ANNALS OF PHYSICS **246**, 446–458 (1996)
ARTICLE NO. 0033

Turbulent Two-Dimensional Magnetohydrodynamics and Conformal Field Theory

M. R. RAHIMI TABAR AND S. ROUHANI

Department of Physics, Sharif University of Technology, P.O. Box 11365-9161, Tehran, Iran; and Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran

Received June 29, 1995

We show that an infinite number of non-unitary minimal models may describe two dimensional turbulent magnetohydrodynamics (MHD), both in the presence and absence of the Alf'ven effect. We argue that the existence of a critical dynamical index results in the Alf'ven effect or equivalently the equipartition of energy. We show that there are an infinite number of conserved quantities in $2D-MHD$ turbulent systems both in the limit of vanishing the viscocities and in force free case. In the force free case, using the non-unitary minimal model $M_{2,7}$ we derive the correlation functions for the velocity stream function and magnetic flux function. Generalising this simple model we find the exponents of the energy spectrum in the inertial range for a class of conformal field theories.

1. INTRODUCTION

There has been some recent activity towards the application of Conformal Field Theory (CFT) to the theory of turbulence in two dimensions [1–7]. The main point is that the energy spectrum and higher correlation functions can be derived by means of some nonunitary minimal model of CFT. Polyakov [1, 2] derives a few criteria for a CFT which can be a possible candidate for describing turbulence, and finds a candidate CFT which gives the value of $-25/7$ for the exponent of the energy spectrum (experimental results give an exponent between 3 and 4 [11–14]). Expanding on Polyakov's method others [3, 6] argue that there are a large number of CFT's which satisfy Polyakov's constraints but have more than one primary field. The role of extra primary fields is not clear, but it has been suggested that they may have to do with passive scalar and magnetic fields [4]. Briefly Polyakov's method is as follows. To describe turbulent behaviour at high Reynold's numbers, we interpret the Navier-Stokes equation from a statistical mechanical point view. That is we consider the correlation functions of the stream function with respect to some stationary probability density:

$$\langle \Phi_{i_1}(x_1)\, \Phi_{i_2}(x_2) \cdots \Phi_{i_N}(x_N) \rangle.$$

0003-4916/96 $18.00

as correlators of primary fields of some CFT. The requirement that these correlations be stationary is the main prerequisite. This condition is referred to as the Hopf equation [9]. The fact that the correlation functions have a power law behaviour, indicates that conformal invariance may be at work. Indeed in the limit of vanishing viscosity ($\mu \to 0$) the Navier–Stokes equations are scale invariant (see [8] for more details). Also in a plasma one can use hydrodynamic equations to describe the collective degrees of freedom, however the more conventional method uses the kinetic equations to describe turbulent processes [10]. As in the case of pure hydrodynamics, the two dimensional magnetohydrodynamic system (2D-MHD) differs from its three dimensional version in that it has an infinite number of conserved quantities in the limit of zero viscocity $\mu \to 0$ and zero molecular resistivity $\eta \to 0$ and in this limit the equations are scale invariant. Therefore 2D-MHD is also a good candidate where CFT may be applicable. Such a system was first considered by Ferretti and Yang [4]. Here we shall extend their arguments and find non-unitary models which can give the correlation functions and the energy spectrum index. It is worth noting that the requirement that the critical dynamical index be consistent, is equivalent to the Alf'ven effect, this means that there is equipartion of energy between kinetic and magnetic components and this requirement greatly reduces the number of possible minimal models of CFT solutions.

This paper is organised as follows: in Section 2 we first describe the equations governing two dimensional magnetohydrodynamic systems (2D-MHD) in the inertial range. We then discuss its scaling properties. In Section 3 we go on to describe a conformal field theory with two primary fields pretaining to (2D-MHD). We then generalise to CFT's with more primary fields and give a table of possible solutions in the general case and in the Alf'ven region. In the appendix we describe a method of deriving OPE coefficients, for any Minimal Models in CFT and give the coefficients up to the third level. The results are summarized in Section 4.

2. Properties of Two-Dimensional MHD

The incompressible two dimensional magnetohydrodynamic (2D-MHD) system has two independent dynamical variables, the velocity stream function φ, related to the velocity field V_α

$$V_\alpha = e_{\alpha\beta}\, \partial_\beta \varphi \tag{2.1}$$

and the magnetic flux function ψ related to the magnetic field B_α via:

$$B_\alpha = e_{\alpha\beta}\, \partial_\beta \psi \tag{2.2}$$

here $e_{\alpha\beta}$ is antisymmetric and $e_{12} = +1$, $e_{21} = -1$.

