Conformal invariance of isoheight lines in a two-dimensional Kardar-Parisi-Zhang surface

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(Received 25 November 2007; revised manuscript received 2 March 2008; published 23 May 2008)

The statistics of isoheight lines in the (2+1)-dimensional Kardar-Parisi-Zhang (KPZ) model is shown to be conformally invariant and equivalent to those of self-avoiding random walks. This leads to a rich variety of exact analytical results for the KPZ dynamics. We present direct evidence that the isoheight lines can be described by the family of conformally invariant curves called Schramm-Loewner evolution (or SLE_κ) with diffusivity $\kappa=8/3$. It is shown that the absence of the nonlinear term in the KPZ equation will change the diffusivity κ from 8/3 to 4, indicating that the isoheight lines of the Edwards-Wilkinson surface are also conformally invariant and belong to the universality class of domain walls in the $O(2)$ spin model.

DOI: 10.1103/PhysRevE.77.051607 PACS number(s): 81.10.−h, 68.35.Ct, 61.43.Hv, 68.37.−d

Recently, it was shown that the statistics of the zero-vorticity lines in an inverse cascade of two-dimensional (2D) Navier-Stokes turbulence is conformally invariant and belongs to the percolation universality class [1]. The same issue has been studied for zero-temperature isolines in the inverse cascade of surface quasigeostrophic turbulence [2], domain walls of spin glasses [3], and nodal lines of random wave functions [4]. Moreover, it has been shown recently that the statistics of the isoheight lines on the experimental WO_3-grown surface is the same as domain walls statistics in the critical Ising model as well as those of the ballistic deposition (BD) model [5].

Evidence of conformal invariance in the geometrical features of such complex nonlinear systems have been provided, in the continuum limit, by stochastic Schramm-Loewner evolution—i.e., SLE_κ, where κ is the diffusivity [6,7]. Schramm and Sheffield showed that the contour lines in a two-dimensional discrete Gaussian free field are statistically equivalent to SLE_4 [8]. Moreover, it is shown that the restriction property only applies in the case for $\kappa=8/3$ [9]. Since self-avoiding random walks (SAWs) satisfy the restriction property, it is conjectured in the scaling limit to fall in the SLE class with $\kappa=8/3$ [10]. The scaling limit of SAWs in the half-plane has been proven to exist [11], but there is no general proof of its existence.

In this paper we investigate numerically the isoheight lines of the (2+1)-dimensional Kardar-Parisi-Zhang (KPZ) model [12] and study their possible conformal invariance. It is shown that the KPZ isoheight lines are equivalent to self-avoiding walks and that the isoheight lines in the 2D KPZ surface are $\mathrm{SLE}_{8/3}$ curves. For the Edwards-Wilkinson (EW) interface (the KPZ model without the nonlinear term) the isoheight lines fall in the universality class of the interfaces in the $O(2)$ model and can be described by SLE_4.

The KPZ equation is given by

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}|\nabla h|^2 + \eta(\mathbf{x},t). \qquad (1)$$

The first term on the right-hand side describes relaxation of the interface caused by a surface tension ν, and the nonlinear term is due to the lateral growth. The noise η is uncorrelated Gaussian white noise in both space and time with zero average—i.e., $\langle\eta(\mathbf{x},t)\rangle=0$ and $\langle\eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle =2D\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t')$. The KPZ equation is invariant under translations both along the growth direction and perpendicular to it, as well as time translation and rotation [13]. Rescaling the variables $h=\tilde{h}\sqrt{2D/\nu}$ and $t=\tilde{t}/\nu$ changes Eq. (1) to $\partial\tilde{h}(\mathbf{x},\tilde{t})/\partial\tilde{t}=\nabla^2\tilde{h}+\sqrt{\epsilon}|\nabla\tilde{h}|^2+\tilde{\eta}(\mathbf{x},\tilde{t})$, where $\epsilon=\lambda^2 D/2\nu^3$ and $\langle\tilde{\eta}(\mathbf{x},\tilde{t})\tilde{\eta}(\mathbf{x}',\tilde{t}')\rangle=\delta^d(\mathbf{x}-\mathbf{x}')\delta(\tilde{t}-\tilde{t}')$. In the following, we work with the single parameter ϵ and drop all the tildes for simplicity.

