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Multifractal analysis of light scattering-intensity fluctuations

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We provide a simple interpretation of non-Gaussian nature of the light scattering-intensity fluctuations from an aging colloidal suspension of Laponite using the multiplicative cascade model, Markovian method, and volatility correlations. The cascade model and Markovian method enable us to reproduce most of recent empirical findings: long-range volatility correlations and non-Gaussian statistics of intensity fluctuations. We provide evidence that the intensity increments $\Delta x(\tau)=I(t+\tau)-I(t)$, upon different delay time scales τ, can be described as a Markovian process evolving in τ. Thus, the τ dependence of the probability density function $p(\Delta x,\tau)$ on the delay time scale τ can be described by a Fokker-Planck equation. We also demonstrate how drift and diffusion coefficients in the Fokker-Planck equation can be estimated directly from the data.

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I. INTRODUCTION

As shown by most recent empirical studies on the huge amount of data set, the light scattering-intensity time series from higly interacting colloidal systems and inhomogeneous media are characterized by several "universal" features [1–3]: volatilities strong correlation and non-Gaussian probability distribution function (PDF) on the small time scales. The PDF's shape changes from almost Gaussian at large time scales to fat tails at fine scales [3]. Many authors recently aim at proposing simple, discrete, or continuous time models that are able to account for similar observations [4,5]. Among all the proposed models, one can distinguish several streams from the simplest Brownian process, which constitutes the main tool used by practitioners, to the class of "heteroskedastic" nonlinear processes as proposed in [6].

Recently, an interesting method has been introduced by Ghashghaie *et al.* [7], which is known as Markovian method. It has turned out that this method can be successfully applied to fluctuating time series, such as fluid turbulence [8,9], characterization of rough surfaces [10], finance [11], etc. (see [12] for more details and applications). In the Markovian method one can derive a Fokker-Planck (FP) equation for describing the evolution of the probability distribution function of stochastic properties of given time series. As shown in [7], the conditional probability density of the *increments* of a stochastic field (for example, the increments in the velocity field in turbulent flow) satisfies the Chapman-Kolmogorov (CK) equation, even though the velocity field itself contains long-range nondecaying correlations. As is well known, satisfying the CK equation is a necessary condition for any fluctuating data to be a Markovian process over the relevant length (or time) scales (Markovian time scale). Hence, one has a way of analyzing stochastic phenomena in terms of the corresponding FP and CK equations. Here, we provide two complementary points of view to understand the "multiscaling" and non-Gaussian nature of scattered light intensity fluctuations from a non-equilibriumm aging colloidal system [1,2] and propose a simple multifractal "stochastic volatility" model that captures very well the above-mentioned features of the intensity fluctuations.

The paper is organized as follows. In Sec. II we give a review on multifractal processes and the cascade model. We introduce the related notions of multiscaling, scale invariance, cascade process, and self-similarity kernel. In Sec. III, we use the multifractal random walk (MRW) as a stochastic volatility model and derive multifractal exponents ξ_q for general case with arbitrary well-known Hurst exponent. Section IV is devoted to a brief summary of the most important theorems on Markovian processes and their application to the analysis of empirical data. We estimate a Langevin equation to describe the fluctuations of light scattering-intensity in Sec. V. In Sec. VI volatility and magnitude correlation function were presented. The last section is devoted to summary and conclusions.

II. MULTIFRACTAL PROCESSES AND CASCADE MODEL

In this section we briefly discuss the related notions of multifractality and multiplicative cascade model. Most of the ideas and concepts that we recall below have been introduced in the field of fully developed turbulence [4].

A. Multifractal process and extended self-similarity

Let $\{I(t)\equiv x(t)\}$ be the intensity fluctuations time series and consider its statistics over a certain time scale τ, which is defined as

$$\Delta x(\tau)=x(t+\tau)-x(t). \tag{1}$$

Let us denote $M(q,\tau)$ the order q absolute moment of intensity fluctuations,

$$M(q,\tau)=\langle|\Delta x(\tau)|^q\rangle. \tag{2}$$

We will say that the process is *scale invariant* if the scale behavior of the absolute moment $M(q,\tau)$ has a power-law

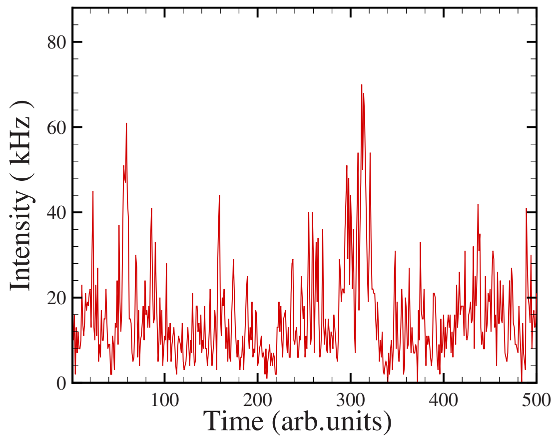

FIG. 1. (Color online) Typical light scattering-intensity fluctuations measured from an interacting colloidal suspension of Laponite with concentration 3.2 wt % as a function of time. In this paper we express intensities in terms of the detector count rate in kHz.