The dynamics is given by the pair of equations [10]:

$$\frac{\partial w}{\partial t} = -e_{\alpha\beta}\,\partial_\alpha \varphi\,\partial_\beta w + e_{\alpha\beta}\,\partial_\alpha \psi\,\partial_\beta J + \mu\,\nabla^2 w \tag{2.3}$$

$$\frac{\partial \psi}{\partial t} = -e_{\alpha\beta}\,\partial_\alpha \varphi\,\partial_\beta \psi + \eta J \tag{2.4}$$

where

$$w = \nabla^2 \varphi \qquad J = \nabla^2 \psi$$

here μ is the molecular kinematic viscosity and η is the molecular resistivity. Note that normalisation is chosen such that the magnetic field assumes the same dimensions as velocity. In the inertial range η and μ can be ignored then it follows from Eqs. (2.3) and (2.4) that there exist three global, quadratic, conserved quantities.

$$E = \tfrac{1}{2} \int (V^2 + B^2)\, d^2x \tag{2.5}$$

$$H = \int V \cdot B\, d^2x \tag{2.6}$$

$$A = \int \psi^2\, d^2x \tag{2.7}$$

which are the total energy, the cross helicity and the mean square magnetic potential. In fact this system has an infinite number of conserved quantities such as:

$$R_n = \int \psi^n\, d^2x \tag{2.8}$$

where n is any real number. The time evolution of the E, H and A are given as:

$$\frac{dE}{dt} = -\eta \int (\nabla^2 \psi)^2\, d^2x - \mu \int (\nabla^2 \phi)^2\, d^2x \tag{2.9}$$

$$\frac{dH}{dt} = -(\mu + \eta) \int (\nabla^2 \psi)(\nabla^2 \varphi)\, d^2x \tag{2.10}$$

$$\frac{dA}{dt} = -\eta \int (\nabla \psi)^2\, d^2x \tag{2.11}$$

a similar expression for the R_n can be written as follows

$$\frac{dR_n}{dt} = -\eta n(n-1) \int \psi^{n-2} (\nabla\psi)^2 \, d^2x \tag{2.12}$$

But when η and μ are negligible, the system of Eqs. (2.3) and (2.4) display the scale invariance

$$x \to \lambda x \qquad \varphi \to \lambda^{1-h}\varphi \qquad t \to \lambda^{1+h} t \qquad \psi \to \lambda^{1-h}\psi \tag{2.13}$$

Now if we impose Kolmogorov's idea of constant flux of energy we get $h = \frac{1}{3}$ [15], so that the behaviour of fields under scaling are:

$$x \to \lambda x \qquad \varphi \to \lambda^{2/3}\varphi \qquad t \to \lambda^{3/4} t \qquad \psi \to \lambda^{2/3}\psi \tag{2.14}$$

Following Polyakov [24] we believe that this scale invariance signals conformal symmetry of this system. Simple scaling arguments [16–21], show that in turbulent 2D-MHD the energy spectrum behaves as:

$$E(k) \sim k^{-3/2} \tag{2.15}$$

Unfortunately this result is in poor agreement with recent numerical simulations [29]. But our results show the deviation from this spectrum and is in agreement with simulation [29]. Another important aspect to consider is the Alf'ven effect [22–23]. This effect essentially amounts to equipartition of energy between the kinetic and the magnetic components of the energy. Thus V_k^2 and B_k^2 should have the same spectrum in the inertial range. We shall later show how this effect bears on the critical dynamical index, and limits our choice of CFT. If Polyakov's ideas were applicable here, a conformal field theory may exist such that its correlation functions coincide with those of the 2D-MHD system. However such a system has to be non-unitary in order to give the power law behaviour suggested by the scaling relations (i.e. Eq.(2.14)). First of all, let us show that to describe turbulence by means of CFT, we have to use non-unitary minimal models. Any local conformal field $A_j(x)$ have associated with it an anomalous dimension d_j, i.e. under a transformation $x \to \lambda x$ we have [24, 25]:

$$A_j(x) \to \lambda^{-d_j} A_j(x) \tag{2.16}$$

We therefore observe that if associated with the fields of 2D-MHD system i.e. the φ and the ψ, they must have negative anomalous dimensions. On the other hand negative anomalous dimensions are possible only in non-unitary minimal models which in turn non-unitary minimal models result in infrared divergences. We shall deal with this problem later.

