Three-dimensional forced Burgers turbulence supplemented with a continuity equation

Jahanshah Davoudi,[1,2] A. A. Masoudi,[2,6] M. Reza Rahimi Tabar,[2,3,4] A. Reza Rastegar,[2,5] and F. Shahbazi[2,6]
[1]Max-Planck Institute for Complex Systems, Noethnitzer Strasse 38, 01187 Dresden, Germany
[2]Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5531, Tehran, Iran
[3]Department of Physics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran
[4]CNRS UMR 6529, Observatoire de la Côte d'Azur, Boîte Postal 4229, 06304 Nice Cedex 4, France
[5]Department of Physics, Tarbiat-Moallem University, Tabriz, Iran
[6]Department of Physics, Sharif University of Technology, P.O. Box 11365-9161, Tehran, Iran
(Received 20 September 2000; published 20 April 2001)

We investigate turbulent limit of the forced Burgers equation supplemented with a continuity equation in three dimensions. The scaling exponent of the conditional two-point correlation function of density, i.e., $\langle\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)|\Delta\mathbf{u}\rangle\sim|\mathbf{x}_1-\mathbf{x}_2|^{-\alpha_3}$, is calculated self-consistently in the nonuniversal region from which we obtain $\alpha_3=3$. Also we derive an equation governing the evolution of the probability density function (PDF) of longitudinal velocity increments in length scale, from which a possible mechanism for the dependence of the inertial PDF to one-point u_{rms} is developed.

DOI: 10.1103/PhysRevE.63.056308 PACS number(s): 47.27.Gs, 47.40.Ki

I. INTRODUCTION

Burgers equation describes a variety of nonlinear wave phenomena arising in the theory of wave propagation, acoustics, plasma physics, surface growth, charge density waves, dynamics of the vortex lines in high-T_c superconductors, dislocations in the disordered solids, and formation of large-scale structures in the universe [1–9]. The problem of forced and unforced Burgers turbulence has been attacked recently by various methods [1–27]. It is well known that both decaying and forced Burgers equation develop singular structures. In one dimension, nonlinearity in the advection term develops the so-called shock structures. Heuristic arguments [3,6,11,19,23] show that shock structures are responsible for the extreme intermittency. So in the structure functions defined as

$$S_n(r)\equiv\langle[u(x+r)-u(x)]^n\rangle\sim r^{\xi_n}, \tag{1}$$

the exponents are $\xi_n=1$. At the same time, energy cascade has a simple picture in terms of shock structures in the stationary Burgers turbulence. Forcing at large scales the injected energy is advected from large scales down to the scales of typical shock width where the energy is dissipated. In multidimensional Burgers turbulence the presence of large scale structures forming a d-dimensional frothlike pattern is believed to be responsible for extreme case of intermittency, causing the saturation of the intermittency exponent to $\xi_n=1$. Similarity of intermittency in stationary multidimensional Burgers problem to one dimension is motivated by the replica calculations [11] in infinite dimensions and simulations [19]. Recently the nature of singularities in multidimensional decaying Burgers turbulence with density has been elaborated [22]. However, in one dimension there are an infinity of conserved currents in the inviscid and unforced equation, while the multidimensional Burgers problem ceases to have such conservation laws. According to recent theoretical [11–28] and numerical work [10,16], it is known that the probability density function (PDF) for the velocity difference behaves differently in universal and nonuniversal regions. In the universal region, i.e., the interval $|\Delta u|\ll u_{rms}$ and $r\ll L$, the PDF can be represented by the universal scaling form

$$P(\Delta u,r)=\frac{1}{r^z}F\left(\frac{\Delta u}{r^z}\right), \tag{2}$$

where $F(x)$ is a normalizable function and the exponent z is related to the exponent of random-force correlation η as $z=(\eta+1)/3$. For $x=|\Delta u|/r^z\gg1$ the universal scaling function $F(x)$ is given by the expression $F(x)\sim\exp(-\alpha x^3)$, where α is some constant in one-dimension and it depends on the cosine of angle between the vectors $\Delta\mathbf{u}$ and $\mathbf{r}$ in the higher dimensions. On the other hand, the PDF in the interval $|\Delta u|\gg u_{rms}$ behaves as

$$P(\Delta u,r)=rG\left(\frac{\Delta u}{u_{rms}}\right), \tag{3}$$

where the argument depends on the single-point nonuniversal u_{rms}. Analytic supports for any one of the observations starting from the dynamical equations is the major challenge of theoretical understanding of intermittent statistics of Burgers equation.

We will study the three-dimensional Burgers equation supplemented with a continuity equation in the inviscid limit. Contrasting the fact that there are infinity of conserved currents in one dimension to the lack of such conserved currents in multidimensional Burgers problem we aim to extract some information about the intermittency and probability density of longitudinal velocity increments. Providing a mean-field-like approximation for the conditional two-point correlation of density, i.e., assuming $\langle\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)|\Delta u\rangle\sim|\mathbf{x}_1-\mathbf{x}_2|^{-\alpha_3}$, the right tail of probability density of longitudinal velocity increment is shown to behave as $P(\Delta u,r)\sim(1/r)\exp[-(\Delta u/r)^3]$ in the universal region. Positivity of the PDF indicates that the exponent of two-point correlation of density for the inviscid case fixes to $\alpha_3=7/2$ in the universal regime. Unlike the Burgers problem [11–23] we corroborate that when continuity equation is coupled it is possible to find a positive PDF even in the *strict inviscid* case, i.e., $\nu=0$. However, due to lack of control on the dissipation anomaly we cannot provide any information regarding *inviscid limit*. Relying on the same mean field analysis we derive an equation governing the evolution of stationary probability density of longitudinal velocity increments in the nonuniversal region. The results indicate that the information of one-point u_{rms} is transferred from integral scales down to inertial-scale PDF. This is also observed numerically in one-dimensional Burgers problem [11,19]. Within our approximations the intermittency exponents are derived to saturate to a constant and from there we obtain $\alpha_3=3$.

The paper is organized as follows: In Sec. II we define the generating function and comment on its relevance to conservation laws. In Sec. III we derive the right tail of the probability density of longitudinal velocity increments in the universal region and obtain the exponent of density-density correlator in inviscid case. In Secs. III and IV we obtain stationary relations for some of the structure functions in which the viscose terms are not relevant and determine the small-scale statistics of longitudinal velocity difference by finding an evolution equation for the PDF of longitudinal velocity difference in the nonuniversal regime. The picture for nonskewed part of the PDF is consistent and we confirm that intermittency exponents saturate to a constant.

II. GENERATING FUNCTION EQUATION IN THREE DIMENSIONS

Our starting point is the 3D Burgers equation supplemented with a continuity equation

$$\mathbf{u}_t+(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}=\nu\boldsymbol{\nabla}^2\mathbf{u}+\mathbf{f}(\mathbf{x},t), \tag{4}$$

$$\rho_t+\partial_\alpha(\rho u_\alpha)=0, \tag{5}$$

for the Eulerian velocity $\mathbf{u}(\mathbf{x},t)$ and viscosity ν and density ρ, in three-dimensions. The force $\mathbf{f}(\mathbf{x},t)$ is the external stirring force that injects energy into the system on a typical length scale L. More specifically, we take a Gaussian distributed random force that is identified as

$$\langle f_\mu(\mathbf{x},t)f_\nu(\mathbf{x}',t')\rangle=k(0)\,\delta(t-t')k_{\mu\nu}(\mathbf{x}-\mathbf{x}'), \tag{6}$$

where $\mu,\nu=x,y,z$. The correlation function $k_{\mu\nu}(r)$ is normalized to unity at the origin and decays rapidly enough where r becomes larger or equal to integral scale L, i.e., we suppose that

$$k_{\mu\nu}(\mathbf{x}_i-\mathbf{x}_j)=k(0)\left[1-\frac{|\mathbf{x}_i-\mathbf{x}_j|^2}{2L^2}\,\delta_{\mu,\nu}+\frac{(\mathbf{x}_i-\mathbf{x}_j)_\mu(\mathbf{x}_i-\mathbf{x}_j)_\nu}{L^2}\right], \tag{7}$$

with $k(0),L=1$. The quantity $k(0)$ measures the energy injected into the turbulent fluid per unit time and unit volume. $\mathbf{f}(\mathbf{x},t)$ provides also the energy flux in the kth shell as $\Pi_k=\Pi(r=k^{-1})\simeq\int_{1/L}^{k}\langle|\mathbf{f}(\mathbf{k})|^2\rangle$, where r belongs to the inertial range. Equations (4) and (5) exhibit special type of nonlinear interactions, hidden in the nonlinear term $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$. The advective term couples any given scale of motion to the large scales where large scales contain most of the energy of the flows. This means that large-scale fluctuations of turbulence production in the energy-containing range couple to the small-scale dynamics of turbulence flow. In other words, the details of the large-scale turbulence production mechanism are important, leading to the nonuniversality of probability distribution function of velocity difference. However, in the case of one-dimensional forced Burgers equation when $|u(x)-u(x')|\ll u_{rms}$ it is believed that the PDF for the velocity difference is not dependent on u_{rms} and therefore one-point u_{rms} does not appear in the velocity difference PDF. This region is known as the Galilean invariant (GI) region. The problem is to understand the statistical properties of velocity and density fields that are the solutions of Eqs. (4) and (5). Before starting the statistical analysis of these coupled equations we wish to remind some basic differences between Burgers equation and Burgers and density equations in higher dimensions. It is well known that Burgers equation in one dimension has infinity of conserved currents in inviscid and unforced case, i.e., for purely convective dynamics we have

$$\frac{\partial u^n}{\partial t}+\frac{n}{n+1}\frac{\partial u^{n+1}}{\partial x}\sim 0. \tag{8}$$

