbf{x}_2)$.

Let us rescale the variables by writing $x' = x/x_0$, $t' = ct/(x_0\sqrt{k})$, $E' = E/E_0$, $B' = cB/(E_0\sqrt{k})$, and $\eta'(x') = \eta(x)/\bar{\epsilon}$, which yields the following dimensionless equations: $\nabla' \times \mathbf{B}' = [1 + \eta'(x')]\partial \mathbf{E}'/\partial t'$, $\nabla' \times \mathbf{E}' = -\partial \mathbf{B}'/\partial t'$. The variance of η which is random in 1D is also rescaled as $W'^2 = W^2/(\bar{\epsilon}^2 x_0)$, so that $\langle \eta'(\mathbf{x}'_1)\eta'(\mathbf{x}'_2) \rangle = (W'^2/d')\delta_{\mathbf{x}'_1, \mathbf{x}'_2} = \sigma^2 \delta_{\mathbf{x}'_1, \mathbf{x}'_2}$, where d' is the lattice spacing in the rescaled system. Thus, the noise η' is characterized by the width $\sigma = (W'^2/d')^{1/2} = (W^2/W_0^2)^{1/2}$, with $W_0^2 = \bar{\epsilon}^2 x_0 d' = \bar{\epsilon}^2 d$.

To carry out the computations, we use finite-difference discretization and the Yee algorithm [25]. For simplicity we use the indices (i, j, k) to denote (the center of) one basic cubic block of the Yee mesh (see Figure 1), and then the discretized rescaled equations for, for example, the E_x component of the electric field $\mathbf{E}$ are given by

$$\begin{aligned} &E_x|_{i,j,k}^{t+1/2} - E_x|_{i,j,k}^{t-1/2} \\ &= \frac{\mathrm{d}t}{\frac{640}{3} d(1 + \eta_{(i,j,k)})} \Bigg[\Bigg(B_z|_{i,j+3,k}^t - \frac{125}{9} B_z|_{i,j+2,k}^t + 250 B_z|_{i,j+1,k}^t - 250 B_z|_{i,j,k}^t + \frac{125}{9} B_z|_{i,j-1,k}^t \\ &\quad - B_z|_{i,j-2,k}^t \Bigg) - \Bigg(B_y|_{i,j,k+3}^t - \frac{125}{9} B_y|_{i,j,k+2}^t + 250 B_y|_{i,j,k+1}^t - 250 B_y|_{i,j,k}^t + \frac{125}{9} B_y|_{i,j,k-1}^t \\ &\quad - B_y|_{i,j,k-2}^t \Bigg) \Bigg] \end{aligned} \tag{2}$$

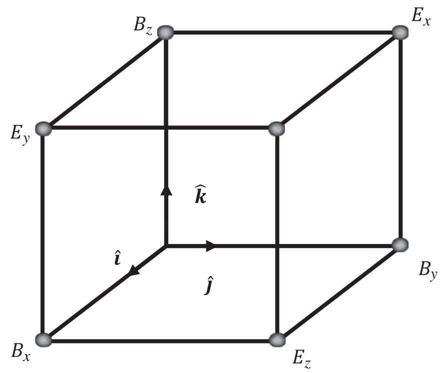

Figure 1. The basic cell of the Yee mesh. The length of the cube is $d/2$.

where all the primes have been omitted for simplicity, and dt and d are the time step and lattice spacing, respectively. To ensure the stability of the algorithm, we set d$t = d/8$, and monitor the time variation of the total electromagnetic energy of the system and its fluctuation.

We also used the TM method [5,6] in order to calculate the localization length of electromagnetic waves. It is well known that to estimate the localization length using the TM method in higher dimensions, one must utilize the finite-size scaling method [5,6,26,27]. But, the resulting estimate cannot be compared directly with what is obtained with the dynamic method. Therefore, to make a proper comparison we compare the results obtained with both methods in a disordered medium that is effectively one dimensional. We note that the estimation of the localization length using the dynamic method does not need a finite-size scaling analysis in higher dimensions, as it is applicable to any disordered medium of any space dimensionality.