3. Conformal 2D-MHD Turbulence

The simplest model we consider is $M_{2,7}$ with three primary fields I, the identity φ, with anomalous dimension $(-2/7, -2/7)$, and ψ, with anomalous dimension $(-3/7, -3/7)$. The central change is $C=-68/7$. Now we can derive the small scale behaviour of the two points functions.

$$\langle \varphi_{(x)} \varphi_{(0)} \rangle \sim |x|^{8/7} \tag{3.1}$$

$$\langle \psi_{(x)} \psi_{(0)} \rangle \sim |x|^{12/7} \tag{3.2}$$

and the correlation function of φ and ψ vanishes since they have different anomalous dimensions. However the above expressions are clearly unphysical since they grow with distance. To avoid this problem one has to introduce an infrared cutoff [2]. Let us look at this problem in the momentum space

$$\langle \varphi_i(k)\, \varphi_i(-k) \rangle \sim C_i\, |k|^{-2-4\,|\varDelta\varphi_i|} \tag{3.3}$$

where φ_i can be either φ or ψ and

$$C_i = 2^{4\,|\varDelta\varphi_i|+1}\, \frac{\Gamma(4\,|\varDelta\varphi_i|)}{\Gamma(4\,|\varDelta\varphi_i|+2)} \tag{3.4}$$

To avoid the problem of infrared divergence we can restrict ourselves to the inertial range i.e. $1/a \gg K \gg 1/R$, where R is the large scale boundary of the system and a is the dissipation range. Thus

$$\langle \varphi(x)\, \varphi(0) \rangle \simeq C_i \int_{k>(1/R)}^{\infty} k^{-2-4\,|\varDelta\varphi|}\, e^{ik\cdot x}\, dk \tag{3.5}$$

which results in

$$\langle \varphi(x)\, \varphi(0) \rangle \sim R^{4\,|\varDelta\varphi_i|} - x^{4\,|\varDelta\varphi_i|} + \sum_{m=1}^{\infty} (\alpha_m) \left(\frac{x}{2R}\right)^m x^{4\,|\varDelta\varphi_i|} \tag{3.6}$$

where

$$\alpha_m = \frac{1}{m!}\, \frac{\Gamma(4\,|\varDelta\varphi_i|+1)}{\Gamma(2\,|\varDelta\varphi_i|+1+m/2)}\, \frac{\sin(\frac{1}{2}(1-4\,|\varDelta\varphi_i|-m)\,\pi)}{\Gamma(\frac{1}{2}(4\,|\varDelta\varphi_i|+m+2))} \tag{3.7}$$

Here the IR problem just as in the case of pure turbulence [2, 3, 26], and it may be removed by considering of turbulent 2D-MHD with boundary.

On the other hand, there exists a dissipation scale 'a', so that the inertial range was defined by $k \ll 1/a$. Thus fully developed turbulence is equivalent to letting 'a' tend to zero, or equivalently letting the Reynold's number tend to infinity. However the existence of an ultraviolet cutoff means that we have to be careful when products of operators at the same point are involved. To this end we handle such products using the point splitting technique. Consider the right hand side of equation (2.3, 2.4)

$$e_{\alpha\beta}\,\partial_\alpha\psi\;\partial_\beta\,\nabla^2\psi=\overline{limit}_{a\to 0}\;e_{\alpha\beta}\,\partial_\alpha\psi\left(x+\frac{a}{2}\right)\partial_\beta\,\nabla^2\psi\left(x-\frac{a}{2}\right) \tag{3.8}$$

$$e_{\alpha\beta}\,\partial_\alpha\varphi\;\partial_\beta\psi=\overline{limit}_{a\to 0}\;e_{\alpha\beta}\,\partial_\alpha\varphi\left(x+\frac{a}{2}\right)\partial_\beta\psi\left(x-\frac{a}{2}\right) \tag{3.9}$$