In higher dimensions such conserved currents do not exist. However, purely convected dynamics in the coupled density and Burgers equations have infinite conserved currents both in one dimension or higher dimensions. Following [12] we demonstrate these infinity of conserved equations in terms of $e_\lambda(x)=\rho(x)e^{\lambda_k u_k(x)}$ as

$$\frac{\partial}{\partial t}e_\lambda(x)+\frac{\partial}{\partial x_k}[u_k e_\lambda(x)]\sim 0. \tag{9}$$

Expanding the above relation in powers of λ shows that generally all the tensors $T_{\alpha_1\cdots\alpha_n}=\rho u_{\alpha_1}\cdots u_{\alpha_n}$ are conserved in purely convective case. For example, components of momentum and energy satisfy some conservation laws as following:

$$\partial_t(\rho u_i)+\partial_j(\rho u_i u_j)=0, \tag{10}$$

$$\partial_t(\rho\mathbf{u}\cdot\mathbf{u})+\partial_j(\rho\mathbf{u}\cdot\mathbf{u}u_j)=0. \tag{11}$$

Qualitatively these conservation laws indicate that in the inviscid case the fluctuations of density and velocity should be interrelated. In the driven case fundamental quantities that emerge through calculations are mixed correlations of velocity and density and the statistics of density and velocity alone would be extracted from those mixed correlations. For studying the driven Burgers equation in three dimensions constrained with continuity, we consider the following two-point generating function:

$$Z_2(\lambda_1,\lambda_2,\mathbf{x}_1,\mathbf{x}_2)=\langle\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\exp[\boldsymbol{\lambda}_1\cdot\mathbf{u}(\mathbf{x}_1)+\boldsymbol{\lambda}_2\cdot\mathbf{u}(\mathbf{x}_2)]\rangle. \tag{12}$$

Basically we have written the object $e_\lambda(x)$ in two different points and the symbol $\langle \cdots \rangle$ means an average over various realizations of the random force. To derive an equation for Z_2, we write Eqs. (4) and (5) in two points $\mathbf{x}_1$ and $\mathbf{x}_2$ for different components of velocity vector u_1, u_2, u_3, and $\rho(x)$ and multiply the equations in $\rho(\mathbf{x}_2)$, $\lambda_{1x}\rho(\mathbf{x}_1)\rho(\mathbf{x}_2), \ldots, \lambda_{1z}\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)$ and $\rho(\mathbf{x}_1)$, $\lambda_{2x}\rho(\mathbf{x}_1)\rho(\mathbf{x}_2), \ldots$, and $\lambda_{2z}\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)$, respectively. After adding the equations and multiplying the result by $\exp[\boldsymbol{\lambda}_1 \cdot \mathbf{u}(\mathbf{x}_1) + \boldsymbol{\lambda}_2 \cdot \mathbf{u}(\mathbf{x}_2)]$ we average with respect to external random force, so

$$\partial_t Z_2 + \sum_{\{i=1,2\}\mu=x,y,z} \frac{\partial}{\partial \lambda_{i,\mu}} \partial_{\mu_i} Z_2 - \sum_{\{i,j=1,2\}\mu,\nu=x,y,z} \times \lambda_{i,\mu}\lambda_{j,\nu} k_{\mu\nu}(\mathbf{x}_i - \mathbf{x}_j) Z_2 = D_2, \tag{13}$$

where D_2 is given by

$$D_2 = \langle \nu\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)[\boldsymbol{\lambda}_1 \cdot \nabla^2 \mathbf{u}(\mathbf{x}_1) + \boldsymbol{\lambda}_2 \cdot \nabla^2 \mathbf{u}(\mathbf{x}_2)] \exp[\boldsymbol{\lambda}_1 \cdot \mathbf{u}(\mathbf{x}_1) + \boldsymbol{\lambda}_2 \cdot \mathbf{u}(\mathbf{x}_2)] \rangle. \tag{14}$$

Second and third terms in the left-hand side are respectively related to convective terms and random forcing. It is one of the advantages of this method that all the nonlinearities due to convection can be written in a closed form. The Gaussianity of the forcing statistics also helps to write its contribution in terms of generating function according to a typical trick in Gaussian random variables [1]. On dropping the dissipation terms, the generating function defined above satisfies a closed equation for Gaussian random forcing. However, it is already emphasized [12,19] that the role of dissipation term in the turbulent limit can be understood by looking at the statistics of $\langle u^n \rangle$ in Burgers equation. In one dimension convective terms cannot offset the pumped rate of $\langle u^n \rangle$ since $\langle \partial u^n / \partial x \rangle = 0$. Therefore the stationary state may be maintained by nonzero limit of dissipation terms. Finite contribution of the dissipation terms in the driven turbulent limit resembles the notion of *anomaly*. In the problem of Burgers and continuity the convective terms can be written as conserved currents even in higher dimensions. Hence anomalous behavior of the dissipation terms in the multidimensional Burgers and continuity equation are important in maintaining a statistical stationary state. Recently [21,23] it has been shown that regularizing the convective terms in a precise way is equivalent to imposing anomalous contribution of dissipation terms in the turbulent limit not only for the forced Burgers equation but even for decaying case in one dimension. Hence one can in principle determine the anomaly by appropriate regularization of the derivatives in order to take care of the singularities. However, we believe that proposing the regularization scheme when one is working with momentum equation is not a trivial task. The scheme introduced in [23] is not applicable to this case since it relies upon a special separation of velocity equation and density continuity. In other words, naive coupling of Burgers and continuity equations is not equivalent to momentum and continuity equations unless one assumes the smoothness of the fields. As soon as the singularities are developed, the equivalence is no more valid. In this sense up to the time scale of singularity formation Eq. (10) and Eq. (11) are valid and after that one needs to regularize the convection terms in them in a way that produces the anomaly terms established by the presence of the singularities.

Hereafter we change the variables as: $\mathbf{x}_\pm = \mathbf{x}_1 \pm \mathbf{x}_2$, $\boldsymbol{\lambda}_+ = \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_- = (\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2/2)$ and write Eq. (8) in terms of $\boldsymbol{\lambda}_+$ and $\boldsymbol{\lambda}_-$ so we will have

$$\sum_{\mu=1}^{d} \left(\frac{\partial}{\partial \lambda_{+\mu}} \frac{\partial}{\partial x_{+\mu}} + \frac{\partial}{\partial \lambda_{-\mu}} \frac{\partial}{\partial x_{-\mu}} \right) Z_2 - \sum_{\mu,\nu=1}^{d} \lambda_{+\mu}\lambda_{+\nu} k(0) Z_2 + \sum_{\mu,\nu=1}^{d} \left(\frac{1}{2}\lambda_{+\mu}\lambda_{+\nu} - 2\lambda_{-\mu}\lambda_{-\nu} \right) \times k(0) \left(\frac{r^2}{2L^2}\delta_{\mu\nu} + \frac{x_{-\mu}x_{-\nu}}{L^2} \right) Z_2 = D_2, \tag{15}$$

Where D_2 term is the dissipation contribution and is written as

$$D_2 = \left\langle \nu\rho(\mathbf{x}_1)\rho(\mathbf{x}_2) \left[\frac{\lambda_{+\mu}}{2} \{ (\nabla_-^2 + \nabla_+^2) u_{+\mu} + 2(\nabla_- \cdot \nabla_+) u_{-\mu} \} + \lambda_{-\mu} \{ (\nabla_-^2 + \nabla_+^2) u_{-\mu} + 2(\nabla_- \cdot \nabla_+) u_{+\mu} \} \right] e^{\lambda_{+\mu}u_{+\mu} + \lambda_{-\mu}u_{-\mu}} \right\rangle. \tag{16}$$