behavior. Let us call ξ_q the exponent of power law, i.e.,

$$M(q,\tau) \simeq N_q \tau^{\xi_q}, \tag{3}$$

where N_q is a prefactor. The process is called *monofractal* if ξ_q is a linear function versus q and *multifractal* if ξ_q is a nonlinear function of q. To check the estimated scaling exponent ξ_q [Eq. (3)] with original time series, we use the extended self-similarity (ESS) method [13,14].

In the ESS method, we rely on the scaling behavior of $S_q(\tau)$ with respect to the specific order of structure function, namely, $S_3(\tau)$ as

$$S_q(\tau) \sim S_3(\tau)^{\zeta_q}. \tag{4}$$

For any Gaussian process, the exponent in the above equation is given by $\zeta_q = q/3$ [13,14]. Any deviation from this relation can be interpreted as a deviation from Gaussianity.

Multifractality has been introduced in the context of fully developed turbulence in order to describe the spatial fluctuation of the fluid velocity at very high Reynolds number [15]. As suggested by recent studies [7,16–19], multifractality is likely to be a pertinent concept to account for the fluctuation in complex systems. We use this concept here for analyzing the fluctuation of time series of light scattering intensity from an aging colloidal suspension of Laponite clay with a concentration of 3.2 wt % measured at a certain aging time. Here we only focus on the general stochastic behavior of light intensity fluctuations. The evolution of the statistical properties on aging time will be presented in a following work. The typical intensity fluctuation is plotted in Fig. 1. The details of experimental setup are given in [2].

B. Multiplicative cascade model

Multifractality is a notion that is often related to an underlying multiplicative cascading process. In the context of deterministic function the situation is rather clear since the analyticity of the ξ_q spectrum is deeply connected to the self-similarity properties of the function [20,21]. A process $x(t)$ is called self-similar with exponent H if $\forall\lambda > 0, \lambda^{-H}\Delta x(\lambda\tau)$ is the same process as $\Delta x(\tau)$. Define $P_\tau(\Delta x)$ to be the PDF of $\Delta x(\tau)$. The process $x(t)$ is self-similar with an exponent H if its PDF satisfies [22,23]

$$P_\tau(\Delta x) = \lambda^H P_{\lambda\tau}(\lambda^H \Delta x). \tag{5}$$

Then, the moments at scale τ and $T=\lambda\tau$ are related by

$$M(q,\tau) = N_q\left(\frac{\tau}{T}\right)^{qH} \tag{6}$$

with $N_q = M(q,T)$. Therefore one has a monofractal process with $\xi_q = qH$. In order to account for multifractality, one has to generalize this classical definition of self-similarity. This can be done by introducing a weaker notion, as originally proposed in the field of fully developed turbulence by Castaing *et al.* [24]. According to Castaing's definition of self-similarity, a process is self-similar if the increment's probability density functions at scales τ and $T=\lambda_0\tau$ are related by the relationship [24]

$$P_\tau(\Delta x) = \int G_{\tau,T}(u) e^{-u} P_T(e^{-u}\Delta x) du, \tag{7}$$

where the *self-similarity kernel* $G_{\tau,T}$ depends only on τ/T. We note that this definition generalizes the Eq. (6) that corresponds to the "trivial" case $G_{\tau,T}(u) = \delta(u - H\ln(\tau/T))$. This equation basically states that the probability density function P_τ can be obtained through a "geometrical convolution" between the kernel $G_{\tau,T}$ and P_T. A simple argument shows that the logarithm of the Fourier transform of the kernel $G_{\tau,T}$ can be written as $F_{\tau,T}(k) = \ln G_{\tau,T}(k) = F(k)\ln(\tau/T)$. Thus, from Eq. (7), one can easily show that the q order absolute moments at scales τ and T are related by [24,25]

$$M(q,\tau) = G_{\tau,T}(-iq) M(q,T) = M(q,T)\left(\frac{\tau}{T}\right)^{F(-iq)}, \tag{8}$$

so that $N_q = M(q,T)$ and $\xi_q = F(-iq)$. A nonlinear ξ_q spectrum implies that F is nonlinear and thus that G is different from a Dirac delta function. For example, the simplest nonlinear case is the so-called logarithmic-normal model that corresponds to a parabolic ξ_q function and thus to a function G that is Gaussian [24].

