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A Langevin equation for the rates of currency exchange based on the Markov analysis

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Abstract

We propose a method for analyzing the data for the rates of exchange of various currencies versus the U.S. dollar. The method analyzes the return time series of the data as a Markov process, and develops an effective equation which reconstructs it. We find that the Markov time scale, i.e., the time scale over which the data are Markov-correlated, is one day for the majority of the daily exchange rates that we analyze. We derive an effective Langevin equation to describe the fluctuations in the rates. The equation contains two quantities, $D^{(1)}$ and $D^{(2)}$, representing the drift and diffusion coefficients, respectively. We demonstrate how the two coefficients are estimated directly from the data, without using any assumptions or models for the underlying stochastic time series that represent the daily rates of exchange of various currencies versus the U.S. dollar.

Keywords: Markov process; Rates of exchange of various currencies; Fluctuations; Drift and diffusion coefficients

1. Introduction

The complex statistical properties of economic systems have attracted the attention of many researchers, as a result of which an extensive literature has evolved for modeling such systems and, in particular, fluctuations of financial markets (see, e.g., Ref. [1] and references therein). Traditionally, fluctuations in various price indices were viewed and modeled as random variables. Well-known examples are the ARCH-type (see, for example, Ref. [2]) and stochastic volatility models [3]. Since advances in the computer technology have made it

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possible to have high frequency (large-volume) data, many physicists have joined the field of analyzing financial systems, adapting methods from statistical physics, which has given rise to the field of econophysics. One line of studies within econophysics focusses on the statistical properties of financial time series, such as stock prices, stock market indices, and currency exchange rates. Rather than comparing the predictions of models with the various aspects of empirical data (which is the traditional approach), physicists try to extract information about the stochastic processes that govern financial markets by analyzing the data. Several recent articles review the recent developments [4–6].

At the same time, financial time series are not the only type of data that exhibit stochastic fluctuations. In fact, many natural or man-made phenomena, as well as the morphology of many physical systems, are characterized by a degree of stochasticity. Turbulent flows, seismic recordings, the internet traffic, pressure fluctuations in chemical reactors, and the surface roughness of many materials and rock [7,8] are but a few examples of such phenomena and systems. A long standing problem has been the development of an effective reconstruction method for such phenomena. That is, given a set of data for certain characteristics of a phenomenon, one would like to develop an effective equation that can reproduce the data with an accuracy comparable to the measured data. If such a method can be developed, one may utilize it to (1) *reconstruct* the original process with similar statistical properties, (2) understand the nature and properties of the stochastic process, and (3) *predict* the phenomenon's future behavior, if it is time-dependent, or its behavior over larger (or smaller) length scales, if it is length scale-dependent.

In this paper we use a novel method to address this general problem. The proposed method utilizes a set of data for a phenomenon which contains a degree of stochasticity and constructs a simple equation that governs the phenomenon for which the data have been measured. The method is quite general; it is capable of providing a rational explanation for complex features of the stochastic phenomenon under study; it requires no scaling feature, and it enables us to accomplish the three tasks listed above. As an example, we apply the method to analyze the rates of exchange of various currencies versus the U.S. dollar, which normally fluctuate stochastically and in a complex manner. We develop an effective reconstruction method for the fluctuations in the exchange rates; that is, given a set of data for certain characteristics of the rates, we develop an effective equation that reproduces the data with an accuracy comparable to the original data. The effective equation is then utilized to not only reconstruct the original time series with similar statistical properties, but also to *predict* its future dynamic evolution.

The rest of this paper is organized as follows. Section 2 is devoted to a brief summary of the most important notions on Markov processes, which we use for the reconstruction, and their application to the analysis of the empirical data for the foreign exchange rates. Section 3 contains the main results of our analysis. A physical interpretation of the method and the results is presented in Section 4.