Because of statistical homogeneity all the terms proportional to $\partial Z_2 / \partial x_{+\mu}$ are vanished. The above equation can be expanded in powers of $\lambda_{-\mu}$ and $\lambda_{+\mu}$ so in each order of expansion one would obviously get an equation that is governed over different mixed moments of ρ, $u_{-\mu}$, and $u_{+\mu}$. In general, the generating function satisfying the above dynamical equation is a compact way of writing the dynamical equations of all the structure functions. Its solution would be a function of $\lambda_{-\mu}$ and $\lambda_{+\mu}$; however, it is easy to check that the dependence of Z_2 on $\lambda_{-\mu}$ and $\lambda_{+\mu}$ can be separated self-consistently so that

$$Z_2(\mathbf{x}_-, \boldsymbol{\lambda}_-, \boldsymbol{\lambda}_+) = \delta(\boldsymbol{\lambda}_+) F_2(\mathbf{x}_-, \boldsymbol{\lambda}_-). \tag{17}$$

It is remarked [12] that in the *inviscid limit*, the proposed ansatz is the only consistent form for Z_2 in which its dependence on $\boldsymbol{\lambda}_+$ and $\boldsymbol{\lambda}_-$ can be separated. However, one may see that in the inviscid problem when $\nu = 0$ it is possible to find more general forms of separation in which the dependence on $\boldsymbol{\lambda}_+$ is not a delta function. In the case of Burgers equation in one dimension the proposed separation is valid only when $u \ll u_{rms}$ and $r \ll L$. As far as convective terms are concerned in the present problem the validity of this ansatz is not necessarily restricted to $u \ll u_{rms}$. Actually the conservation of density cancels the terms of the type $(1/\lambda_{i,\mu})(\partial Z_2 / \partial x_{i,\mu})$ in the convective contributions [12,17–19] that complicates the problem by mixing $\lambda_{-\mu}$ and $\lambda_{+\mu}$ in the equation of $Z_2(\mathbf{x}_-, \boldsymbol{\lambda}_-, \boldsymbol{\lambda}_+)$. However, we emphasize that it might happen that effective closure for dissipation generates such complicated operators that limit the validity of the ansatz. Hence putting $\boldsymbol{\lambda}_+=0$ and considering the spherical coordinates, i.e., $\mathbf{x}_-:(r,\theta,\varphi)$ and $\boldsymbol{\lambda}_-:(\mu,\theta',\varphi')$. It becomes clear that on inserting $\boldsymbol{\lambda}_+=0$ in Eq. (15) the remaining will involve velocity *increments*. We find that $F_2(\mathbf{x}_-,\lambda_-)$ satisfies the following equation for homogeneous and isotropic case:

$$\left[s\partial_r\partial_\mu-\frac{s(1-s^2)}{r\mu}\partial_s^2+\frac{1+s^2}{r\mu}\partial_s+\frac{1-s^2}{\mu}\partial_r\partial_s+\frac{1-s^2}{r}\partial_\mu\partial_s -r^2\mu^2(1+2s^2)\right]F_2=D_2, \tag{18}$$

where $s=\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')$.

In this case the dissipation term is the limit of $\nu\to 0$ and then $r\to 0$ of the following:

$$D_2=\langle\nu\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)\lambda_{-\mu}(\nabla_1^2u_{1\mu}-\nabla_2^2u_{2\mu})e^{\lambda_{-\mu}u_{-\mu}}\rangle. \tag{19}$$

The same kind of master equation was first derived by Polyakov [12] in the problem of forced Burgers equation in $d=1$ and then generalized by Boldyrev [17]. This equation is not closed due to the dissipation term and many proposals have been suggested for treating D_2 in the case of one-dimensional Burgers equation that have given rise to different results [11–22].

Adapting $\nu=0$ in one-dimensional Burgers equation converts the original problem to Riemann equation [20,21]. Structure of nonlinearity in Riemann equation leads to multivalued solutions so the complete statistical analysis of the problem would be very complicated. However, in one dimension it is shown that the closed master equation of Riemann equation gives some upper bounds for the tails of the velocity increment PDF in the corresponding Burgers problem. Here although we do not discard the viscosity term but we will justify that it would be fruitful to study the inviscid case, $\nu=0$, since some of the details of longitudinal PDFs are not sensitive to dissipation contributions. Hence we aim to extract as much information about the problem as possible without considering the difficulties related to anomalous contributions of dissipation terms. The limitations of validity of the results will be discussed later.

Even forgetting the dissipation the generating function equation involves some correlations between velocity increments and density. Vaguely speaking, since the density field is advected by velocity field, one expects that there would be strong correlation between density and velocity increments. So extracting information about density and velocity fields alone from such mixed correlations is a nontrivial task. Proceeding further we propose the following ansatz for $F_2(\mu,r,s)$

$$F_2(\mu,r,s)=r^{-\alpha_3}F(\mu,r,s). \tag{20}$$

In order to clarify the meaning of $F(\mu,r,s)$ we write $F_2(\mu,r,s)$ explicitly as follows:

$$F_2(\mu,r,s)=\int\langle\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)|\Delta\mathbf{u}\rangle \times\exp(\lambda_-\cdot\Delta\mathbf{u})P(\Delta\mathbf{u},\mathbf{r})d\Delta\mathbf{u}. \tag{21}$$

Assuming scaling invariance for density fluctuations, the conditional density-density correlation appearing in the integrand would be a scaling function of *r*. In addition, since density-density correlation is conditioned on a fixed value of velocity increment, it should be a nontrivial function of velocity increment fluctuations too. Having the idea of mean-field analysis we assume that conditional average is functionally dependent on fluctuations of velocity increments just through different values of scaling exponent in universal and nonuniversal regimes, i.e.,

$$\langle\rho(\mathbf{x}_1)\rho(\mathbf{x}_2)|\Delta\mathbf{u}\rangle\sim r^{-\alpha_3}. \tag{22}$$

Although an approximation, this assumption simplifies the form of generating function so one can identify the function $F(\mu,r,s)$ in Eq. (20) as the generating function of velocity increments so that

$$F(\mu,r,s)=\int\exp(\lambda_-\cdot\Delta\mathbf{u})P(\Delta\mathbf{u},\mathbf{r})d\Delta\mathbf{u}. \tag{23}$$

Therefore we expect that the mean field kind of reasoning would fix the value of the parameter α_3 self-consistently. We would notice that even fixing α_3 within this approximation becomes very promising due to dissipation anomaly.

III. UNIVERSAL PROBABILITY DENSITY FUNCTION

We propose that in homogeneous and isotropic problem, with stirring correlation as $k(r)\sim 1-r^\eta$, where in the present case $\eta=2$, there exists a universal scale-invariant solution of Eq. (18) in the following form:

$$F_2(\mu,r,s)=g(r)F(\mu r^\delta,s),\quad g(r)=r^{-\alpha_3} \tag{24}$$

This ansatz first introduced by Boldyrev [18] in the problem of turbulence with pressure in one dimension. Substituting the following form for the generating function fixes the exponent δ as $\delta=(\eta+1)/3$, so from Eq. (7) we find $\delta=1$. Invoking to the scaling invariance of the inviscid Burgers equation and continuity equation, we assume the existence of conditional density-density correlation with the scaling form introduced in Eq. (21). The scaling exponent of two-point correlation of density, i.e., α_3, can be found by taking the limit of generating function when $\mu\to 0$. Therefore it is necessary to find such a solution for $F(\mu r^\delta,s)$ that tends to a constant in the limit of $\mu\to 0$. Now the proposed scale invariant argument as $F(\mu r,s)$ can be interpreted as if we are seeking those solutions for velocity-increment PDFs that behave as $P(\Delta u/r,s)$. Based on this mean-field-like calculation the parameter α_3 enters in the dynamical equations of velocity-increment generating function. The goal would be to fix this parameter according to the general consistency conditions like positivity or normalizability of velocity-increment PDF. Proceeding further we focus our attention on the longitudinal velocity components, i.e., $s=1$. Hence we assume the scaling ansatz $F(\mu r,s)=F(\mu rs)$ *after* which we put $s=1$. The proposed form of the arguments will dictate that $S_n(r,s)\sim s^n S_n(r)$ for $s\rightarrow 1$ when $n<1$. However, in Appendix A we rationalize the solution in more detail. Rewriting Eq. (18) in terms of the variable $z=\mu rs$, the following equation is obtained in the case of $s=1$:

$$z\partial_z^2 F(z)+(3-\alpha_3)\partial_z F(z)-3z^2F(z)=D_2. \tag{25}$$

This is the projection of the three-dimensional master equation on separation line between the two observation points. *Neglecting* D_2 term it is interesting that the above equation is formally similar to the master equation first derived by Polyakov [12] for the problem of one-dimensional Burgers equation in the *inviscid limit*. Treating more carefully the origin of different terms in the master equation would reveal the fact that although we are essentially looking at the projection of fluctuations on one line but at the same time the fluctuations of transverse components contribute to the equation. The exponent of density correlation appears in the resulting equation that is necessary in order to find a positive and finite PDF. One can readily deduce some information about the tails of PDF by Laplace transforming of Eq. (25), that is