We have studied the rescaled KPZ equation on a square lattice with periodic boundary conditions. The numerical integration was done using the Runge-Kutta-Fehlberg scheme of orders $O(4)$ and $O(5)$ [14]. This scheme controls automatically the integration time step δt, such that the resulting height error δh [which is estimated by comparing the results obtained from the $O(4)$ and $O(5)$ integrations] can be ignored at each time step. We took the error to be less than 0.1, and we checked that smaller values of δh do not improve the precision of the computed quantities. The noise η was generated by the Box-Muller method. To avoid the instabilities that may appear during the growth, we used the algorithm introduced in [15], where the term $(1-c^{-1}e^{-cf})$ is used instead of the nonlinear term in Eq. (1)—i.e., $f=|\nabla h|^2$. Since $f\ll w_L^2(\infty)$, where $w_L(t)$ is the interface width of the system with size L at time t, by keeping $c\ll w_L^{-2}(\infty)$, one can control the possible numerical divergences that may appear during the integration. Clearly for very small c this term converges to f.

We have checked that the growth exponents for (1+1) dimensions are obtained correctly (both roughness and growth exponents $\alpha=1/2$ and $\beta=1/3$, respectively), and the (2+1)-dimensional results are given in Fig. 1, which are in good agreement with previous studies [16].

We now consider the saturated 2D KPZ surface and set its mean height to zero and attribute the same sign to the points that have positive or negative heights. The same-sign regions (clusters) and their boundaries (loops) were identified by the Hoshen-Kopelman algorithm (Fig. 2).

To investigate the scaling behavior of such loop ensembles in the 2D KPZ interface, a set of scaling exponents associated with cluster and loop statistics were computed and

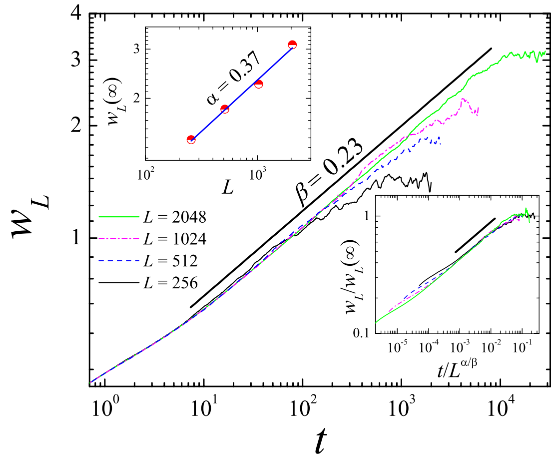

FIG. 1. (Color online) Main frame: Interface width $w_L(t)$ vs time t of the KPZ equation in (2+1) dimensions and for $\epsilon=10$, for different square lattice sizes L. The slope of the straight line yields the growth exponent $\beta=0.23\pm0.01$. Upper left inset: Saturation width $w_L(\infty)$ for systems of different sizes L. The slope of the solid line fit yields the roughness exponent $\alpha=0.37\pm0.01$. Lower right inset: Rescaled w_L vs rescaled t.