We begin with deriving the TM of the effective 1D model. The discretized equations are given by

$$\begin{aligned} B_x(k+1) &= B_x(k) - \omega d\mu[1+\eta(k)]E_y(k) \\ E_y(k+1) &= E_y(k) + \omega d B_x(k+1). \end{aligned} \tag{3}$$

Thus, through numerical simulations values of B_x in the slice $k+1$ are computed using values of E_y and B_x, in the slice k, while values of E_y in the slice $k+1$ are calculated knowing E_y in the slice k and B_y in the slice $k+1$, which we compute in the previous step. In the numerical simulations we set $d = 1$ and distribute the random variable η, representing the disorder, uniformly in the interval $[-\sigma, \sigma]$. To formulate the TM computations, the equations for the fields E_y and B_x in 1D are written in the following form,

$$\begin{pmatrix} E_y \\ B_x \end{pmatrix}_{k+1} = \mathbf{T}_k \begin{pmatrix} E_y \\ B_x \end{pmatrix}_k , \tag{4}$$

where $\mathbf{T}_k$ is the TM for the slice k:

$$T_k = \begin{pmatrix} 1 - d^2\omega^2\mu[1+\eta(k)] & \omega d \\ -\omega d\mu[1+\eta(k)] & 1 \end{pmatrix}. \tag{5}$$

3. Results

We consider transmission of light through a lossless random medium that cannot absorb the wave's energy. The medium is constructed by layers (blocks) of random dielectric constant ϵ, arranged in series. Instead of studying the transmission of a plane wave with a given frequency, we investigate propagation of a pulse in the medium in which the pulse's amplitude and the frequency content vary. The wave disperses during its propagation and loses energy by multiple scattering caused by the fluctuations of the background dielectric constant. Thus, the frequency spectrum of the pulse changes, such that the amplitude of the higher frequencies decreases faster than that of the low frequencies. We show below that the spectrum's amplitude

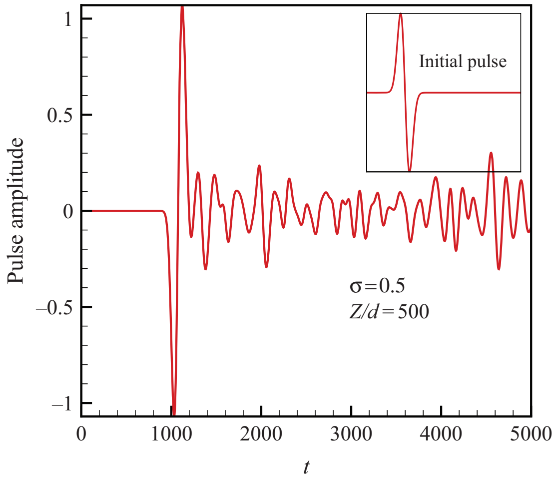

Figure 2. (Color online) The transmitted pulse for $\sigma = 0.5$ at distance $Z = 500$ from the source. Inset: initial pulse at $Z = 0$.

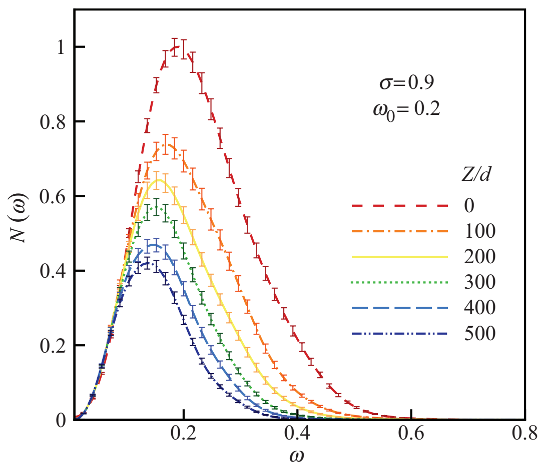

Figure 3. (Color online) The initial spectral density $\mathcal{N}$ of the pulse at $Z = 0$ and the measured ones at locations Z from the source. There is a redshift in the spectral density due to the localization of various frequencies at different length scales.

decays exponentially and, therefore, we may define a localization length for every frequency. The inset of Figure 2 presents the initial pulse in time, while the main part of the figure depicts the transmitted pulse.

We calculated the SD of the electromagnetic field from the Fourier transform of the field, which is given by $\mathcal{N}(\omega) = \frac{1}{2}\epsilon \mathbf{E}^2(\omega) + \frac{1}{2\mu}\mathbf{B}^2(\omega)$. Figure 3 shows the SD of the electromagnetic wave versus the distance Z from the source, for the disorder strength $\sigma = 0.9$, where ω_0 is the peak frequency of the initial spectrum at $Z = 0$. At a given distance Z there is a receiver and the results represent averages over 500 realizations