$$3\frac{\partial^2 P}{\partial y^2}-y^2\frac{\partial P}{\partial y}+(1-\alpha_3)yP=D_2, \tag{26}$$

where $y=\Delta u/r$. Right tail of the PDF, i.e., when $\Delta u/r\rightarrow+\infty$ (for $s=1$), in three dimensions, is insensitive to dissipation terms. So we neglect the dissipation terms in the right hand of the PDF equation and it is immediately observed that the asymptotic of PDF behaves as $(1/r)\exp[-(\Delta u/r)^3]$. This form has been confirmed by several other approaches [10–21]. It is believed that the same functional behavior of right tail is valid also when viscose effects are present. The reason is based on our intuition about Burgers equation in one dimension. Actually right tail is just built in by the contribution of ramps with positive gradients much larger than the typical gradient imposed by the forcing hence shock structures do not contribute to this part of the PDF. Since almost all the dissipation is occurred by shock structures in one dimension, neglecting the dissipation term somehow is equivalent to neglecting the effects of shocks that are corresponding to the large negative gradients and therefore the right tail of PDF would not be affected. Although the one-dimensional simple shock structures are changed in higher dimensions to more complicated objects [5,22], however, we think that the same ideas would be applicable in higher dimensions. The left tail of PDF strongly depends on the structure of D_2 terms. Therefore one would resolve the anomalous contribution of dissipation terms in the PDF equation. In Polyakov's work the effect of viscous term is found in the limit of $\nu\rightarrow 0$ and $r\ll L$ by appealing to the self-consistent conjecture of operator product expansion. It is found that consistent with the symmetries of the problem, two terms would be generated by the viscous term. These two anomaly terms modify the master equation governing the generating function in such a way that a positive, finite, renormalizable PDF is found [12,16,17]. A simple comparison between Eq. (25) and Polyakov's result will reveal that the structure of *b anomaly* is similar to the term proportional to the scaling exponent of density-density correlation. In the problem of one-dimensional Burgers equation and in the zero-viscosity limit the presence of the *b anomaly* generated by viscosity term ensures the existence of a positive PDF for velocity increments in the universal regime and the requirement of positivity will fix the value of anomaly coefficient [12]. Boldyrev [17] shows that one can find a family of solutions for different values of the *b anomaly* coefficient if one relaxes the homogeneity condition for the universal part of the PDF. The value of this coefficient is related to the algebraic decay of the left tail of PDF in the universal regime. Determination of the decay exponent has been a controversial subject for which other methods have been developed. Among them recent rigorous methods should be mentioned within which the exponent of the algebraic decay is fixed to 7/2 [20,21]. Since we are not able to give a closure for dissipation terms we cannot argue about the left tail in the inviscid limit. However, the interesting point is that our calculations in three dimensions show that when density fluctuations are taken into account, even in the inviscid problem when $\nu=0$, it is possible to find a positive solution for the longitudinal velocity increment PDF. It is easy to show that the requirement of the positivity on the PDF will fix the density-density scaling exponent to $\alpha_3=7/2$. Left tail of the PDF in this case is sensitive to the scaling exponent of the density-density correlator and is given by $1/(\Delta u)^{(\alpha_3-1)}$ when $\Delta u/r\rightarrow-\infty$. As we mentioned, in one-dimensional Burgers equation neglecting the dissipation term does not result in finding positive solutions for PDF while density fluctuations play such a role that even in inviscid case, i.e., $\nu=0$, one can in principle find a positive solution. Because density is advected passively by velocity, this result may seem strange and one expects though that the statistics of density would not affect the statistics of velocity. The resolving point is already mentioned that conservation laws connect density and velocity dynamically so that there should be a back reaction of density on velocity too. In our analysis the self-consistent determination of adjustable mean-field parameter α_3 reflects this interrelation. We think that the exponent for left tail is valid until the typical time scale of the singularity development. After singularity formation the anomaly terms would be considered that would surely change the exponent. Still we think that the mixed interrelation between density and velocity after singularity formation makes the result different from what one would get for Burgers equation that is coupled naively to density continuity. However, we are not able to resolve the dissipation effects to include the anomaly in the calculations yet.

IV. STRUCTURE FUNCTIONS IN NONUNIVERSAL REGION

In this section we consider the three-dimensional Burgers turbulence supplemented with a continuity equation in the nonuniversal region, i.e., $|u(x)-u(x')|\gg u_{rms}$. The force-free Burgers equation is invariant under space-time translation, parity, and scaling transformation. Also, it is invariant under Galilean transformation, $x \to x+Vt$ and $v \to v+V$, where V is the constant velocity of the moving frame. Both boundary conditions and forcing can violate some or all of the symmetries of force-free Burgers equation. However, it is usually assumed that in the high-Reynolds-number flows all symmetries of the dynamical equation are restored in the limit $r \to 0$ and $r \gg \eta$. So in this limit the root mean square of velocity fluctuations $u_{rms} = \sqrt{\langle v^2 \rangle}$ which is not invariant under a constant shift V, cannot enter the relations describing the moments of velocity difference. Therefore the effective equations for velocity correlation functions in the inertial range must have the symmetries of the original Burgers equations. Recent understandings of Burgers turbulence [10–21] indicate that in the non-universal region the PDF of velocity difference depends on the one-point u_{rms} and therefore is not universal that is meant to be sensitive on the details of large-scale forcing. This phenomenon is called breakdown of Galilean invariance in the nonuniversal region.

Possible generalization of these ideas for Navier-Stokes turbulence are developed by Yakhot recently [19]. In the following we aim to give a possible analytic mechanism within which one-point u_{rms} enters in argument of the PDF in nonuniversal region.

We shall be interested in the moments of velocity increments. As we emphasized in the previous section, because of the structure of convective terms we put $\lambda_1 + \lambda_2 = 0$ without any restriction in the phase space. Although correct, we cannot discard other solutions in which other complex dependencies on $\lambda_{-\mu}$ and $\lambda_{+\mu}$ exist but at least the results derived under this assumption are consistent.

Specifically, we first aim to find the behavior of *longitudinal* components of some of the structure functions, so following [19] it is more convenient to change the variables to $\eta_2 = (\lambda_- \cdot \mathbf{r})/r$ and $\eta_3 = \sqrt{\lambda^2 - \eta_2^2}$. Decomposing velocity increment as $\mathbf{u}_{\|} = u$ and $\mathbf{u}_{\perp} = v$, then η_2 and η_3, would be respectively the sources of longitudinal and transverse components of velocity increments. In terms of these variables, we obtain the following differential equation for $F_2(\eta_2, \eta_3, r)$:

$$\left[\frac{\partial^2}{\partial r \partial \eta_2} + \frac{d-1}{r} \frac{\partial}{\partial r} + \frac{\eta_2}{\eta_1} \left(\frac{2-d}{\eta_3} \right) \frac{\partial}{\partial \eta_3} + \frac{\eta_3}{r} \frac{\partial^2}{\partial \eta_2 \partial \eta_3} - \frac{\eta_2}{r} \frac{\partial^2}{\partial \eta_3^2} \right] F_2 - r^2 (3\eta_2^2 + \eta_3^2) F_2 = D_2. \tag{27}$$

Invoking the proposed mean-field interpretation we adapt the ansatz $F_2(\eta_2, \eta_3, r) = r^{-\alpha_d} F(\eta_2, \eta_3, r)$ from which one obtains the equation of velocity-increment generating function. All the information regarding the moments of velocity increments can be determined just by suitable differentiation of $F(\eta_2, \eta_3, r)$ with respect to η_2 and η_3, i.e., $\langle u^n v^m \rangle = \partial^n_{\eta_2} \partial^m_{\eta_3} F|_{\eta_2 = \eta_3 = 0}$. Unlike the universal part we do not restrict the solutions of F to the ones with scale invariant arguments. Laplace transforming Eq. (27) one easily determines the equation satisfied by the joint probability density of longitudinal and transverse components of velocity increments $P(u,v,r)$, that is

$$\frac{\chi}{r} \frac{\partial}{\partial v} [u P(u,v)] - \frac{\partial}{\partial r} \frac{\partial}{\partial v} [u P(u,v)] + \frac{d-2}{r} \frac{\partial}{\partial u} [v P(u,v)] + \frac{1}{r} \frac{\partial^2}{\partial v^2} [u v P(u,v)] - \frac{1}{r} \frac{\partial}{\partial u} \frac{\partial}{\partial v} [v^2 P(u,v)] = \frac{k(0)}{L^2} \left[-3r^2 \frac{\partial}{\partial v} \frac{\partial^2}{\partial u^2} P(u,v) + r^2 \frac{\partial^3}{\partial v^3} P(u,v) \right] + D_2. \tag{28}$$

The parameter χ is defined as $\chi = \alpha_d - d + 1$ and D_2 *black box* is resembling all the nonzero contributions buried in dissipation term. Since dissipation contributions cannot be written in terms of generating function itself, the equations are not closed. However, we extract some valuable information about some specific moments of velocity increments in which the dissipation terms are not relevant.