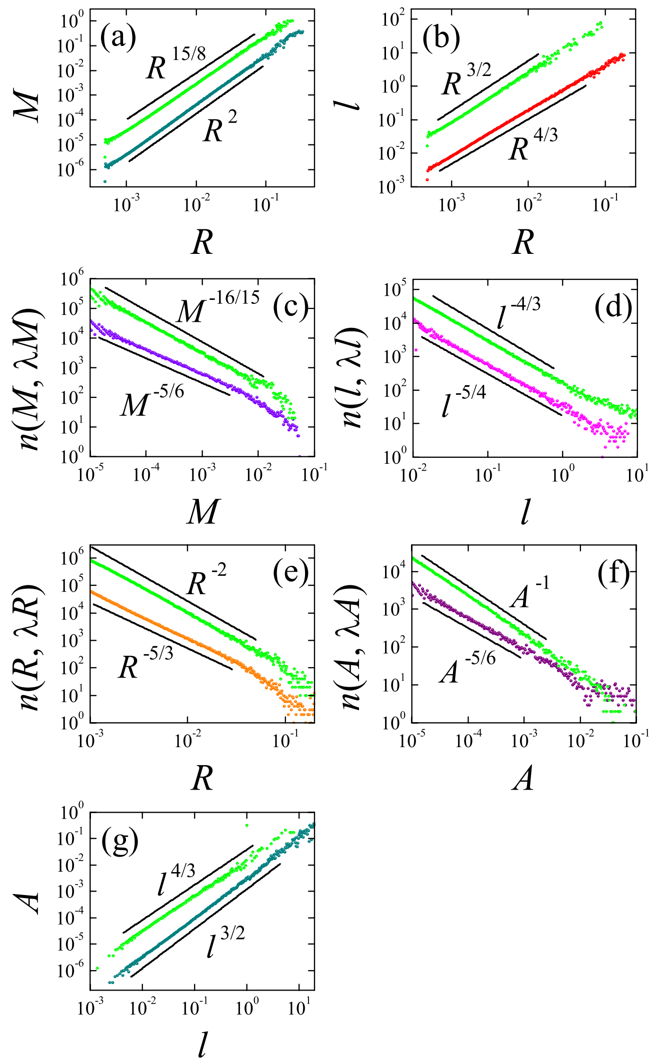

FIG. 3. (Color online) Cluster and loop statistics for the iso-height lines of the 2D KPZ and EW surfaces are shown by the lower (different colors) and upper (green) graphs, respectively. The results for the EW have been shifted by a constant of 1 in order to distinguish them from the 2D KPZ results. (a) The average area M vs the radius of gyration R. (b) The length of a loop l vs the radius of gyration R. (c) Number of clusters of area between M and λM. (d) Number of loops of length between l and λl. (e) Number of loops of radius of gyration between R and λR. (f) Number of loops of area between A and λA. (g) The average area of loops A vs the length l. In all figures $\lambda \simeq 1.05$. Solid lines show the corresponding analytical results for the SAWs (bottom) and $O(2)$ model (top). The error bars are almost the same size as the symbols in the scaling regions and have not been drawn. The change in the exponents relative to the SAWs in (c), (d), (e), and (f) is due to the roughness exponent [17].

checked, and are shown in Fig. 3 [17–19]. The estimated exponents are in good agreement with the corresponding analytical results for SAWs in the scaling limit. The fractal dimension D_f of contour lines obtained from the scaling relation between their length l and radius of gyration, R—i.e., $l \sim R^{D_f}$—is given by $D_f=1.33\pm0.01$, which is in agreement with the one obtained by the box-counting method for the largest contour lines, $D_f=1.33\pm0.02$. Comparing with the known fractal dimension of SLE$_\kappa$ curves $D_f=1+\kappa/8$ for $0 \leq \kappa \leq 8$, the contour lines may have a conformally invariant scaling limit according to SLE$_{8/3}$.

The quantity that can confirm the SAW property of the contour lines is the restriction property. Suppose that S is a hull in the upper-half plane $\mathbb{H}$, which is bounded away from the origin, and γ is a simple SLE curve in $\mathbb{H}$, with $\kappa \leq 4$. Let Ψ_S be a unique conformal map of $\mathbb{H}\backslash S$ onto $\mathbb{H}$, such that $\Psi_S(0)=0$, $\Psi_S(\infty)=\infty$, and $\Psi_S'(\infty)=1$. The restriction property states that the distribution of curves conditioned not to hit S is the same as the distribution of curves in the domain $\mathbb{H}\backslash S$. This happens only for $\kappa=8/3$, and it is shown that [9] the probability that a curve does not hit the hull S is