Let us now make a link between the multiplicative cascade model and Castaing equation. This can be easily done if one consider discrete scales $t_n = 2^{-n}T$. Let us suppose the local variation of the process $\Delta_{t_n}x$ at scale t_n is obtained from the variation at scale T as

$$\Delta_{t_n}x(t) = \left(\prod_{i=1}^{n} W_i\right)\Delta_T x(t), \tag{9}$$

where W_i are random positive factors. This is the cascade paradigm. Realizations of such processes can be constructed using orthonormal wavelet bases as discussed in [26]. Using Eq. (9) one can prove immediately the Castaing equation, i.e., Eq. (7).

III. SIMPLE SOLVABLE MULTIFRACTAL MODEL

In this section our aim is to build a simple solvable model based on multiplicative cascade model and employ it to fit the experimental data of light scattering fluctuations. Multiplicative cascading processes [26] consist of writing Eq. (9), starting from some "coarse" scale $\tau=T$, and then iterating it toward finer scales using an arbitrary fixed scale ratio, e.g., $\lambda=1/2$. Such processes can be constructed rigorously using, for instance, orthonormal wavelet bases [26]. However, these processes have fundamental drawbacks: they do not lead to stationary increments and they do not have continuous dilation invariance properties. Indeed, they involve a particular arbitrary scale ratio, i.e., Eq. (3) holds only for the discrete scales $t_n=\lambda_0^n T$. We first build a discretized version $x_{\Delta t}(t=k\Delta t)$ of this process. Let us note that the limit process $x(t)=\lim_{\Delta t\to 0} x_{\Delta t}$ is well defined [4]. It is shown that different quantities (q order moments, increment correlation, etc.) converge when $\Delta t\to 0$ [4]. We rewrite Eq. (6) at the smallest scale so $\Delta x_{\Delta t}(k\Delta t)=\epsilon_{\Delta t}(k)W_{\Delta t}(k)$, where $\epsilon_{\Delta t}$ and $W_{\Delta t}(k)=e^{\omega_{\Delta t}(k)}$ are Gaussian and logarithmic-normal variables, respectively, i.e.,

$$x_{\Delta t}(t)=\sum_{k=1}^{t/\Delta t}\epsilon_{\Delta t}(k)e^{\omega_{\Delta t}(k)}, \tag{10}$$

where $\omega_{\Delta t}(k)$ is the logarithm of the stochastic variance. More specifically, we will choose $\epsilon_{\Delta t}$ to be a Gaussian white noise independent of ω and of variance $\sigma^2\Delta t$. This choice for the process $\omega_{\Delta t}$ is introduced in [4] and dictated by the cascade picture. It corresponds to a Gaussian stationary process where its covariance can be written as

$$\langle\omega_{\Delta t}(k_1)\omega_{\Delta t}(k_2)\rangle=\lambda_0^2\ln\rho_{\Delta t}(|k_1-k_2|). \tag{11}$$

Here, $\rho_{\Delta t}$ is chosen in order to mimic the correlation structure observed in cascade models with an integral time scale T,

$$\rho_{\Delta t}(k)=\begin{cases}\dfrac{T}{|k|+1} & \text{for} \quad |k|\le T/\Delta t-1\\ 1 & \text{otherwise}.\end{cases} \tag{12}$$

$\omega_{\Delta t}$'s are correlated up to the time scale T and their variance $\lambda_0^2\ln(T/\Delta t)$ goes to infinity when Δt approaches zero. Direct computation shows that we need to choose the following relations [4]:

$$\langle\omega_{\Delta t}(k)\rangle=-r\,\mathrm{Var}(\omega_{\Delta t}(k))=-r\lambda_0^2\ln(T/\Delta t) \tag{13}$$

with $r=1$ and $\mathrm{Var}(x(t))=\sigma^2 t$. This computation builds "MRW" process $x(t)$ [4]. We can build MRWs with correlated increments by just replacing the white noise $\epsilon_{\Delta t}$ with a fractional Gaussian noise

$$\epsilon_{\Delta t}^{(H)}=B_H([k+1]\Delta t)-B_H(k\Delta t), \tag{14}$$

where $B_H(t)$ is a fractional Brownian motion with the so-called Hurst exponent H and of variance $\sigma^2 t^{2H}$ [choosing $r=1/2$ in Eq. (13)].

The qth moment of light scattering-intensity time series can be computed directly from experimental data and the cascade model. In cascade model, we construct the increments of the model as $x_{\Delta t}(t+\tau)-x_{\Delta t}(t)$, which does not depend on t and is the same law as $x_{\Delta t}(\tau)$. It has been proven that the moments of $x(\tau)\equiv x_{\Delta t\to 0^+}(\tau)$ can be expressed as [4]

$$\langle x(\tau)^{2p}\rangle=\frac{\sigma^{2p}(2p)!}{2^p p!}\int_0^l (u_1)^{2H-1}du_1\cdots\int_0^l (u_p)^{2H-1}du_p \times\prod_{i<j}\rho(u_i-u_j)^{\lambda_0^2}. \tag{15}$$