Starting with the PDF equation, the structure functions $S_{n,m} = \langle u^n v^m \rangle$ generally satisfy the following equation:

$$m \left(\frac{\partial}{\partial r} + \frac{m-1-\chi}{r} \right) S_{n+1,m-1} - \frac{n(d-2+m)}{r} S_{n-1,m+1} = 3 \frac{k(0)}{L^2} m n (n-1) r^2 S_{n-2,m-1} - \frac{k(0)}{L^2} m(m-1)(m-2) r^2 S_{n,m-3} + D_2. \tag{29}$$

The equation of third moment of longitudinal velocity increment $S_{3,0}(r)$ is readily found since assuming the stationarity one can estimate the contribution of dissipation. Actually it is simpler to think of generating function equation for evaluating the contribution of dissipation. To obtain the equation for $S_{3,0}$ one needs the terms of the order of $O(\eta_2^2 \eta_3)$. Fortunately dissipation contribution at this order is proportional to average energy-dissipation rate, i.e., $\epsilon(x) = \langle (\partial u_i / \partial x_j)^2 \rangle$ when $\nu \to 0$. However, in the statistical stationary state the average dissipation rate would be equal to rate of energy pumping injected by forcing term. Due to the Gaussianity of the forcing statistics the rate of energy pumping is $\langle f_\mu(x) u_\mu(x) \rangle \sim k(0)$. So the equation of $S_{3,0}$ becomes

$$\left(\frac{d}{dr} - \frac{\chi}{r} \right) S_{3,0} - 2 \frac{(d-1)}{r} S_{1,2} = -4k(0) + 6 \frac{k(0)}{L^2} r^2. \tag{30}$$

Since the above equation is coupled to the $S_{1,2}$ one should determine it beforehand. Again the equation for $S_{1,2}$ can be read from $S_{n,m}$ equation, so we have

$$3 \left(\frac{d}{dr} - \frac{\chi}{r} \right) S_{1,2} + 6 \frac{1}{r} S_{1,2} = -4k(0) + 6 \frac{k(0)}{L^2} r^2, \tag{31}$$

where the viscous term again is of the same order as in previous case. Substituting the solution for $S_{1,2}$ back in Eq. (30), we can solve the equation of $S_{3,0}(r)$. In the inertial ranges the order of $O(r^2)$ forcing term is negligible in comparison with $O(1)$ dissipation terms. Since we will be interested in the integral-scale effects for structure function scalings we do not discard the forcing contributions. The general solution of $S_{3,0}$ is

$$S_{3,0}(r)=\left[\frac{2(d-1)}{3(3-\chi)}+1\right]\frac{4k(0)}{1-\chi}r+\frac{2(d+2)k(0)}{L^2(3-\chi)}r^3+Cr^{\chi}. \tag{32}$$

The unknown coefficient C is determined by the *boundary condition* imposed by the statistics in integral scales. Although we have no precise way for determination of integral scale L it is pragmatically defined [11,19,28] by appealing to the idea that in the integral scales the longitudinal PDF is nearly Gaussian so approximately all the odd-order moments would vanish in that scale, i.e., $S_{2n+1,0}(r=L)=0$. The coefficient of homogeneous solution consequently is found to be $\sim k(0)L^{1-\chi}$ apart from some numerical coefficients. Requiring that the coefficient of third-order structure function becomes L independent the exponent χ is immediately fixed to $\chi=1$. In the next section we will clear out the role of parameter χ in our study. Thanks to the invariance of the basic dynamical equations under simultaneous operations $x\rightarrow -x$ and $u\rightarrow -u$, the underlying PDF equations would be invariant under the same operations too. Due to the rotational invariance it is interesting to note that combinations like $\langle v^{2n}\nabla^2 v\rangle$ do not contribute to expansion of dissipation terms in powers of η_2 and η_3. However still the combinations $\langle u^{2n}\nabla^2 u\rangle$ may contribute. It is straightforward to see that equations for all the even-order moments of longitudinal velocity increments will involve such combinations in their corresponding dissipation terms. Recently it is shown experimentally that such contributions are zero [29] in the Navier-Stokes turbulence. We think that such terms would be small compared to the other terms in the equation because they involve the odd part of the PDF that is orders of magnitude smaller than the even part, so in principle one can approximately neglect such terms at least in the equation of the even-order moments of the longitudinal structure functions. Therefore we study strict inviscid equations for investigating the behavior of $S_{2n,0}$, that is

$$\left(\frac{-\chi}{r}+\frac{d}{dr}\right)S_{2n,0}-\frac{(2n-1)(d-1)}{r}S_{2n-2,2}$$
$$=3r^2\frac{k(0)}{L^2}(2n-1)(2n-2)S_{2n-3,0}. \tag{33}$$

V. PROJECTION OF THE DYNAMICS ON A LINE AND LONGITUDINAL PDF

Back to variables introduced in Eq. (18) we aim to find the structure functions of velocity increment that are defined as

$$S_n(r,s)=\int u^n P(u,r,s)du,$$

where u is now the modulus of velocity increment vector and s is the angle between velocity increment and r. Switching to these variables it is straightforward to show that in the nonuniversal region the PDF for the velocity difference in three dimensions satisfies the following equation:

$$\frac{\alpha_d s}{r}\partial_u uP-s\partial_u u\partial_r P-\frac{s(1-s^2)}{r}\partial_s^2 P+\frac{d-2+s^2}{r}\partial_s P$$
$$-\frac{\alpha_d}{r}(1-s^2)\partial_s P+(1-s^2)\partial_s\partial_r P-\frac{(1-s^2)}{r}\partial_u u\partial_s P$$
$$+r^2(1+2s^2)\partial_u^3 P=D_2, \tag{34}$$

where $s=\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')$ in three dimensions and again D_2 term is dissipation contributions. Assuming all of the moments of velocity difference exist, the structure functions S_n for given angle γ [or $s=\cos(\gamma)$] satisfies the following equation:

$$[sn+(1-s^2)\partial_s]r\partial_r S_n-ns\alpha_d S_n-s(1-s^2)\partial_s^2 S_n$$
$$+(d-2+s^2)\partial_s S_n-\alpha_d(1-s^2)\partial_s S_n+n(1-s^2)\partial_s S_n$$
$$+r^3 n(n-1)(n-2)(1+2s^2)S_{n-3}=D_2. \tag{35}$$

In order to study longitudinal PDF we must consider the above equation in the limit of $s\rightarrow 1$. This limit is not trivial and needs to be considered more carefully. The contributions of different terms of the PDF equation in the limit when $s\rightarrow 1$ is determined by the corresponding terms in the equation of the structure functions. Again due to dissipation the equation is not at all trivial to be analyzed, however, as we argued before one can safely drop the D_2 term in even-order moments of longitudinal components. Hence the $S_{2n}(r,s)$ satisfies a closed equation. The forcing contribution to the above equation is the last term, i.e., $r^3 n(n-1)(n-2)(1+2s^2)S_{n-3}$ and this term does not have any contribution to the exponent of structure function. However, the amplitude of the structure functions does depend on the details of forcing. It means that the exponents of multiscaling in the structure functions are not changed by the forcing term and they are determined by the structure of nonlinearity and the transverse contributions to the Burgers equation. For solving Eq. (35) we examine the solutions in which their angular and scale-dependent parts in $S_n(r,s)$ are separated when $s\rightarrow 1$, i.e., the structure functions have the following form:

$$S_n(r,s)\rightarrow f_n(s)S_n(r), \tag{36}$$

where $S_n(r)=\langle[u(x+r)-u(x)]^n\rangle\sim r^{\xi_n}$. Factorizing the angle and scale dependences in the limit of $s\rightarrow 1$ is known for the *N-S* turbulence too [19]. Plugging the ansatz for $S_n(r,s)$ in the structure function equation it is easy to see that the intermittency exponent fixes to

$$\xi_n=\alpha_d-d+1, \tag{37}$$

and $f_n(s) \propto s^n$ (details in Appendix B). So this observation encodes the information that ξ_n's are constant. It is seen that the expression of scaling exponent is exactly the unknown parameter χ that appeared in $S_{3,0}(r)$ in the previous section. The assumption of independence of the amplitude of the longitudinal third-order structure function of the integral scale fixed $\chi=1$ hence $\xi_n=1$. Heuristic arguments [3,4,6,10,19] about one-dimensional Burgers equation based on the shock singularities also suggest to us that the exponents would saturate to $\xi_n=1$. Experiences with d-dimensional Burgers problem based on instanton analysis tell us that at least the structure of PDF in d dimensions is very similar to one dimensions and the angular dependencies indicate as if $F(\mu,r,s)=F(\mu s,r)$ in the limit when $s\rightarrow 1$ [11,14,19]. Other methods like replica analysis in infinite dimensions [11] also give the same picture regarding the saturation of scaling exponents. In spite of the obvious fact that the nature of singularities in three dimensions is much more complicated because $S_n(r,s)$ separates as $\sim s^n S_n(r)$ *conditioned* to Eq. (37), it leads to $F(\mu,r,s)=F(\mu s,r)$. This similarity also led us to accept that the saturation value of intermittency exponents is $\xi_n=1$. On the other hand, as soon as ξ_n is fixed because of the consistency condition we get