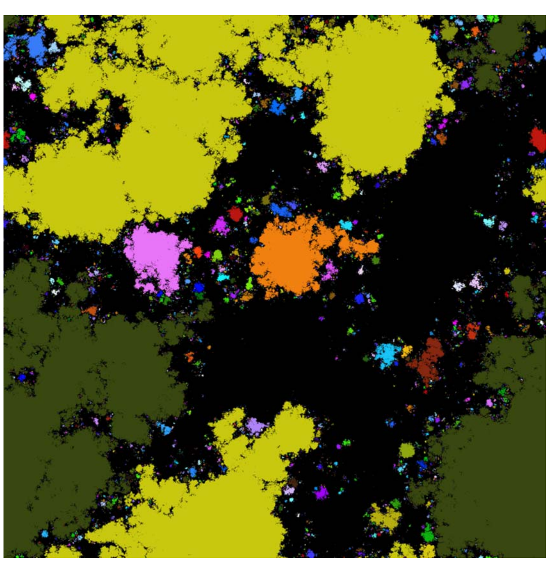

FIG. 2. (Color online) The clusters with positive heights are shown for the 2D KPZ interface with different colors. Negative-height regions are colored with black.

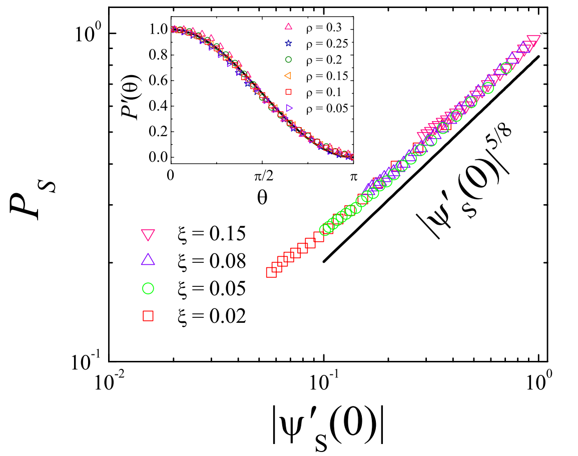

FIG. 4. (Color online) Main frame: The probability that a contour line of the 2D KPZ surface in the upper half-plane does not hit the slits of height $0.05 \leq h \leq 0.5$ placed at ξ from origin on the real axis vs $|\Psi'_S(0)| = |\xi|(\xi^2+h^2)^{-1/2}$. The solid line is the corresponding analytical prediction for $\mathrm{SLE}_{8/3}$ curves. The error bars are almost the same size as the symbols in the scaling region. Inset: The probability that such a contour line passes to the left of a point $z=\rho e^{i\theta}$ for ρ=0.05, 0.1, 0.15, 0.2, 0.25, and 0.3. The solid line shows the prediction of SLE for $\kappa=8/3$.

$$P[\gamma \cap S = \emptyset] = |\Psi'_S(0)|^{5/8}. \quad (2)$$

To examine this property directly for contour lines on the saturated 2D KPZ surface, we proceed as follows. First, we identify all the cluster boundaries (contour lines): for each cluster an explorer walks on the zero-height line as keeping the sites with positive height on the right. Then, we consider an arbitrarily placed straight line for each curve as a real axis and cut the portion of the curve above it. Using this procedure, we obtain an ensemble of contour lines in the half-plane which start at the origin and end on the real axis x_∞. To obtain curves whose size is of order 1, we rescale them by a factor of N^ν, where $\nu=1/D_f$. Second, we consider the hull S as a slit placed at various distances ξ from the origin and various heights h, for which the map Ψ_S is defined by $\Psi_S(z)=\xi+\sqrt{(z-\xi)^2+h^2}$. After mapping the curves by $\varphi(z) = x_\infty z/(x_\infty - z)$,[1] we checked Eq. (2) for the contour lines of the 2D KPZ surface. As shown in Fig. 4, the result is consistent with Eq. (2) and implies a connection between the contour lines and both the SAWs and $\mathrm{SLE}_{8/3}$.