2. Stochastic time series as Markov processes

Complete information about any stochastic process would be available, if one has all the possible n-point joint probability density functions (PDF) $p(r_1, t_1; r_2, t_2; \ldots; r_n, t_n)$, describing the probability of finding simultaneously the return r_1 on the time t_1, r_2 on the time t_2, and so forth up to r_n on the time t_n. The returns are defined by, $r_i = \ln[x(t_{i+1})/x(t_i)]$, where $x(t_i)$ is the datum at time t_i. Without loss of generality, we take $t_1 < t_2 < \ldots < t_n$. The n-point joint PDF is expressed by multiconditional pdf

$$p(r_1, t_1; r_2, t_2; \ldots; r_n, t_n) \\ = p(r_1, t_1|r_2, t_2; \ldots; r_n, t_n)p(r_2, t_2|r_3, t_3; \ldots; r_n, t_n) \ldots p(r_{n-1}, t_{n-1}|r_n, t_n)p(r_n, t_n). \quad (1)$$

Here, $p(r_i, t_i|r_j, t_j)$ denotes the conditional probability of finding the return r_i on the time t_i under the condition that on a larger time t_j the return r_j is found. It is defined with the help of the joint probability $p(r_i, t_i; r_j, t_j)$ by

$$p(r_i, t_i|r_j, t_j) = \frac{p(r_i, t_i; r_j, t_j)}{p(r_j, t_j)}. \quad (2)$$

An important simplification arises if

$$p(r_i, t_i|r_{i+1}, t_{i+1}; \ldots; r_n, t_n) = p(r_i, t_i|r_{i+1}, t_{i+1}). \tag{3}$$

Eq. (3) is the defining feature of a Markov process evolving from r_{i+1} to r_i. Thus, for a Markov process the n-point joint PDF factorizes into n conditional PDF

$$p(r_1, t_1; \ldots; r_n, t_n) = p(r_1, t_1|r_2, t_2) \cdots p(r_{n-1}, t_{n-1}|r_n, t_n)p(r_n, t_n). \tag{4}$$

The Markov property implies that the t-dependence of the returns r can be regarded as a stochastic process evolving in t. It should be noted here that, if Eq. (3) holds, it would be true for a process evolving in time from the large down to the small times, as well as the opposite, from the small to the large times [9]. Eq. (4) also expresses the fundamental property of the conditional probabilities for Markov processes, since they determine any n-point joint PDF and, thus, the complete statistics of the process.

As is well known, a given process with a degree of randomness or stochasticity may have a finite or an infinite Markov time scale [10–12]. The proposed method utilizes a set of data for a phenomenon which contains a degree of stochasticity. We begin by describing the procedure that leads to the development of an effective Langevin equation based on the (stochastic) data set [11,12]. As the first step, we check whether the returns for the data, as defined above, follow a Markov chain and, if so, measure the Markov time scale t_M. To determine the Markov scale t_M for the returns, we note that a complete characterization of the statistical properties of stochastic fluctuations of a quantity r in terms of a parameter t requires the evaluation of the joint PDF $p_n(r_1, t_1; \ldots; r_n, t_n)$ for an arbitrary n, the number of the data points. If the data represent a Markov process, an important simplification can be made, as the n-point joint PDF, p_n, is generated by the product of the conditional probabilities $p(r_{i+1}, t_{i+1}|r_i, t_i)$, for $i = 1, \ldots, n-1$. A necessary condition for a stochastic phenomenon to be a Markov process is that the Chapman–Kolmogorov (CK) equation [13],

$$p(r_2, t_2|r_1, t_1) = \int \mathrm{d}r' \; p(r_2, t_2|r', t')p(r', t'|r_1, t_1) \tag{5}$$

should hold for any value of t' in the interval $t_2 < t' < t_1$. Thus, one should check the validity of the CK equation for different r_1 by comparing the directly evaluated conditional probability distributions $p(r_2, t_2|r_1, t_1)$ with the ones calculated according to right side of Eq. (5). The simplest way to determine t_M for the data is the numerical calculation of the quantity,

$$S = |p(r_2, t_2|r_1, t_1) - \int \mathrm{d}r' p(r_2, t_2|r', t')p(r', t'|r_1, t_1)| \tag{6}$$

for given r_1 and r_2, in terms of, for example, $t' - t_1$ and considering the possible errors in estimating S. Then, $t_M = t' - t_1$ for that value of $t' - t_1$ for which either S vanishes, or reach its minimum [12].