Where $\overline{limit}_{a\to 0}$, expresses angle averaging. Now to evaluate the above, we take advantage of the operator product expansion. We have the general form for the fusion rule of (p, q) minimal model [27, 28]:

$$[\psi_{n_1m_1}][\psi_{n_2,m_2}]=\sum_{k=|n_1-n_2|+1}^{min(n_1+n_2-1,\,2p-n_1-n_2-1)}\;\sum_{l=|m_1-m_2|+1}^{min(m_1+m_2-1,\,2q-m_1-m_2-1)}[\psi_{(k,l)}] \tag{3.10}$$

Where variables k, l run over odd or even numbers if they are bounded by odd or even numbers respectively. For our particular choice of $(q, p)=(2, 7)$ we have two primary fields φ and ψ and their families $[\varphi]$ and $[\psi]$ satisfy:

$$[\varphi]\times[\varphi]=[I]+[\psi] \tag{3.11}$$

$$[\varphi]\times[\psi]=[\psi]+[\varphi] \tag{3.12}$$

$$[\psi]\times[\psi]=[I]+[\psi]+[\varphi] \tag{3.13}$$

Note that by the family of φ, we mean all operators which can be constructed from φ using the Virasoro generators L_n, such as:

$$L_{-n_1}L_{-n_2}\cdots L_{-n_k}\varphi \tag{3.14}$$

In order to explicitly calculate the rhs. of Eqs. (3.8) and (3.9) by means of the fusion rules, we use the following relations for the OPE of field operators:

$$\begin{aligned}\varphi_n\left(x+\frac{a}{2}\right)\varphi_m\left(x-\frac{a}{2}\right)\sim|a|^{2(\Delta_p-\Delta_n-\Delta_m)}(\varphi_p+\alpha_1 aL_{-1}\varphi_p+a^2(\alpha_2L_2+\alpha_3L_{-1}^2)\,\varphi_p\\ +a^3(\alpha_4L_{-1}^3+\alpha_5L_{-1}L_{-2}+\alpha_6L_{-3})\,\varphi_p+\cdots\end{aligned} \tag{3.15}$$

where $n, m, p=1, 2$ and $\varphi_1=\varphi$, $\varphi_2=\psi$. In the appendix we have given a method of deriving the coefficients $\alpha_1, \alpha_2, \alpha_3$ etc. and have derived these coefficients up to the third level. By differentiation of lhs. of Eq. (3.15) we will find the leading term in product of φ and ψ in the limit $a\to 0$ as follows:

$$e_{\alpha\beta}\,\partial_\alpha\varphi\;\partial_\beta\psi=C_1\,|a|^{11/7}\,(\alpha\bar{L}_{-1}(L_{-2}+sL_{-1}^2)-C\cdot C)\,\psi \tag{3.16}$$

where s is a constant, determined by the operator product expansion and α is a constant. It is clear that the antisymmetry of rhs. of Eq. (3.16) under complex conjugation is just the consequence of the e-tensor at the lhs. Similar calculation shows that the other terms in Eqs. (2.3) and (2.4) by means of OPE and point splitting procedure can be written in terms of Virasoro generators in the *limit* $a \to 0$ as follows:

$$e_{\alpha\beta}\, \partial_\alpha \psi\, \partial_\beta\, \partial^2 \psi = C_2\, |a|^{6/7}\, (L_{-2}\bar{L}^2_{-1} - \bar{L}_{-2} L^2_{-1})\, \psi \tag{3.17}$$

$$e_{\alpha\beta}\, \partial_\alpha \varphi\, \partial_\beta\, \partial^2 \varphi = C_4\, |a|^{1/7}\, (L_{-2}\bar{L}^2_{-1} - \bar{L}_{-2} L^2_{-1})\, \psi \tag{3.18}$$

Before proceeding further let us look at the various constants of motion R_n, A_2, H and E:

$$\frac{dR_n}{dt} = -\eta(n-1) \int \psi^{n-1}\, \nabla^2 \psi\, d^2 X \tag{3.19}$$