$$\alpha_d=d.$$

Albeit we should emphasize that all our rationales are based on the fact that dissipation is irrelevant to the equation of even-order moments, so we are not claiming that the value for α_d is valid in all regions. For obtaining the proposed form of structure functions in the limit when $s\rightarrow 1$, it is sufficient to have the scaling form $P(r,u,s) \rightarrow (1/s)P(r,u/s)$ for probability distribution of velocity increments. Imposing this form in the PDF equation it is easy to verify that when $s\rightarrow 1$ the following equation governs the nonskewed part of the PDF in three-dimensions:

$$\left[-\frac{\partial}{\partial u}u-B\right]\frac{\partial}{\partial r}P+\frac{A}{r}\frac{\partial}{\partial u}uP+3r^2\frac{\partial^3}{\partial u^3}P=0, \tag{38}$$

where $P(u,r)$ is the longitudinal velocity difference PDF, and B approaches zero as $O(1-s^2)$ and $A=\xi_n=1$. The A coefficient in Eq. (38) is responsible for the scaling of the structure functions while the B term is an infinitesimal coefficient that is zero for the longitudinal components and its value is responsible for n independence of the scaling exponents. The same form of PDF equation has been conjectured recently by Yakhot for the *N-S* turbulence [19]. The forcing contribution in the above equation is $3r^2\partial_u^3 P$ and it is irrelevant in the small scale $r\rightarrow 0$. We will take into account the forcing contribution by imposing a matching condition for PDF in the large scales with a distribution that is approximately Gaussian. Accordingly this boundary condition induces the breakdown of Galilean invariance. Equivalently, the probability density and as a result the conditional probability density of longitudinal velocity increments satisfies a Kramers-Moyal (KM) evolution equation in terms of logarithmic length scale $\lambda=\ln L/r$ [28]:

$$-\frac{\partial P}{\partial\lambda}=L_{KM}(u,\lambda)P,$$

$$L_{KM}=\sum_{n=1}^{\infty}(-1)^n\frac{\partial^n}{\partial u^n}[D^{(n)}(u,\lambda)P], \tag{39}$$

where $D^{(n)}(\lambda,u)=\sigma_n u^n$ [28]. We find that the coefficients σ_n depend on A and B through the relation

$$\sigma_n=(-1)^n\frac{A}{(B+1)(B+2)(B+3)\cdots(B+n)}.$$

Now it is easy to see that the solution of Eq. (39) can be written as a scale-ordered exponential:

$$P(u,\lambda)=\mathcal{T}\left[\exp_+\int_{\lambda_0}^{\lambda}d\lambda' L_{KM}(u,\lambda')P(u,\lambda_0)\right].$$

Using the properties of scale-ordered exponentials the conditional probability density will satisfy the Chapman-Kolmogorov equation. The same equation [i.e., Eq. (39)] obviously governs the conditional PDF too but with another boundary condition, i.e., $P(u,\lambda|u',\lambda)=\delta(u-u')$. For a simple case we proceed to find velocity difference PDF $P(u,\lambda)$, that is

$$P(u,\lambda)=\int P(u,\lambda|u',0)P(u',0)du'. \tag{40}$$

Since we know that in integral scale PDF is Gaussian with a good approximation so $P(u,0)\sim\exp(-u^2/2u_{rms}^2)$, invoking the fact that conditional probability can be written as

$$P(u,\lambda_1|u',\lambda_2)=\mathcal{T}\left[\exp_+\int_{\lambda_2}^{\lambda_1}d\lambda' L_{KM}(u,\lambda_1)\delta(u-u')\right], \tag{41}$$

and the property of the proposed KM operator, i.e.,

$$L_{KM}^{\dagger}u^m=\xi_m u^m, \tag{42}$$

where $\xi_m=1$ is the scaling exponent of the longitudinal velocity difference S_m. Substituting the scale-ordered form of the conditional PDF in Eq. (40), and expanding the assumed Gaussian form for $P(u,0)$ we get

$$P(u,\lambda)=\sum_{m=0}^{\infty}\exp(\xi_{2m}\lambda)\left(\frac{u}{u_{rms}}\right)^{2m}\frac{(-1)^m}{m!}$$
$$=\left(\frac{r}{L}\right)\exp\left(-\frac{u^2}{2u_{rms}^2}\right),$$

where we have used $\xi_m=1$. This result is consistent with the proposed form of the PDF in the inner scales where $\eta\ll r\ll L$ and consistent with numerical simulations in the one dimensional Burgers turbulence [3,4,6,10,19], where the nonuniversal part of the PDF fits with $P(\Delta u,r)=rG(\Delta u/u_{rms})$. The interesting point with respect to the possible GI-breaking mechanism is that because the variance of the velocity-increment PDF in the integral scales L is in the order of the variance of the *one-point* PDF matching between PDFs of inertial range and integral scale gives rise to appearance of one-point u_{rms} in the inertial range PDF. Once again we would stress that the results are based on the irrelevance of dissipation that is valid for even part of PDF. While dissipation terms do not matter for even part of the PDF, the odd part of PDF is obviously sensitive to dissipation effects. In case of even order moments, although the forcing terms are negligible with respect to the terms that give the main scaling, however, their accounting would cause nonuniversal behaviors of the amplitudes in the velocity-increment structure functions, i.e., they depend on forcing correlation. In our case the forcing contribution to structure functions when $r\to 0$ give some corrections as,

$$S_n(r)=A_n r^{\xi_n}+3A_{n-3}\frac{n(n-1)(n-2)}{n+B}\frac{r^{3+\xi_{n-3}}}{3+\xi_{n-3}-\xi_n}, \tag{43}$$

where $\xi_n=1$. This leads to the nonuniversality of the PDF shapes in the inertial range [10,12–19]. For odd-order structure functions dissipation contributions are $O(1)$ while forcing are $O(r^2)$ so the dissipation corrections are order of magnitudes more important. Although we cannot overcome the difficulty of dissipation terms we still think that the leading term in the scaling of structure functions for odd-order moments are not sensitive to dissipation contributions. To find an intuition about the correction of dissipation terms we turn back to longitudinal third-order structure function $S_{3,0}(r)$ since at least in this case we could estimate the dissipation effects due to stationarity. Returning to the expression of $S_{3,0}(r)$ in previous section and plugging $\xi_n=1$ it is immediately found that

$$S_{3,0}(r)=A_3 r+\frac{(d+5)k(0)}{2}\frac{r^3}{L^2}+\frac{4(d-1)}{3}k(0)r\ln r. \tag{44}$$

So the dissipation contribution gives rise to logarithmic corrections but still the dominant term is given by first scaling term when $r\to 0$. This picture is very interesting since returning to Eq. (30) we see that scaling terms r^{χ} are homogeneous solutions of the structure function equations in stationary state. Putting $m=1$ one reaches the equation of longitudinal structure function $S_{n,0}$ as

$$\left(\frac{\partial}{\partial r}-\frac{\chi}{r}\right)S_{n+1,0}=3\frac{k(0)}{L^2}n(n-1)r^2 S_{n-2,0}+D_2. \tag{45}$$

Homogeneous solutions of these equations are behaving as r^{χ} while source terms included in the right-hand side are forcing and dissipation contributions. Fixing $\chi=1$ it is clear that inhomogeneous solutions for even-order moments become negligible. This picture preserves also in the projection of the dynamics when $s\to 1$. In this sense, L_{KM} without GI-breaking terms is responsible for intermittency. The other parts of the L_{KM} operator are consisted of forcing and dissipation contributions. All the information regarding the skewness of the PDF in the inertial ranges are inherited in GI-breaking terms. This would be so, because the skewness is related to finite energy flux from large scales to small scales but the information of large-scale eddies or one-point information $k(0)=u_{rms}/L$ are just contained in GI-breaking terms.