3. Analysis of the rates of exchange of various currencies versus U.S. dollar

We now apply the method to construct the fluctuations in the rates of exchange of various currencies versus the U.S. dollar, by calculating the Markov time scale t_M. For this purpose, we first construct the return series r_i. Then, to determine t_M, the validity of the CK equation for different r_1 is checked by comparing the directly evaluated conditional probability distributions $p(r_2, t_2|r_1, t_1)$ with those calculated according to the right side of Eq. (5). For example, we show in Fig. 1 the computed S values for the hourly closing prices of EURO versus the U.S. dollar, along with their statistical errors, for different time scales. The data were taken from the source, http://finance.yahoo.com/, and are all related to the same period: 31 December, 1979 – 31 December, 1998. Except for the hourly price of EURO over a period of 4 months (30 January – 31 May, 2003), the rest have been recorded for each trading day. In Table 1 we report the Markov time scale t_M for different currencies versus the U.S. dollar.

Since the price index returns can be represented by a Markov process, we can derive an effective stochastic equation that describes the fluctuations of the returns r_n. The CK equation yields an evolution equation for the distribution function $p(r_n)$ across the time step n which, when formulated in differential form, yields a master

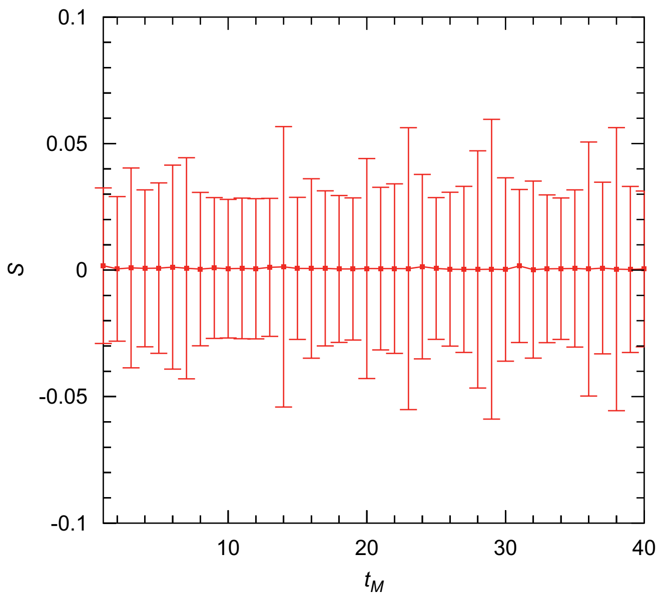

Fig. 1. The S values of the hourly prices of EURO versus the U.S. dollar, along with the statistical errors for the returns.

Table 1
Markov time scale t_M, and the drift and diffusion coefficients for the various exchange rates versus the U.S. dollar

Currency	t_M	$D^{(1)}$	$D^{(2)}$	Period
FRNFR/US	1	$-1.0328r$	$0.0081 + 0.0096r + 0.4133r^2$	12/31/1979–12/31/1998
GERDM/US	1	$-0.0082 - 0.9474r$	$0.0078 + 0.0110r + 0.5942r^2$	12/31/1979–12/31/1998
DTCHG/US	1	$-0.0033 - 1.0071r$	$0.0043 - 0.0007r + 0.7134r^2$	12/31/1979–12/31/1998
SWISF/US	1	$-0.0051 - 1.0142r$	$0.0067 + 0.0087r + 0.4736r^2$	12/31/1979–12/31/1998
JAPYN/US	1	$-0.0033 - 1.0571r$	$0.0029 - 0.0088r + 0.6716r^2$	12/31/1979–12/31/1998
AUSTR/US	1	$-0.0014 - 1.1629r$	$0.0080 + 0.0126r + 0.6385r^2$	12/31/1979–12/31/1998
BRITP/US	1	$-0.0013 - 0.9439r$	$0.0053 + 0.0117r + 0.5853r^2$	12/31/1979–12/31/1998
CDNDL/US	1	$-0.0008 - 1.0460r$	$0.0039 + 0.0091r + 0.7006r^2$	12/31/1979–12/31/1998
EURO/US(hourly)	1	$-0.04242 - 0.0287r$	$0.00069 + 0.00123r + 0.00045r^2$	30/01/2003–30/05/2003

The results are, from top to bottom, for French, German (Mark), Dutch, Swiss, Japanese, Austrian, British, and Canadian currencies.