For R_n, A_2, H and E to be constants of motion, we must have $\mu, \eta = 0$ ($\eta = 1/\mu_0 \sigma$ where μ_0 and σ are permeability and conductivity, respectively) which require the conductivity σ to be infinite. On the other hand it is obvious that the above quantities are conserved in the limit of μ and $\nabla^2 \psi \to 0$, that is for finite conductivity we have the same conserved quantities. But, letting $\nabla^2 \psi$ tend to zero is equivalent to vanishing of J. This situation is well known as force free MHD [10]. Let us return to conformal field theory and look at $\nabla^2 \psi \to 0$, it is evident that;

$$\nabla^2 \psi = 4 L_{-1} \bar{L}_{-1} \psi = 0 \tag{3.20}$$

With this condition we can proceed further with the rhs. of Eqs. (3.16), (3.17), and (3.18) and we find that they are all zero. In non-unitary minimal model $M_{2,7}$ with $C = -68/7$, ψ is degenerate in the second level that is:

$$(L_{-2} - \tfrac{21}{2} L^2_{-1})\, \psi = 0, \qquad \Delta_\psi = -3/7 \tag{3.21}$$

Thus the rhs.s of Eqs. (3.16), (3.17), and (3.18) vanish. The more general N-point correlation function satisfies the well known BPZ equation:

$$\left(\frac{-21}{2} \frac{\partial^2}{\partial z^2} - \sum_{j=1}^{n-1} \frac{-3/7}{(z - z_j)^2} + \frac{1}{(z - z_j)} \frac{\partial}{\partial z_j} \right) \langle \psi(z_1) \cdots \psi(z_{n-1}) \rangle = 0 \tag{3.22}$$

Now for a generalisation of our simple model let us postulate fusion rules as below:

$$[\varphi] \times [\psi] = [X_1] + \cdots \tag{3.23}$$

where X_1, is the primary field of the lowest dimension in the OPE of φ and ψ. Equation (3.16) changes to

$$\frac{\partial \psi}{\partial t} \sim \lim_{a \to 0} |a|^{2(\Delta X - \Delta\varphi - \Delta\psi)} (\alpha \bar{L}_{-1} (\beta L^2_{-1} + L_{-2}) - C.C.)\, X_1. \tag{3.24}$$

Let us now use the conserved quantities to find the constraints on dimensions of the fields. Similar to the arguments used in [4], we consider the cascade of mean square magnetic potential:

$$A = \int \psi^2 \, d^2x \tag{3.25}$$

this requires that $\langle \dot{\psi}_{(x)} \psi_{(x)} \rangle$ be scale invariant.

Thus, using Eqs. (3.24), we have

$$\Delta X_1 + 2 + \Delta\psi = 0 \tag{3.26}$$

$$\Delta X_1 \geqslant \Delta\varphi + \Delta\psi \tag{3.27}$$

where the second condition comes from requiring non-singularity of the rhs. of (3.24) in the limit $a \to 0$. Table I gives a list of models which satisfy these conditions.

Let us now consider the implications of the Alf'ven effect [22, 23]. It is well known that [30] the excitations of the Elsasser's fields propagate as the Alf'ven waves in opposite directions along the lines of force of the 'B' field at speeds of order 'B'. What is meant by the 'inertial range' in this context is, the region where the wavenumber of the Alf'ven waves lie whitin the inertial range and the energy

TABLE I

Solution Satisfying the Constant Mean Square Magnetic Potential Flux Condition

(p, q)	Φ	Ψ	X	$4\Delta_\varphi + 1$	$4\Delta_\psi + 1$
(2, 13)	$\Psi_{1,2}$	$\Psi_{1,4}$	$\Psi_{1,5}$	-2.69	-0.53
(2, 17)	$\Psi_{1,6}$	$\Psi_{1,2}$	$\Psi_{1,7}$	-0.64	-4.88
(2, 19)	$\Psi_{1,5}$	$\Psi_{1,2}$	$\Psi_{1,6}$	-0.68	-4.47
(2, 23)	$\Psi_{1,2}$	$\Psi_{1,3}$	$\Psi_{1,4}$	-2.30	-0.73
(2, 27)	$\Psi_{1,4}$	$\Psi_{1,2}$	$\Psi_{1,5}$	-0.77	-3.88
(3, 29)	$\Psi_{2,12}$	$\Psi_{1,3}$	$\Psi_{2,14}$	-2.17	-1.20
(3, 34)	$\Psi_{1,2}$	$\Psi_{1,3}$	$\Psi_{1,4}$	-2.29	-0.73
(3, 37)	$\Psi_{1,5}$	$\Psi_{2,14}$	$\Psi_{2,18}$	-0.18	-5.05
(3, 40)	$\Psi_{1,4}$	$\Psi_{1,2}$	$\Psi_{1,5}$	-0.77	-3.87
(3, 46)	$\Psi_{2,17}$	$\Psi_{1,3}$	$\Psi_{2,19}$	-2.47	-0.21
(3, 50)	$\Psi_{1,3}$	$\Psi_{2,19}$	$\Psi_{2,21}$	-1.4	-2.52
(3, 52)	$\Psi_{2,21}$	$\Psi_{1,2}$	$\Psi_{2,22}$	-0.82	-3.61
(3, 62)	$\Psi_{1,5}$	$\Psi_{2,22}$	$\Psi_{2,26}$	-0.37	-5.83
(3, 77)	$\Psi_{2,27}$	$\Psi_{1,3}$	$\Psi_{2,29}$	-2.68	-0.36
(3, 79)	$\Psi_{1,2}$	$\Psi_{2,29}$	$\Psi_{2,30}$	-2.10	-0.88
(3, 80)	$\Psi_{2,28}$	$\Psi_{1,3}$	$\Psi_{2,30}$	-2.7	-0.36
(3, 82)	$\Psi_{1,2}$	$\Psi_{2,30}$	$\Psi_{2,31}$	-2.10	-0.89
(3, 89)	$\Psi_{2,33}$	$\Psi_{1,2}$	$\Psi_{2,34}$	-0.89	-3.35
(3, 92)	$\Psi_{2,34}$	$\Psi_{1,2}$	$\Psi_{2,35}$	-0.90	-3.33