Since the restriction property only holds for $\mathrm{SLE}_{8/3}$ curves, we test the probability that an SLE curve passes to the left of a given point $z=\rho e^{i\theta}$, where θ is the angle between the point and the origin, and ρ is the distance from the origin inside the upper half-plane. Given scale invariance, this probability is independent of ρ and the theory of SLE predicts [20] that

[1]We have used this map to ensure that the curves begin at the origin and end at infinity, the so-called chordal SLE_κ. Also, we have used these chordal curves to measure the left-passage probability. To avoid numerical errors, only the parts of the curves corresponding to capacity $t \leq 0.3$ were used.

$$P'_\kappa(\theta) = \frac{1}{2} + \frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{8-\kappa}{2\kappa}\right)} {}_2F_1\left(\frac{1}{2},\frac{4}{\kappa};\frac{3}{2};-\cot^2(\theta)\right)\cot(\theta). \quad (3)$$

Here, ${}_2F_1$ is the hypergeometric function. The computed $P'_\kappa(\theta)$ for the contour lines of the 2D KPZ surface is also consistent with the analytical form with $\kappa=8/3\pm1/10$ (Fig. 4).

These results strongly suggest that the isoheight lines might be, in the scaling limit, conformally invariant, giving rise to the SLE curves with $\kappa=8/3$. To examine this suggestion directly, we can extract the Loewner driving function ζ of the curves using successive conformal maps. We use the algorithm introduced by Bernard *et al.* [2] based on the approximation that driving function is a piecewise constant function. Each curve is parametrized by a dimensionless parameter t, to be distinguished from time in (1). The procedure is based on applying the map

$$\begin{aligned} G_{t,\zeta} = x_\infty\{ & \eta x_\infty(x_\infty - z) \\ & + [x_\infty^4(z-\eta)^2 + 4t(x_\infty - z)^2(x_\infty-\eta)^2]^{1/2}\}/\{x_\infty^2(x_\infty - z) \\ & + [x_\infty^4(z-\eta)^2 + 4t(x_\infty - z)^2(x_\infty-\eta)^2]^{1/2}\} \end{aligned}$$

on all the points z of the curve approximated by a sequence of $\{z_0=0, z_1, \ldots, z_N=x_\infty\}$ in the complex plane, where $\eta = \varphi^{-1}(\zeta)$ and again $\varphi(z)=x_\infty z/(x_\infty - z)$. At each step, by using the parameters

$$\eta_0 = \varphi^{-1}(\zeta_0) = [\mathrm{Re}\, z_1 x_\infty - (\mathrm{Re}\, z_1)^2 - (\mathrm{Im}\, z_1)^2]/(x_\infty - \mathrm{Re}\, z_1)$$

and

$$t_1 = (\mathrm{Im}\, z_1)^2 x_\infty^4/\{4[(\mathrm{Re}\, z_1 - x_\infty)^2 + (\mathrm{Im}\, z_1)^2]^2\},$$

one point of the curve z_0 is swallowed and the resulting curve is rearranged by one element shorter. This operation yields a set containing N numbers of $\zeta_k(t_k)$ for each curve.

The next step is analyzing the ensemble of the driving functions $\zeta(t)$, which can indicate, within the statistical errors, whether the curves are SLE or not. As shown in Fig. 5, the statistics of the ensemble of $\zeta(t)$ converges to a Gaussian process with variance $\langle \zeta^2(t) \rangle = \kappa t$ and $\kappa=2.6\pm0.1$. This evidence certifies that the isoheight lines of the 2D KPZ interface in the saturation regime appear to be conformally invariant and are described by the $\mathrm{SLE}_{8/3}$. The above results were obtained for $\epsilon=10$; however, the same analysis for growth surfaces with other values of ϵ (which were checked for $\epsilon=5$ and 25) indicates no changes.