equation which takes the form of a Fokker–Planck equation [13]:

$$\frac{\partial}{\partial t}p(r) = -\frac{\partial}{\partial r}[D^{(1)}(r,t)p(r,t)] + \frac{\partial^2}{\partial r^2}[D^{(2)}(r,t)p(r,t)]. \tag{7}$$

The drift and diffusion coefficients, $D^{(1)}(r,t)$ and $D^{(2)}(r,t)$, are estimated directly from the data and the moments $M^{(k)}$ of the conditional probability distributions:

$$D^{(k)}(r,t) = \frac{1}{k!}\lim_{\Delta t\to 0} M^{(k)},$$

$$M^{(k)} = \frac{1}{\Delta t}\int \mathrm{d}r'(r'-r)^k p(r', t+\Delta t|r,t). \tag{8}$$

The quantities $D^{(k)}(r,t)$ are known as the Kramers–Moyal coefficients. The drift and diffusion coefficients $D^{(1)}$ and $D^{(2)}$ are displayed in Figs. 2 and 3 (the data used are the hourly prices of EURO versus the U.S. dollar).

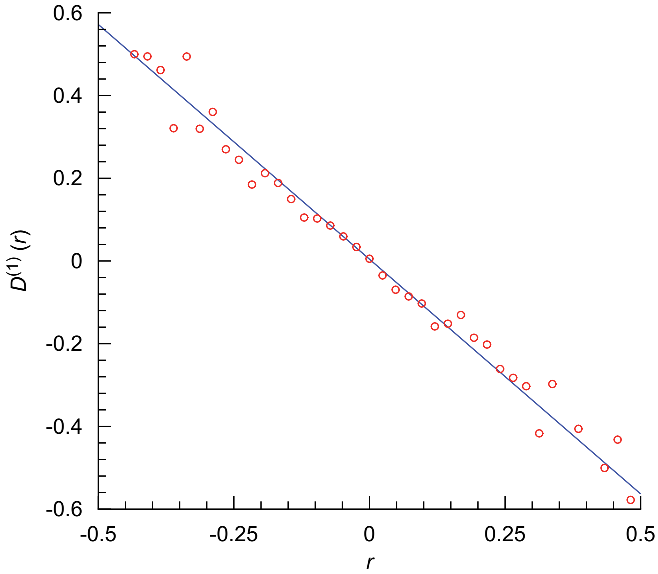

Fig. 2. The drift coefficient for EURO versus the U.S. dollar.

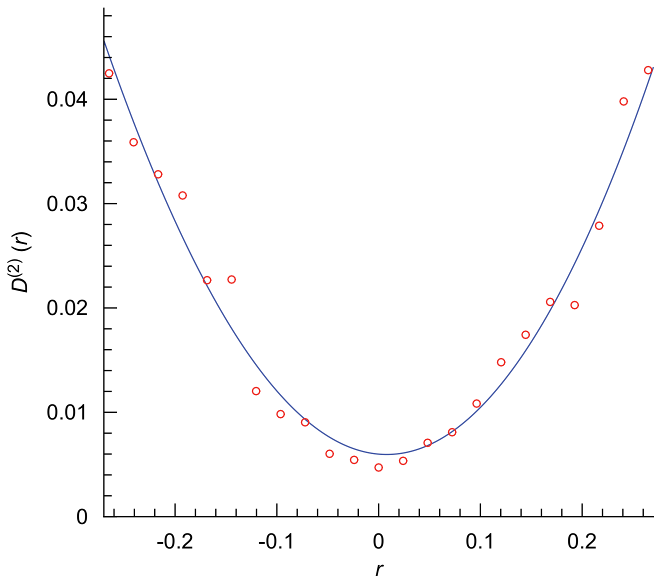

Fig. 3. The diffusion coefficient for EURO versus the U.S. dollar.

It turns out that the drift coefficient $D^{(1)}$ is (to a good degree of approximation) a linear function of r, whereas the diffusion coefficient $D^{(2)}$ is a quadratic function in r. For large values of r, the estimates become poor and, thus, the uncertainty increases. From the analysis of the data set we obtain the approximants that are presented in Table 1. To estimate the drift and diffusion coefficients, we measured the returns in units of the 10σ, where σ is the standard deviation of r.