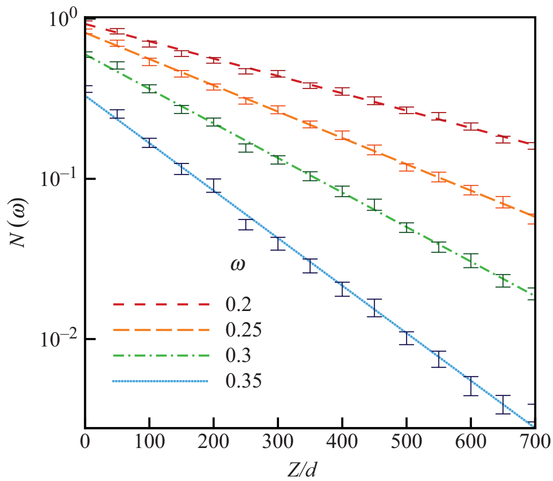

Figure 4. (Color online) The spectral density as a function of the distance Z from the source and the frequency.

of the disorder. For a given frequency, the waves' amplitudes at the receivers that are far from the source decay faster than those measured at the receivers that are near the source. The decay also depends on the frequency, which is due to the scattering of the different frequency content of the transmitted pulse over different length scales. As shown in Figure 3, the high-frequency modes in a random medium scatter more strongly than the low-frequency modes.

The important feature of Figure 3 is the fact that, because the pulse loses its high-frequency contents much faster than those at low frequency, we obtain a shift in the SD to the small frequencies that represents a redshift. The redshift enables us to estimate the localization length for light propagation in disordered media. If the wave's amplitude with a given frequency ω decays as $\psi(\omega, Z) \sim \exp[-Z/\xi(\omega)]$ at large distances from the source, then its SD will decay as $\mathcal{N}(Z) \sim \exp[-2Z/\xi(\omega)]$. Hence, one obtains the following relation for the localization length:

$$\xi^{-1}(\omega) = -\lim_{Z\to\infty}(2Z)^{-1}\ln\left[\frac{\mathcal{N}(\omega, Z)}{\mathcal{N}(\omega, 0)}\right]. \tag{6}$$

Figure 4 presents the semi-log graph of the SD for a given frequency, as a function of the receivers' distance from the source. Its straight-line behavior indicates an exponential decay of the electromagnetic wave. The localization length is computed using the slopes of the lines in Figure 4, repeated for many frequencies.

The results for the localization length calculated based on Figure 4 are shown in the left diagram of Figure 5. The localization length decreases at high frequencies. We find that for low frequencies the localization length scales as $\xi(\omega) \sim \omega^{-\alpha}$, where the exponent α is estimated to be 1.96 ± 0.15 and 1.90 ± 0.15 for disorder strength $\sigma = 0.5$ and 0.9, respectively.

We also checked the above results using the TM method. The results obtained with the TM are also shown in the right diagram of Figure 5. It is clear that values of

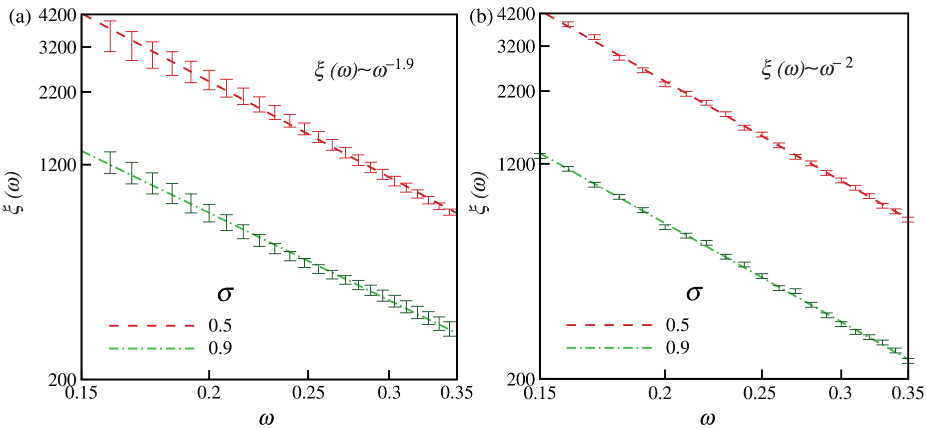

Figure 5. (Color online) Logarithmic plot of localization length as a function of the frequency for several intensities of disorder σ, calculated by (a) the dynamic method, and (b) the TM method. The results obtained with both methods are the same, and show that for low frequencies the localization length is proportional to the inverse of the square of the frequency, $\xi \sim \omega^{-2}$.