Using this expression, Eq. (6) becomes

$$M(2p,\tau)=K_{2p}\left(\frac{\tau}{T}\right)^{pH-2p(p-1)\lambda_0^2}, \tag{16}$$

where K_{2p} is given by

$$K_{2p}=T^p\sigma^{2p}(2p-1)\,!\,!\int_0^1 u_1^{2H-1}du_1\cdots\int_0^1 u_p^{2H-1}du_p \times\prod_{i<j}|u_i-u_j|^{\lambda_0^2}. \tag{17}$$

It is worth to note that K_{2p} is nothing else but the moment of order $2p$ of the random variable $x(t)$. From the above expression, we thus obtain

$$\xi_{2p}=2pH-2p(2p-1)\frac{\lambda_0^2}{2}. \tag{18}$$

The corresponding ξ_q spectrum is thus the parabola

$$\xi_q=qH-q(q-1)\frac{\lambda_0^2}{2}. \tag{19}$$

Let us suppose the MRWs with variance $\mathrm{Var}(x_{\omega\Delta t})=\sigma^2 t$, then the spectrum of the MRW $x(t)$ is [4]

$$\xi_q=\frac{q}{2}-q(q-1)\frac{\lambda_0^2}{2}. \tag{20}$$

To find the relation between ξ_q and λ_0^2, we illustrated the scaling behavior of the moments $M(q,\tau)$ and corresponding extended self-similarity exponents in Fig. 2. The upper panel of Fig. 2 indicates the log-log plot of structure function, Eq. (3) versus τ. In the lower left panel, we show the scaling exponent ξ_q as a function of q. Filled circle symbols have been directly computed by the experimental light intensity time series. The solid line in this panel corresponds to a monofractal process with the same Hurst exponent as our data set. The value of the Hurst exponent of underlying data set has been determined by detrended fluctuation analysis and is equal to $H=0.92\pm0.02$ [27]. The long-dashed curve corresponds to Eq. (19) with the value of $\lambda_0=0.077\pm0.054$, which is determined by multiplicative cascade model [28].

The lower right panel shows ζ_q versus q for light scattering data set (filled circle symbols) and a Gaussian process (solid line). It appears that light scattering-intensity data represented by $I(t)$ are a multifractal process with continuous dilation invariance properties. As shown in the lower panel of Fig. 2, the fitting formula for ξ_q (solid line) derived by multifractal analysis is in agreement with experimental data in an acceptable confidence level. We note that due to the smallness of λ_0, deviation from monofractality will appear

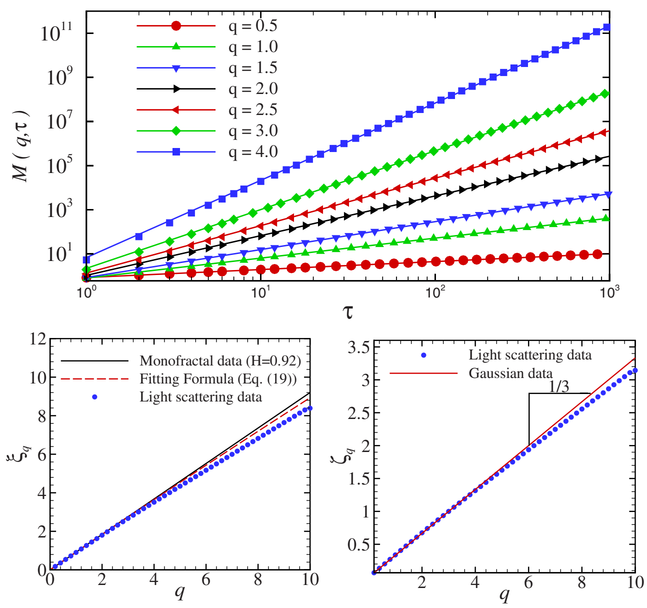

FIG. 2. (Color online) Upper panel shows the scaling behavior of structure function $M(q,\tau)$ for various values of q. Lower left panel corresponds to the scaling exponent ξ_q of light scattering-intensity data (filled circle symbol) as a function of moment order q. In this panel, solid lines are given for a monofractal process with Hurst exponent equal to $H=0.92$. Long-dashed line is given by Eq. (19). Lower right panel indicates ζ_q versus q (the exponents derived using the extended self-similarity method). Solid line shows the exponents for a Gaussian process, $\zeta_q=q/3$.

for large q's. We confirm this observation with direct estimation of PDF of increments and show that they have deviation from Gaussian distribution (see below).

In the cascade picture, the total number of data points is 2^m $(i=1,\ldots,2m)$, where m is the total number of cascade steps. Also, the time series $\Delta x(\tau)$ can be described by the form of logarithmic-normal cascade-type multiplicative process which has been introduced in Ref. [29] and with the probability density function given by Eq. (7), with $\lambda^2=m\lambda_0^2$. As illustrated in Fig. 3, Eq. (7) accounts very well for the evolution of the probability density function of the increments. This shows that the smaller the scale τ, the fatter the tails of the probability density function of $\Delta x(t)$. These findings are also in agreement with the previous results [3].