VI. DISCUSSION

We have studied the problem of three-dimensional Burgers equation supplemented with continuity. Because of many conservation laws mixing density with different components of velocity we think that the problem is basically different from Burgers equation alone. Based on a mean field analysis an adjusting parameter emerges through the calculations that is the scaling exponent of conditional density-density correlations. We show that there are two kinds of solutions for velocity-increment PDF. In the universal scale invariant regime we find the right tail of longitudinal velocity increments. It is argued that unlike the inviscid Burgers problem the inviscid case of the present problem develops a positive solution for the PDF. When $\nu=0$ positivity of PDF is the consistency relation that fixes the scaling exponent of density correlation to $\alpha_3=\frac{7}{2}$. This is one of the interesting results in this paper that resembles the back reaction of density fluctuations to velocity fluctuations through conservation laws. We could not determine the exponent when dissipation anomaly is accounted for and that would call for more rigorous methods to be developed in future. In the nonuniversal part we relax the scaling invariant form for the PDF argument and as far as the nonskewed part of the velocity cascade is concerned we develop the solution of PDF equation for longitudinal velocity increments. The result shows that the information of large scale forcing enters the argument of longitudinal PDF in the inertial ranges by matching condition in the integral scale. We analyzed that dissipation contributions change the odd-order moments and contribute to the skewness of the PDF. However, we guess that they do not change the leading scaling contribution of the structure functions in the limit of $r\to 0$. Looking at the longitudinal third-order moment gives an indication that supports the guess. Through the calculation we conclude that scaling exponents in longitudinal-velocity-increment structure functions show the extreme independency to the order of the moment and they saturate to a constant. We conclude that the exponent is saturated to $\xi_n=1$. Fixing ξ_n, the scaling exponent of two-point density correlation function is also determined simply as $\alpha_d=d$.

Numerical analysis of the present problem would be very valuable for clarifying the ideas developed here and presenting a testing ground for the approximate picture in this paper. The issues of density PDF and intermittency in density fluctuations are worth studying too. Thanks to our recent deep understanding of passive scalar theory [30] we know that highly compressible flow advected with white in time Gaussian-correlated velocity does not lead to intermittency of density. However, in our case the intermittent structure of velocity still may cause intermittency in density and it is a challenge to understand this issue. In the case of decaying

Burgers problem added with continuity equation it is shown recently that a careful analysis of density PDF necessitates to turn our attentions to geometrical properties of the underlying singularities in the problem [22]. However, again we emphasize that the picture given in that paper cannot be applied when one starts with momentum and continuity equations. In any case, at this stage we cannot derive the PDF of density fluctuation since we just treat the density fluctuations in an effective way to obtain some information about the velocity PDF and in this approximation detailed information regarding the density PDF cannot be analyzed. In order to enlighten the way toward understanding the density PDF one useful way is simulating the conditional averages like $\langle\rho(\mathbf{x}_1)\cdots\rho(\mathbf{x}_m)|(\Delta\mathbf{u})^n\rangle$ from which the dependence of these conditional averages with respect to longitudinal velocity increments may be deciphered.

ACKNOWLEDGMENTS

We would like to thank Uriel Frisch and Victor Yakhot for useful discussions and S. Rouhani and D.D. Tskhakaya for important comments. The authors would like to thank G. Falkovich and M. Khorami for valuable remarks. M.R.R.T. wishes to acknowledge the support of the French Ministry of Education for his visit to the Observatoire Cote d'Azur and U. Frisch for his kind hospitality.

APPENDIX A: UNIVERSAL REGIME

In the present appendix we will include some of the details of calculations related to universal regime. In the first part we dwell with the form of structure function scaling and in the second part we will remind the proof of positivity of PDF in the strict inviscid case ($\nu=0$) from which the exponent α_3 is fixed.

1. Solutions of the structure function equation in universal regime

Plugging Eq. (24) in Eq. (18) we find the following equation:

$$sz\partial_z^2 F+s\partial_z F-\alpha_3 s\partial_z F-\frac{s(1-s^2)}{z}\partial_s^2 F+\frac{1+s^2}{z}\partial_s F -\alpha_3\frac{1-s^2}{z}\partial_s F+2(1-s^2)\partial_s\partial_z F -z^2(1+2s^2)F=D_2, \tag{A1}$$

where $z=\mu r$ and α_3 is the density-density exponent. It can be proved that $F(z,s)$ satisfying the above equation has the important property that $F(z,s)=F(-z,-s)$, which is in accordance with symmetry properties of the Burgers equation. When $\mu\to\infty$ one can neglect the dissipation term and propose the following solution for $F(z,s)$ as

$$F(z,s)\sim\exp[z^\gamma f(s)], \tag{A2}$$

where $\gamma=3/2$ and $f(s)$ would be simultaneously satisfied in

$$\frac{9}{4}sf^2(s)+3f(s)f'(s)(1-s^2)+f'^2(s)(-s+s^3)=(1+2s^2), \tag{A3}$$

$$-s(1-s^2)f''(s)+[(4-\alpha_3)-(2-\alpha_3)s^2]f'(s) +\left(\frac{9}{4}-\frac{3}{2}\alpha_3\right)sf(s)=0. \tag{A4}$$

Equation (A4) can be converted to hypergeometric differential equation [31] by changing the variable $s^2=w$ so that it is written as

$$w(1-w)\frac{d^2f}{dw^2}+[c-(a+b+1)w]\frac{df}{dw}-abf=0, \tag{A5}$$

where

$$a=\frac{\alpha_3}{2}-\frac{3}{4},$$

$$b=-\frac{3}{4},$$

$$c=\frac{\alpha_3-3}{2}.$$

When c is not an integer there are two independent solutions in the region $|w|<1$ of complex plane so the general solution is a linear combination of them

$$f(w)=C_1\,{}_2F_1(a,b;c;w)+C_2w^{1-c}\,{}_2F_1 \times(a-c+1,b-c+1;2-c;w), \tag{A6}$$

where ${}_2F_1(a,b;c;w)$ is the hypergeometric function [31]. We are interested in $s=1$ and since the hypergeometric function ${}_2F_1(a,b;c;w)$ has branch points in $w=1$ and $w=0$ the parameters in the arguments may be strongly limited. The coefficient C_1 becomes zero since the first function in the linear combination will diverge in $s=1$ and the only finite solution will emerge if

$$b-c+1=0\Rightarrow\alpha=\frac{7}{2}. \tag{A7}$$

In fact this condition will cause the series expansion of the second hypergeometric function to terminate trivially after the first term since

$${}_2F_1(a,b;c;x)=1+\frac{a\cdot b}{c\cdot 1}x+\frac{a(a+1)b(b+1)}{c(c+1)\cdot 1\cdot 2}x^2+\cdots. \tag{A8}$$

Since the solution of $f(s)$ would be consistent with Eq. (A1) simultaneously the undetermined coefficient C_2 is fixed. Essentially the Eq. (A2) put a constraint on $f(1)=2/\sqrt{3}$. The overall behavior of $f(s)$ is

$$f(s)=\frac{2}{\sqrt{3}}s^{3/2}. \tag{A9}$$

Substituting Eq. (A9) functional in Eq. (A2), the proposed scaling in the paper, i.e., $F(\mu r,s)=F(\mu rs)$ recovers. We would stress that *assuming* the proposed ansatz of Eq. (A2) the parameter $\alpha_3=7/2$ already is fixed in the level of positive tail of universal generating function.

2. Positivity of the left tail in universal regime

The calculations in this subsection just remind of some of the general arguments for fixing the α_3 due to positivity of the PDF. Actually in previous subsection we already fixed the value in the inviscid case but we did not argue about the positivity of the corresponding PDF since the ansatz that is used in the previous section just works in the right tail. We want to remind how the arguments of positivity may become consistent in giving the same value as derived in the previous section. Back to Eq. (26) in the paper and following [12,17] we write the Eq. (26) in terms of ψ that is defined as $P(y)=e^{-y^3/18}\psi(y)$. So we get

$$\psi''+\left[\frac{-y^4}{36}+\frac{\alpha_3-2}{3}y\right]\psi=0. \tag{A10}$$

Rewriting the following equation in terms of $\psi=\phi/y$ and then changing the independent variable to $z=y^3/9$ one finds

$$-\phi'+\left(\frac{1}{4}-\frac{k}{z}-\frac{m^2-1/4}{z^2}\right)\phi=0, \tag{A11}$$

where $k=(\alpha_3-2)/3$ and $m=\frac{1}{6}$. This is the well-known Wittaker equation [31] and we can immediately use the independent solutions in terms of hypergeometric functions as follows:

$$\phi(z)=\begin{cases} e^{-z/2}\left[C_1z^{2/3}\mathbf{M}\left(\frac{1}{2}+m-k,1+2m,z\right)+C_2z^{1/3}\mathbf{M}\left(\frac{1}{2}-m-k,1-2m,z\right)\right] & z>0\\ e^{z/2}\left[C_1'(-z)^{2/3}\mathbf{M}\left(\frac{1}{2}+m+k,1+2m,-z\right)+C_2'(-z)^{1/3}\mathbf{M}\left(\frac{1}{2}-m+k,1-2m,-z\right)\right] & z<0.\end{cases} \tag{A12}$$

The functions $\mathbf{M}(a,b,z)$ are hypergeometric functions that are defined as

$$\mathbf{M}(a,b,z)={}_1F_1(a,b,z)=\sum_n\frac{(a)_n}{(b)_n n!}z^n,$$
$$\text{where}\quad (a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}. \tag{A13}$$

Continuity in $x=0$ requires that $C_1=C_2'$ and $C_1=-C_1'$ so that two unknown coefficients remain to be determined by the asymptotic behavior of PDF.