TABLE II

Solutions that Satisfy Both the Constant Mean Square Magnetic Potential and the Alf'ven Constraint

(p, q)	Φ	Ψ	X	$4\Delta_{\varphi}+1$	$4\Delta_{\psi}+1$	Δt
(6, 35)	$\Psi_{2,11}$	$\Psi_{2,12}$	$\Psi_{3,22}$	-2.98	-2.92	2.99
(6, 41)	$\Psi_{3,17}$	$\Psi_{2,10}$	$\Psi_{4,26}$	-2.01	-2.18	2.75

cascade results from the scattering of Elsasser's fields [30]. This implies that, there should be asymptotically exact equipartition of energy in the inertial range, i.e. $V_k^2 = \alpha B_k^2$ where α is of order unity. As discussed in detail by Chandrasekhar [31], for Kolmogorov's hypotheses of similarity to holds here the constant α shuld take value 1.62647. Therefore the equipartition of energy requires α and ψ to have similar scaling behavior:

$$\Delta\varphi = \Delta\psi \tag{3.28}$$

We can derive the dynamical index of Eqs. (2.3) and (2.4) as:

$$\Delta t = -\Delta\varphi + 2 \tag{3.29}$$

and

$$\Delta t = \Delta\varphi - 2\Delta\psi + 2 \tag{3.30}$$

Where the Eqs. (3.29) and (3.30) come from Eqs. (2.4) and (2.3) respectively. Thus again the existence of a single index for temporal scaling require $\Delta\varphi = \Delta\psi$. We can therefore extract models out of the table, which are nearly consistent with the Alf'ven effect. These are given in Table II. By means of the anomalous dimensions of ψ and φ, we can write the energy spectrum as follows:

$$E(k) \sim k^{4\,|\Delta\varphi|\,+\,1} + k^{4\,|\Delta\psi|\,+\,1}$$

Where the exponents of energy spectrum for are given in Tables 1 and 2.

4. Concluding Remarks

In this paper we have derived a number of CFT which are possible candidates for describing 2D-MHD. In the limit of finite conductivity requires 2D-MHD to be force free and increases the number of candidate CFT's. The imposition of the Alf'ven effect is equivalent to requiring a consistent dynamical index and greatly reduces the number of candidates. A direct interpretation of primary fields is as yet, not possible, therefore those candidate CFT's which have more than two primary fields seem as plausible as $M_{2,7}$, which has two primary fields. Adding boundaries is the next step in this theory, which may take it closer to some physical problems such as the quantum wire. Work along this line is already proceeding.