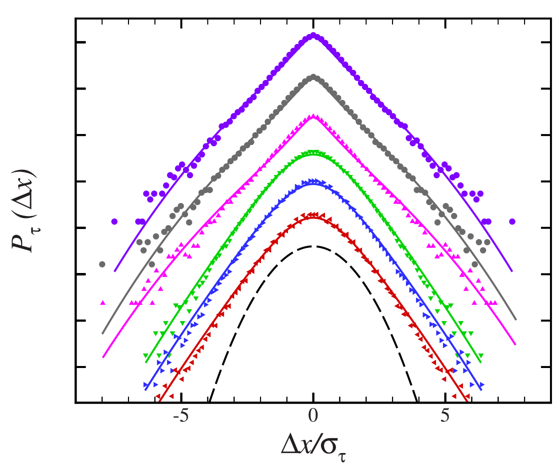

FIG. 3. (Color online) Continuous deformation of the intensity increments' PDFs across scales (from bottom to top) $\tau=2000,1000,500,250,150,25$ (406 μs). Solid lines for each τ's are given by Eq. (7) and symbols are directly computed by data set. Long-dashed curve corresponds to the Gaussian probability density function. Curves are shifted in vertical direction for clarity of presentation.

IV. MARKOVIAN NATURE OF DATA SET

In Ref. [12], it has been demonstrated that the mathematics of stochastic processes is a useful tool for empirical investigations of the time-scale dependence of the PDF $p(\Delta x,\tau)$ of a given time series, namely, the intensity fluctuations on time scale τ. It was shown how the equations governing the underlying stochastic process can be extracted directly from the empirical data, provided that several experimental data obey the Markovian process. In particular, it is possible to derive a partial differential equation, the Fokker-Planck equation, which describes the evolution of the probability density function $p(\Delta x,\tau)$ in the scale variable τ. Hence, the mathematics of Markovian processes yields a complete description of the stochastic process underlying the evolution of the PDFs from Gaussian distributions at large scales τ to the leptokurtic (fat) PDFs at small scales. Here, we show how the existence of a Markovian process can be checked empirically and how the Fokker-Planck equation can be calculated directly from the data set.

In what follows we summarize the notions and theorems which will be important for the statistical analysis of light intensity fluctuations measured in our experiment by Markovian method. For further details on the Markovian processes we refer the reader to Refs. [12,30]. Fundamental quantities related to the Markovian processes are conditional probability density functions. Given the joint probability density $p(x_2,t_2;x_1,t_1)$ for finding the intensity $x_2\equiv x(t_2)$ at time scale t_2 and x_1 at time scale t_1 with $t_1<t_2$, the conditional PDF $p(x_2,t_2|x_1,t_1)$ is defined as

$$p(x_2,t_2|x_1,t_1)=\frac{p(x_2,t_2;x_1,t_1)}{p(x_1,t_1)}, \tag{21}$$

where $p(x_2,t_2|x_1,t_1)$ denotes the conditional probability density for the intensity x_2 at time scale t_2 given x_1 at time scale t_1.

Higher-order conditional probability densities can be defined in an analogous way as follows:

$$p(x_N,t_N|x_{N-1},t_{N-1};\ldots;x_1,t_1)=\frac{p(x_N,t_N;x_{N-1},t_{N-1};\ldots;x_1,t_1)}{p(x_{N-1},t_{N-1};\ldots;x_1,t_1)}. \tag{22}$$

The smaller scales t_i are nested inside the larger scales t_i+1 (with the common reference point t).

The stochastic process in $\tau=t_N-t_{N-1}$ is a Markovian process if the conditional probability densities fulfill the following relations:

$$p(x_N,t_N|x_{N-1},t_{N-1};\ldots;x_1,t_1)=p(x_N,t_N|x_{N-1},t_{N-1}) \quad (23)$$

with $t_1<t_2<t_3<\cdots<t_N$. As a consequence of Eq. (22), each N-point probability density $p(x_N,t_N;x_{N-1},t_{N-1};\ldots;x_1,t_1)$ can be determined as a product of conditional probability density functions,

$$p(x_N,t_N;x_{N-1},t_{N-1};\ldots;x_1,t_1) \\ =p(x_N,t_N|x_{N-1},t_{N-1})\cdots p(x_2,t_2|x_1,t_1)p(x_1,t_1). \quad (24)$$

As Eqs. (23) and (24) indicate, knowledge of $p(x,t|x_0,t_0)$ (for arbitrary scales t and t_0 with $t_0<t$) is sufficient to generate the entire statistics of the underlying fluctuations encoded in the N-point probability density, namely, $p(x_N,t_N;x_{N-1},t_{N-1};\ldots;x_1,t_1)$.