In order to analyze the asymptotic form of the solution we remind that the series expansion in the definition of hypergeometric function $\mathbf{M}$ can be estimated by steepest descent method in a simpler form as

$$\mathbf{M}\sim\frac{\Gamma(b)}{\Gamma(a)}z^{a-b}e^z+O\left(\frac{1}{z}\right). \tag{A14}$$

Now with the above form for $\mathbf{M}$ functions we request that C_1 and C_2 be determined so that when $z\to\infty$ different diverging terms cancel out, i.e.,

$$C_2\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}-k\right)}+C_1\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}-k\right)}=0, \tag{A15}$$

$$C_2\frac{\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}+k\right)}-C_1\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+k\right)}=0.$$

For obtaining nontrivial result for the above system of equations determinant of the coefficients would be zero, that is we get

$$\frac{\Gamma(4/3)\Gamma(2/3)}{\Gamma(2/3+k)\Gamma(1/3-k)}+\frac{\Gamma(4/3)\Gamma(2/3)}{\Gamma(2/3-k)\Gamma(1/3+k)}=0. \tag{A16}$$

This condition, which is obtained by finiteness constraint, quantizes the allowed values of the parameter k as

$$k=n+\frac{1}{2},\quad n=0,1,2,\ldots\,. \tag{A17}$$

However, one can check that all of the integer values of n do not correspond to a positive solution. However, the first value, i.e., $n=0$, does gives rise to a positive solution from which one fixes the parameter α_3 to

$$k=\frac{1}{2}\Rightarrow\alpha_3=\frac{7}{2}. \tag{A18}$$

APPENDIX B: SOLUTION OF STRUCTURE FUNCTION EQUATION IN NONUNIVERSAL REGIME

In this part some straightforward manipulations regarding the scaling of $S_n(r,s)$ with respect to angle variable s in the limit of $s\to 1$ in non-universal regime is reminded. In fact in Eq. (35), neglecting the viscose and forcing contributions we impose the ansatz $S_n(r,s)=r^{\xi_n}f(s)$. From there we find an equation that similarly to the equation of universal structure functions can be converted to a hypergeometric equation after changing the independent variable to $w=s^2$. The obtained equation is again in the form of Eq. (A1) but the parameters a, b, and c are defined as follows:

$$a=-\frac{n}{2},$$

$$b=-\frac{\xi_n-\alpha_d}{2},$$

$$c=\frac{3+\alpha_d-d-\xi_n-n}{2}. \tag{B1}$$

It is standard that the general solutions in $|w|<1$ region can be written as the following linear combination:

$$f(w)=C_{1\,2}F_1\left(\frac{-n}{2},\frac{\alpha_d}{2}-\frac{\xi}{2};\frac{3}{2}+\frac{\alpha_d-\xi_n-n-d}{2};w\right)+C_2w^{[(-1/2)-(\alpha_d-\xi_n-n-d)/2]}{}_2F_1\times\left(\frac{\xi_n+d-\alpha_d-1}{2},\frac{d-\alpha_d-1}{2};\frac{3}{2}+\frac{\alpha_d-\xi_n-n-d}{2};w\right). \tag{B2}$$

We are interested in the point $w=1$ that is pathologic in the sense that it is one of the branch points of hypergeometric function. Since the first hypergeometric function is not finite and real when n gets large, one would choose $C_1=0$. The only way for getting a finite solution in $w=1$ in the second term is terminating the series of hypergeometric function just by trivially putting first or second argument to zero. So we get

$$\xi_n=\alpha_d-d+1. \tag{B3}$$

We note that terminating the series expansion spoils down all the structure of hyper-geometric function and converts it to a constant but this is the only way in which one is able to get a finite and real solution for $f(s)$ in $s=1$ [31].

---

[1] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, England, 1995).

[2] J.M. Burgers, *The Nonlinear Diffusion Equation* (Reidel, Boston, 1974).

[3] J.D. Fournier and U. Frisch, J. Mec. Theor. Appl. **2**, 699 (1983).

[4] U. Frisch, J.D. Fournier, and H. Rose, J. Phys. A **11**, 187 (1978).

[5] S.N. Gurbatov, A.N. Malakhov, and A.I. Saichev, *Non-linear Random Waves and Turbulence in Non-Dispersive Media: Waves, Rays, Particles* (Manchester University Press, Manchester, 1991).

[6] T. Gotoh and R.H. Kraichnan, Phys. Fluids A **5**, 445 (1993); J. Scalo, E. Vasquez-Semadeni, D. Chappel, and T. Passot, Astrophys. J. **504**, 835 (1998).

[7] Ya.G. Sinai, Commun. Math. Phys. **148**, 601 (1992); Z. She, E. Aurell, and U. Frisch, *ibid.* **148**, 623 (1992).

[8] S.F. Shandarin and Ya.B. Zeldovich, Rev. Mod. Phys. **185**, 46 (1989); M. Vergassola, B. Dubrulle, U. Frisch, and A. Noullez, Astron. Astrophys. **289**, 325 (1994).

[9] M. Kardar, G. Parisi, and Y.C. Zhang, Phys. Rev. Lett. **56**, 889 (1986); H.C. Fogedby, Phys. Rev. E **59**, 5065 (1999); **60**, 4950 (1999); e-print cond-mat/9809238; M. Lassig, J. Phys. C **10**, 9905 (1998); D.S. Fisher, e-print cond-mat/9711179.

[10] V. Yakhot and A. Chekhlov, Phys. Rev. Lett. **77**, 3118 (1996); A. Chekhlov and V. Yakhot, Phys. Rev. E **52**, R2739 (1995).

[11] J. Bouchaud, M. Mezard, and G. Parisi, Phys. Rev. E **52**, 3656 (1995).

[12] A. Polyakov, Phys. Rev. E **52**, 6183 (1995).

[13] E. Balkovsky, G. Falkovich, I. Kolokolov, and V. Lebedev, Pis'ma Zh. Éksp. Teor. Fiz. **61**, 980 (1995) [JETP Lett. **61**, 1012 (1995)]; Phys. Rev. Lett. **78**, 1452 (1997); G. Falkovich and V. Lebedev, e-print chao-dyn/9708002.

[14] V. Gurarie and A. Migdal, Phys. Rev. E **54**, 4908 (1996).

[15] J.P. Bouchaud and M. Mezard, Phys. Rev. E **54**, 5116 (1996); M. Mezard, e-print cond-mat/9801029.

[16] T. Gotoh and R.H. Kraichnan, Phys. Fluids **10**, 2859 (1998)..

[17] S. Boldyrev, Phys. Rev. E **55**, 6907 (1997).

[18] S. Boldyrev, Phys. Plasmas **54**, 1681 (1998); Phys. Rev. E **59**, 2971 (1999).

[19] V. Yakhot, Phys. Rev. E **57**, 1737 (1997); **60**, 5544 (1999); e-print chao-dyn/9909017; Phys. Rev. E **63**, 026307 (2001).

[20] W.E., K. Khanin, A. Mazel, and Ya.G. Sinai, Phys. Rev. Lett. **78**, 1904 (1997).

[21] W.E. and E. Vanden Eijnden, Commun. Pure Math. Appl. **53**, 852 (2000); Phys. Fluids **12**, 149 (2000); Phys. Rev. Lett. **83**, 2572 (1999); R. H. Kraichnan, Phys. Fluids **11**, 3738 (1999).

[22] J. Bec and U. Frisch, Phys. Rev. E **61**, 1395 (2000); U. Frisch, J. Bec and B. Villone, Physica D (to be published); J. Bec, U. Frisch, and K. Khanin, J. Fluid Mech. **416**, 239 (2000). chao-dyn/9910001.

[23] D. Bernard and K. Gawedzki, J. Phys A **31**, 8735 (1998); M. Bauer and D. Bernard, *ibid.* **32**, 5179 (1999).

[24] S.N. Gurbatov, S.I. Simdyankin, E. U. Frisch, and G. Toth, J. Fluid Mech. **334**, 339 (1997); L. Frachebourg and Ph.A. Martin, *ibid.* **417**, 323 (2000).

[25] F. Hayot and C. Jayaprakash, Phys. Rev. E **57**, R4867 (1998); Int. J. Mod. Phys. B **14**, 1781 (2000).

[26] A. Rastegar, M.R. Rahimi Tabar, and P. Hawaii, Phys. Lett. A **245**, 425 (1998).
[27] A. Naji, S. Rouhani, and A. Reza Rastegar (unpublished).
[28] J. Davoudi and M.R. Rahimi Tabar, Phys. Rev. Lett. **82**, 1680 (1999).
[29] K. Sreenivasan (private communication).
[30] K. Gawedzki and M. Vergassola, Physica D **138**, 63 (2000).
[31] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, San Diego, 1994).