In the case of $\epsilon=0$, which corresponds to the EW model, comparing Figs. 6 and 2 indicates more "porosity" in the clusters, which is indicative of changes in the cluster boundaries' shape. As presented in Fig. 3, the cluster and loop statistics in this case are most consistent with those for the $O(2)$ model. These lead to the conclusion that if one assumes that the scaling limit of such contour lines exists, it should belong to the SLE_4 curves.

We also checked this directly as above and found that

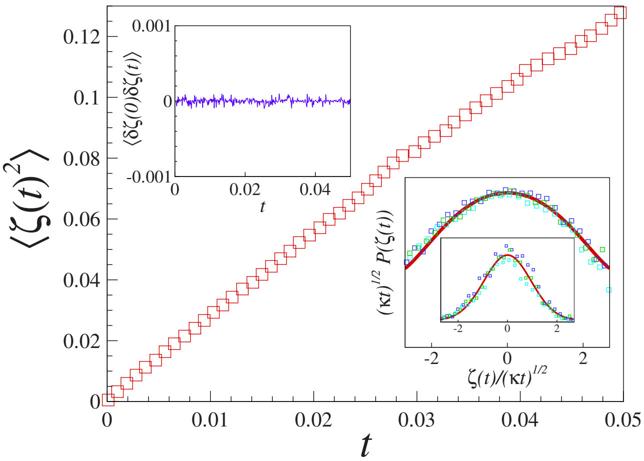

FIG. 5. (Color online) Statistics of the driving function $\zeta(t)$. Main frame: The linear behavior of $\langle\zeta(t)^2\rangle$ with slope $\kappa=8/3\pm1/10$. Lower right inset: The probability density of $\zeta(t)$ rescaled by its variance κ at times $t=0.01,0.015,0.02$ which is Gaussian. Upper left inset: The correlation function of the increments of the driving function $\delta\zeta(t)$. A Kolmogorov-Smirnov (KS) goodness of fit for a normal distribution of the noise $\zeta(t)/\sqrt{\kappa t}$ for $\kappa=8/3$ and $0\leq t\leq 0.05$ yields $K-S=0.017$.

the driving function has Gaussian statistics with variance $\kappa=3.7\pm0.2$.

The height clusters and their boundary statistics, in the manner presented here, can be applied to model experimental grown surfaces by using an ensemble of the samples grown in the saturated regime. Since this analysis is far more accurate, it may be also used to investigate whether a model belongs to a universality class or not. For example, a small difference between the cluster analysis of the BD model [5,21,22] and KPZ equation in two dimensions can be revealed.

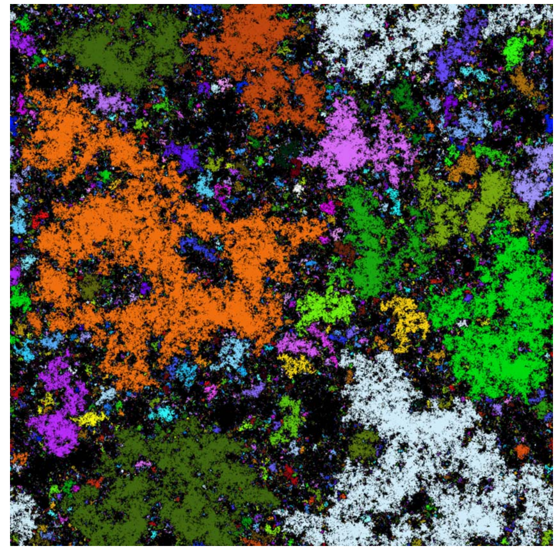

FIG. 6. (Color online) Positive height connected domains of the 2D KPZ interface without the nonlinear term, corresponding to the EW model. Negative height regions are black.

We wish to thank J. Cardy for useful hints on this work. We also thank S. Moghimi-Araghi, M. A. Rajabpour, and M. Sahimi for useful discussions.

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