To investigate whether the underlying signal (or its increments) is a Markovian process, one should test Eq. (24). But in practice, it is beyond the current computational capability for large values of N. For $N=3$ (three points or events), however, the condition will be

$$p(x_3,t_3|x_2,t_2;x_1,t_1)=p(x_3,t_3|x_2,t_2), \quad (25)$$

which should hold for any value of t_2 in the interval $t_1<t_2<t_3$. A process is then Markovian if Eq. (25) is satisfied for a certain time separation t_3-t_2, in which case, we define the Markovian time scale as $t_{Markov}=t_3-t_2$. For simplicity, we let $t_2-t_1=t_3-t_2$. Thus, to compute t_{Markov}, we use a fundamental theory of probability according to which we write any three-point PDF in terms of the conditional probability functions as

$$p(x_3,t_3;x_2,t_2;x_1,t_1)=p(x_3,t_3|x_2,t_2;x_1,t_1)p(x_2,t_2;x_1,t_1). \quad (26)$$

Using the properties of Markovian processes, Eq. (26) can be written as follows:

$$p_{\mathrm{Mar}}(x_3,t_3;x_2,t_2;x_1,t_1)=p(x_3,t_3|x_2,t_2)p(x_2,t_2;x_1,t_1). \quad (27)$$

In order to check the condition for the data being a Markovian process, we must compute the three-point joint PDF through Eq. (26) and compare the result with Eq. (27). One can write Eq. (27) as an integral equation, which is well known as the CK equation

$$p(x_3,t_3|x_1,t_1)=\int dx_2\, p(x_3,t_3|x_2,t_2)p(x_2,t_2|x_1,t_1). \quad (28)$$

The simplest way to determine t_{Markov} is using the well-known Chapman-Kolmogorov equation, which can be written as $K(t_3-t_1)=|p(x_3,t_3|x_1,t_1)-\int dx_2\, p(x_3,t_3|x_2,t_2)p(x_2,t_2|x_1,t_1)|$, for given x_1 and x_3, in terms of, for example, t_3-t_1 and considering the possible errors in estimating K. It is obvious that, for the value of $t_{Markov}=t_3-t_2=t_2-t_1$, the quantity K vanishes or at least is nearly zero (achieves a minimum) [12,31].

Up to now we only showed how one can estimate the Markovian time scale for data set over which time series behaves as a Markovian process. In the next section we will turn to deriving master and stochastic equations governing the evolution of probability density function of intensity fluctuations.

V. LANGEVIAN EQUATION: EVOLUTION EQUATION TO DESCRIBE THE INTENSITY OF LIGHT SCATTERING FLUCTUATIONS

The Markovian nature of the intensity of light scattering fluctuations enables us to derive a master equation, a Fokker-Planck equation, for the evolution of the PDF $p(x,t)$ in terms of time t. The Chapman-Kolmogorov equation, formulated in differential form, yields the following Kramers-Moyal expansion [30]:

$$\frac{\partial}{\partial t}p(x,t)=\sum_{n=1}^{\infty}\left(-\frac{\partial}{\partial x}\right)^n[D_n(x,t)p(x,t)], \quad (29)$$

where $D_n(x,t)$ are called the Kramers-Moyal coefficients. For Markovian processes the conditional probability density fulfills a master equation which can be put into the form of a Kramers-Moyal expansion as follows:

$$\frac{\partial}{\partial t}p(x,t|x_0,t_0)=\sum_{n=1}^{\infty}\left(-\frac{\partial}{\partial x}\right)^n[D_n(x,t)p(x,t|x_0,t_0)]. \quad (30)$$

The Kramers-Moyal coefficients $D_n(x,t)$ are defined as

$$D_n(x,t)=\lim_{\Delta t\to 0}M_n(x,t,\Delta t), \quad (31)$$

where

$$M_n(x,t,\Delta t)=\frac{1}{n!\,\Delta t}\int_{-\infty}^{\infty}(x'-x)^n p(x',t-\Delta t|x,t)dx'. \quad (32)$$

For a general stochastic process, all Kramers-Moyal coefficients are different from zero. According to the Pawula theorem, however, the Kramers-Moyal expansion stops after the second term, provided that the fourth-order coefficient $D_4(x,t)$ vanishes. In that case, the Kramers-Moyal expansion reduces to a Fokker-Planck equation (also known as the backward or second Kolmogorov equation) [30],

$$\frac{\partial}{\partial t}p(x,t|x_0,t_0)=\left\{-\frac{\partial}{\partial x}D_1(x,t)+\frac{\partial^2}{\partial x^2}D_2(x,t)\right\}p(x,t|x_0,t_0). \quad (33)$$

The coefficients D_1 and D_2 are known as drift and diffusion coefficients, respectively. We note that the probability density $p(x,t)$ has to obey the same equation with a different initial condition [12]. The Fokker-Planck equation describes the probability density function of a stochastic process generated by the Langevin equation (we use the Ito definition) [30],

$$\frac{\partial}{\partial t}x(t)=D_1(x,t)+\sqrt{D_2(x,t)}f(t), \quad (34)$$

where $f(t)$ is a Langevin force, i.e., δ-correlated white noise with a Gaussian distribution $\langle f(t)f(t')\rangle=2\delta(t-t')$.