According to Pawula's theorem, the Kramers–Moyal expansion terminates after the second term, provided that the fourth-order coefficient $D^{(4)}(r,t)$ vanishes [13]. In our analysis the fourth-order coefficient $D^{(4)}$ was found to be about $D^{(4)} \simeq 10^{-2} D^{(2)}$. Thus, in this approximation, we may ignore the coefficients $D^{(n)}$ for $n \geqslant 3$. We note that the Fokker–Planck equation is equivalent to the following Langevin equation [13],

$$\frac{\mathrm{d}}{\mathrm{d}t}r(t) = D^{(1)}(r) + \sqrt{D^{(2)}(r)}\, f(t). \tag{9}$$

Here, $f(t)$ is a random force, with zero mean and Gaussian statistics, δ-correlated in t, i.e., $\langle f(t)f(t')\rangle = 2\delta(t-t')$. Furthermore, given Eq. (9), it should be clear that we are able to separate the deterministic and the noisy components of the returns' fluctuations in terms of the coefficients $D^{(1)}$ and $D^{(2)}$. Using Eq. (9), we show in Fig. 4 the reconstructed series for the hourly rates of exchange between the EURO and the U.S. dollar. Compared with the real data (red line), Eq. (9) exhibits the trends perfectly. Moreover, the two series (the actual and the reconstructed ones) exhibit practically the same fluctuations. It is worth mentioning that Eq. (9) should be reconstructed by steps equal to the Markov time scale t_M, since we do not have information at times shorter than the time step (the Markov time scales for all the indices in Table 1 are equal to 1 day).

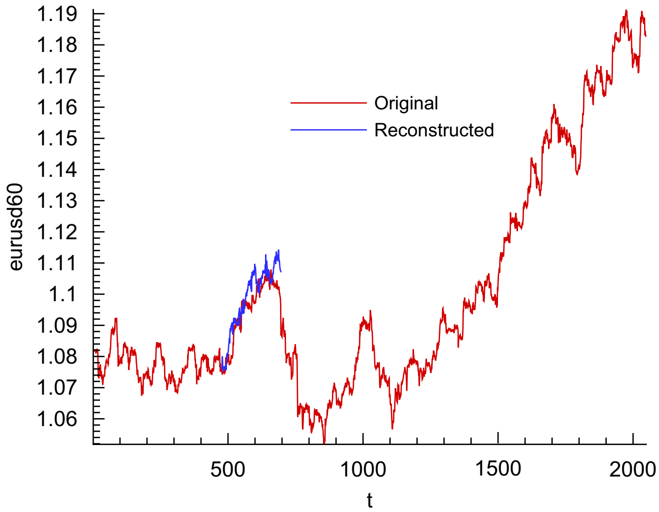

Fig. 4. Comparison of the actual and reconstructed hourly data for EURO.

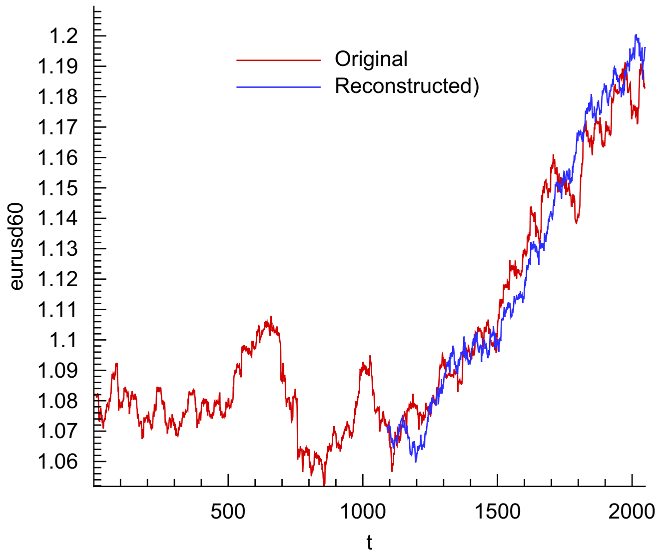

Fig. 5. Comparison of the actual and reconstructed hourly data for EURO.