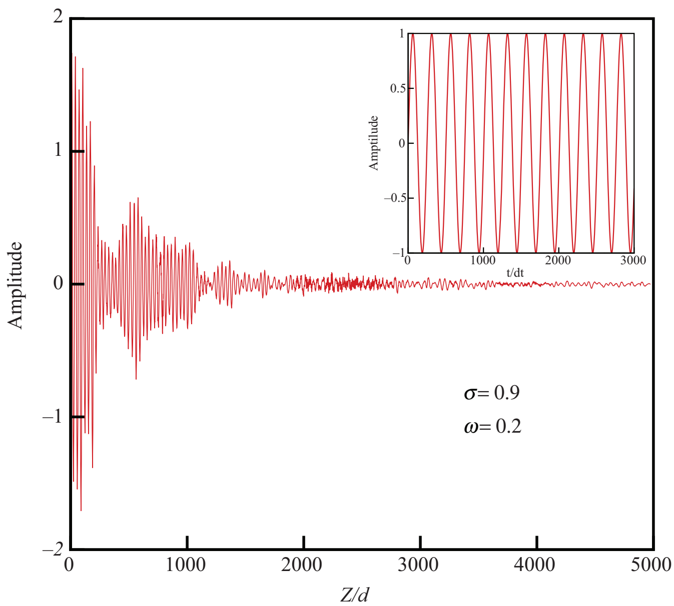

Figure 6. (Color online) The amplitude of the wave in the medium after a long time. The inset shows the sine wave of the source.

the localization length computed with the TM method are the same as those obtained with the dynamic method. Using the TM method, we also calculated the exponent α. The results are 2.04 ± 0.04 and 2.02 ± 0.04, for disorder intensities $\sigma = 0.5$ and 0.9, which are indicated in Figure 5.

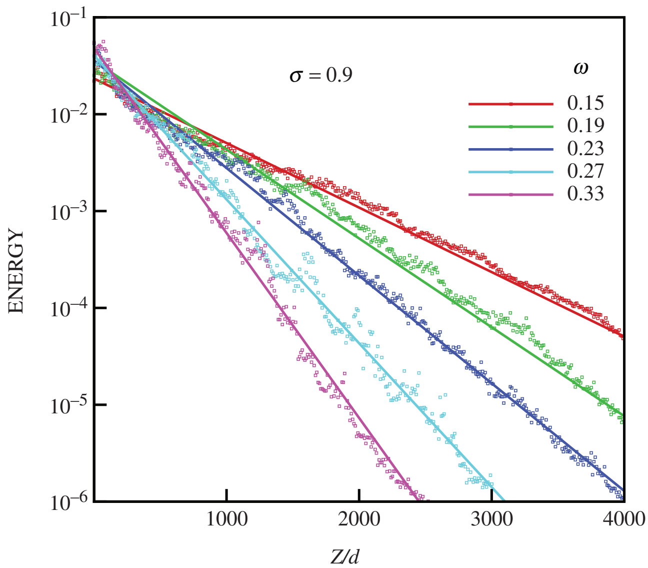

Figure 7. (Color online) Semi-log plot of the energy at various receiver distances from the source and frequencies, to demonstrate the exponential decay. The (solid) lines represent exponential fits of the data.

To compute the decay of a specific frequency (localized modes), we transmit a sinus wave in the medium with frequency ω (shown in the inset of Figure 6), instead of a pulse. Then, a long time after its propagation, we study the energy of the wave in each layer at a distance Z from the source. The energy of the wave at a distance Z is defined by $\frac{1}{2}\epsilon(Z)\mathbf{E}^2 + \frac{1}{2\mu}\mathbf{B}^2$. In the simulations, we took $Z = 5000$ and averaged over 100 realizations of the disorder.

Figure 7 presents the energy versus Z for several frequencies with the disorder strength $\sigma = 0.9$. The exponential decay of the energy with the distance Z is clearly seen in Figure 7. Fitting the numerical results to an exponential decay in Z, we again compute the localization length. For example, the localization lengths computed by this method are $\xi \simeq 1303 \pm 15$, 946 ± 10, and 378 ± 8, for $\omega = 0.15$, 0.19, and 0.31, respectively. These are essentially equal to those computed from the decay of the SD and TM method.

4. Summary

A new method was introduced for calculating the frequency dependence of the localization length $\xi(\omega)$ via measurement of the intensity spectrum of a pulse that propagates in a disordered medium. The new method was illustrated by computing $\xi(\omega)$ for electromagnetic waves in model disordered media, as a function of the disorder strength, and in particular in an effective one-dimensional disordered medium, as functions of the disorder strength and the frequency. The redshift of the spectral density of the incident pulse is observed. The proposed method may also be used in the experimental measurements of the localization length.

To check the results we utilized the transfer-matrix method. The results were found to be in good agreement with those obtained with the dynamic method. We note that, numerically, the dynamic method is much more efficient than the transfer-matrix method. In the dynamic method we can calculate the localization length for a large interval of frequencies in one run of simulations but in the TM method the localization length should be calculated separately for each frequency with a different run of the numerical calculation. The proposed method is general and can be used in *any* disordered medium of any space dimensionality.

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