To check the multifractal nature of time series, we check the Markovian nature of the increments, which is defined by $\Delta x(\tau)=x(t+\tau)-x(t)$, for intensity fluctuations. According to the mentioned procedure, we determine the Markovian time scales for the increments and calculate the Kramers-Moyal coefficients. The Fokker-Planck equation for probability function of the increment is given by [12]

$$-\tau\frac{\partial}{\partial\tau}p(\Delta x,\tau)=\left\{-\frac{\partial}{\partial\Delta x}D_1(\Delta x,\tau)+\frac{\partial^2}{\partial\Delta x^2}D_2(x,\tau)\right\}p(\Delta x,\tau), \quad (35)$$

where the negative sign of the left-hand side of Eq. (35) is due to the direction of the cascade toward smaller time scales τ. The corresponding Langevin equation can be read as

$$-\tau\frac{\partial}{\partial\tau}\Delta x(\tau)=D_1(\Delta x,\tau)+\sqrt{D_2(\Delta x,\tau)}f(\tau), \quad (36)$$

where $f(\tau)$ is the same as random function in Eq. (34). Drift and diffusion coefficients of increment are formulated as [32–34]

$$D_1(\Delta x,\tau)\simeq -H\Delta x,$$

$$D_2(\Delta x,\tau)\simeq b\Delta x^2. \quad (37)$$

Using Eq. (35) we obtain the evolution of structure function as follows:

$$-\tau\frac{\partial}{\partial\tau}\langle|\Delta x(\tau)|^q\rangle=q\langle|\Delta x(\tau)|^{q-1}D_1(\Delta x,\tau)\rangle + q(q-1)\langle|\Delta x(\tau)|^{q-2}D_2(\Delta x,\tau)\rangle. \quad (38)$$

Substituting Eq. (37) into Eq. (38) we find

$$-\tau\frac{\partial}{\partial\tau}\langle|\Delta x(\tau)|^q\rangle=[qH+bq(q-1)]\langle|\Delta x(\tau)|^q\rangle. \quad (39)$$

The above equation implies scaling behavior for the structure function, so that

$$M(q,\tau)\equiv\langle|\Delta x(\tau)|^q\rangle=\langle|x(t+\tau)-x(t)|^q\rangle\sim\tau^{\xi_q}. \quad (40)$$

According to Eqs. (39) and (40), the corresponding scaling exponent can be read as

$$\xi_q=Hq-bq(q-1). \quad (41)$$

As mentioned before, for monofractal and multifractal processes the exponent ξ_q has linear and nonlinear behaviors with respect to q, respectively. We must point out that the exponent H is nothing except the Hurst exponent of time series [35].

Satisfying the Chapman-Kolmogorov (CK) equation confirms that the signal of light scattering-intensity fluctuations is Markovian process. The coefficients $D_1(\Delta x,\tau)$ and $D_2(\Delta x,\tau)$ are estimated as

$$D_1(\Delta x,\tau)=-(0.86\pm0.19)\Delta x, \quad (42)$$

$$D_2(\Delta x,\tau)=-(0.034\pm0.027)\Delta x^2. \quad (43)$$

Consequently, using Eqs. (41)–(43), the scaling exponents will be

$$\xi_q=(0.86\pm0.19)q-(0.034\pm0.027)q(q-1). \quad (44)$$

The scaling exponents ξ_q (for small values of q) derived by the Markovian approach [Eq. (44)] are in agreement with those given by Eq. (19) and as well as with direct computational from time series.

VI. VOLATILITY AND MAGNITUDE CORRELATION FUNCTIONS

As recalled in the Introduction, the scattered light intensity time series are correlated and their amplitude ("local volatilities") possesses power-law correlations. Let us show that our model satisfies these two properties. The increment correlation function is defined by

$$C_q(l,t,\tau)=\langle|x_{\Delta t}(l+\tau)-x_{\Delta t}(l)|^q|x_{\Delta t}(t+\tau)-x_{\Delta t}(t)|^q\rangle, \quad (45)$$

where, for $(\forall|l-t|>\tau)$, the correlation is zero. Let us study the correlation function of the squared increments. Since the increments are stationary, we can choose $t=0$. Thus, we need to compute, in the limit $\Delta t\to 0$, the following correlation function that corresponds to a lag l between increments of size τ:

$$C_q(l,\tau)=\langle|x_{\Delta t}(l+\tau)-x_{\Delta t}(l)|^q|x_{\Delta t}(\tau)-x_{\Delta t}(0)|^q\rangle. \quad (46)$$