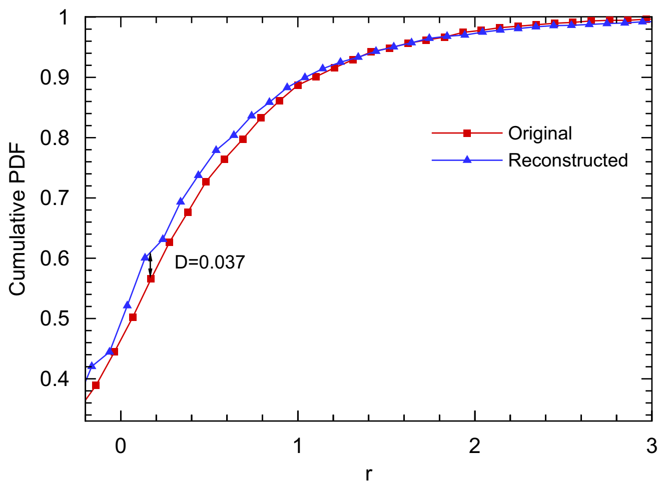

Fig. 6. Comparison of two cumulative PDFs for the original and reconstructed hourly data for EURO versus the U.S. dollar, via the Kolmogorov–Smirnov test.

To make predictions for the future trends, we use the definition of $r(t)$ to write $x(t+1)$ in terms of $x(t)$. But, since the reconstructed data have a variance equal to $\frac{1}{10}$ and a zero mean, we have

$$x(t+1) = x(t)\exp\{10\sigma_r[r(t)+\bar{r}]\}, \tag{10}$$

where $\bar{r}$ and σ_r are the mean and standard deviations of $r(t)$. To use Eq. (10) for predicting $r(t+1)$, we need $[x(t), r(t)]$. We select three consecutive points in the series $r(t)$ and search for three consecutive points in the reconstructed series of $r(t)$ that have the smallest difference with the selected points. We consider the difference to be minimum if it is less than $0.05r_{\max}$ (stricter rules are clearly possible). Wherever this happens is taken to be the time t which fixes $[x(t), r(t)]$. Shown in Figs. 4 and 5 are the actual data and the predictions for some interval in the time series for the hourly rates of EURO versus the U.S. dollar. We also have to say that the figures here are one of the best runs of the stochastic simulation. Since we have different realization of the noise and also different initial conditions, a perfect reconstruction is not always exist, but the more accurate calculation of drift and diffusion coefficients, the better result of reconstruction we have.

We also check to make sure that the original times series and the reconstructed ones have the same statistics. To do this, we computed the cumulative PDF $P(r)$ of the series, based on the data and, in the case of the reconstructed PDF, using the Langevin equation. Fig. 6 presents the comparison of the two PDFs for the EURO. The Kolmogorov–Smirnov test was used to discern the differences between the two PDFs. The maximum difference between the two cumulative PDFs is about 0.037, hence indicating the high precision of the reconstructed PDF.

4. Summary

We analyzed the stochastic time series that represent the rates of exchange of various national currencies versus the U.S. dollar. To convert the series to stationary ones, we analyzed their corresponding return series. The fundamental time scale in the approach is the Markov time scale t_M, which is the minimum time interval over which the series can be considered as constituting a Markov process. Based on the estimates of the Kramers–Moyal coefficients $D^{(k)}(r,t)$ for the series, it was shown that the fourth-order coefficient $D^{(4)}$ is very small, implying that the Kramers–Moyal expansion reduces to a Fokker–Planck equation which, in turn, is equivalent to a Langevin equation. Thus, the probability densities of the fluctuations in the returns r for the rate of exchange of various currencies versus the U.S. dollar satisfy a Fokker–Planck equation, characterized by a drift and a diffusion coefficient, which represent the first two coefficients in the Kramers–Moyal expansion. We computed accurate approximants for the coefficients for the stochastic time series r by using the polynomial ansatz [10–15]. We then used a novel method to utilize the returns' data, which contain a degree of stochasticity, and constructed a simple equation that governs the time series. The resulting equation is capable of providing a rational explanation for complex features of the series. Moreover, it requires no scaling feature. Our approach provides extensive statistical properties of the returns' data, which can help one to precisely check the financial models.

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