From the results of Ref. [4] and for $0\le l<T, 0\le\tau+l<T$, we find

$$C_q(l,\tau)=\sigma^4\int_l^{l+\tau}du\int_0^{\tau}dv\rho(u-v)^{q^2\lambda_0^2}. \quad (47)$$

A direct computation shows that

$$\int_l^{l+\tau}du\int_0^{\tau}dv\rho(u-v)^{q^2\lambda_0^2}=\frac{1}{(1-q^2\lambda_0^2)(2-q^2\lambda_0^2)}[(l+\tau)^{2-q^2\lambda_0^2} + (l-\tau)^{2-q^2\lambda_0^2}-2l^{2-q^2\lambda_0^2}], \quad (48)$$

and consequently

$$C_q(l,\tau)=\mathcal{A}[(l+\tau)^{2-q^2\lambda_0^2}+(l-\tau)^{2-q^2\lambda_0^2}-2l^{2-q^2\lambda_0^2}], \quad (49)$$

where $\mathcal{A}=\sigma^4 t^{q^2\lambda_0^2}/[(1-q^2\lambda_0^2)(2-q^2\lambda_0^2)]$.

For $0\le l\ll\tau$, one gets

$$C_q(l,\tau)\sim\mathcal{A}^2\left(\frac{l}{T}\right)^{2\xi_q}\left(\frac{\tau}{T}\right)^{-q^2\lambda_0^2}. \quad (50)$$

The correlation function for fixed values of l and T behaves as

$$C_q(l,\tau)\sim\tau^{\nu_q}, \quad (51)$$

where the exponent is given by [26,36]

$$\nu_q=-q^2\lambda_0^2. \quad (52)$$

In the upper panel of Fig. 4, we plotted the correlation function of light intensity fluctuations. The lower panel of Fig. 4 indicates the exponent of correlation function for small value of τ for various values of q accompanying with

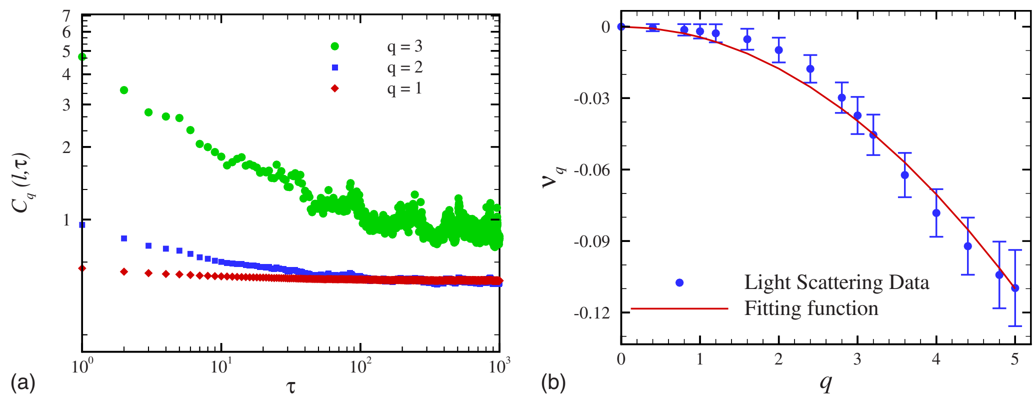

FIG. 4. (Color online) Upper panel corresponds to the correlation function of light intensity increments, $C_q(l,\tau)$ for $l=1$ versus τ for $q=1$ (filled circle), $q=2$ (filled square), and $q=3$ (filled diamond). Lower panel shows the estimation of the power-law exponent $C_q(l,\tau)\sim\tau^{\nu_q}$ derived by direct calculation for small value of τ (filled circle symbol) and theoretical prediction using $\lambda_0=0.077\pm0.054$ and given by Eq. (52).

the theoretical prediction given by Eq. (52). This figure is also another confirmation of the reliability of λ_0 determined by multiplicative cascade model mentioned in Sec. III.

VII. CONCLUSION

In this paper we checked the multifractal nature of the light scattering-intensity time series. We showed how the mathematical framework of cascade modeling and Markovian processes can be applied to develop a successful statistical description of the intensity fluctuations. We characterized the non-Gaussian nature of the light scattering-intensity time series, using a multiplicative model. Also noting to the Markovian nature of fluctuations, we demonstrated that the probability density function of increment fluctuations satisfies a Fokker-Planck equation, which encodes the Markovian property of light intensity fluctuations in a necessary way. We computed the Kramers-Moyal coefficients for the field $I(t+\tau)-I(t)$ and determined their corresponding Langevin equations.

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