APPENDIX: Calculation of OPE Coefficients

The most general expression for the operator product expansion is [27]:

$$\Phi_n(z, \bar{z})\, \Phi_m(o, o) = \sum_p \sum_k C_{nm}^{p, \{k\}, \{\bar{k}\}}\, z^{\Delta_p - \Delta_n - \Delta_m + \sum k_i} \bar{z}^{\bar{\Delta}_p - \bar{\Delta}_n - \bar{\Delta}_m + \sum \bar{k}_i} \Phi_p^{\{k\}, \{\bar{k}\}}(0, 0) \tag{A.1}$$

where the coefficients are

$$C_{nm}^{p, \{k\}, \{\bar{k}\}} = C_{nm}^{p} \beta_{nm}^{p, \{k\}} \bar{\beta}_{nm}^{p, \{\bar{k}\}} \tag{A.2}$$

$$\{k\} = \{k_1, k_2, ..., k_n\} \tag{A.3}$$

Note that we have $\varphi_i(0)\, |0\rangle = |i\rangle$, for the vacuum state $|0\rangle$. Now let equation (A.1) act on $|0\rangle$:

$$\Phi_n(z)\, |\Delta_m\rangle = \sum C_{nm}^{p} z^{\Delta_p - \Delta_n - \Delta_m} \psi_p(z)\, |\Delta_p\rangle \tag{A.4}$$

$$\psi_p(z) = \sum z^{\sum k_i} \beta_{nm}^{p, \{K\}} L_{-k_1} \cdots L_{-k_n} \tag{A.5}$$

$$|z, \Delta_p\rangle = \psi_p\, |\Delta_p\rangle \tag{A.6}$$

Expand $|z, \Delta_p\rangle$ in terms of the complete basis $|N, \Delta_p\rangle$ where $|N, \Delta_p\rangle$ is defined such that the coefficients of expansion are z^N:

$$|z, \Delta_p\rangle = \sum z^N\, |N, \Delta_p\rangle \tag{A.7}$$

By applying of L_j over Eq. (A.4) we have:

$$L_j\, |N + j, \Delta_p\rangle = (\Delta_p - \Delta_m + j\, \Delta_n + N)\, |N, \Delta_p\rangle \tag{A.8}$$

and by solving the recursion relations we can find $\beta_{nm}^{p, \{k\}}$. For level one we have:

$$L_1\, |1, \Delta_p\rangle = (\Delta_p - \Delta_m + \Delta_n)\, |\Delta_p\rangle \tag{A.9}$$

which results is

$$|1, \Delta_p\rangle = \alpha_1 L_{-1}\, |\Delta_p\rangle \tag{A.10}$$

or

$$\alpha_1 = \frac{\Delta_p - \Delta_m + \Delta_n}{2\Delta_p}. \tag{A.11}$$

For the second level by means of Eqs. (A.7) and (A.10) we have:

$$L_1 \, |2, \Delta_p\rangle = \frac{(\Delta_p - \Delta_m + \Delta_n + 1)(\Delta_p - \Delta_m + \Delta_n)}{2\Delta_p} L_{-1} \, |\Delta_p\rangle \tag{A.12}$$

$$L_2 \, |2, \Delta_p\rangle = (\Delta_p - \Delta_m + 2\Delta_n) \, |\Delta_p\rangle \tag{A.13}$$

which results in:

$$|2, \Delta_p\rangle = (\alpha_2 L_{-2} + \alpha_3 L_{-1}^2) \, |\Delta_p\rangle \tag{A.14}$$

where α_2 and α_3 satisfy a system of equations:

$$M_{ij} \alpha_j = A_i \qquad i, j = 2, 3$$

where

$$\begin{gathered} M = \begin{bmatrix} 4\Delta_p + c/2 & 6\Delta_p \\ O & 4\Delta_p + 2 \end{bmatrix} \\ \begin{bmatrix} A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \Delta_p - \Delta_m + 2\Delta_n \\ ((\Delta_p - \Delta_m + \Delta_n + 1)(\Delta_p - \Delta_m + \Delta_n))/2\Delta_p \end{bmatrix}. \end{gathered} \tag{A.15}$$

This system can now be solved to give:

$$\begin{bmatrix} \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 6\Delta_p A_3 - A_2(4\Delta_p + 2) \\ 3A_1 - A_2(4\Delta_p + C/2) \end{bmatrix} \frac{1}{2\Delta_p(5 - 8\Delta_p) - (2\Delta_p + 1)_C}. \tag{A.16}$$

A similar method will work for higher levels. For level N, we have in place of Eq. (A.14), an expansion corresponding to the partition of N. We then find a system of equations by successively applying L_j, and finally the coefficients are derived. For the third level, using Eq. (A.8), we have:

$$L_1 \, |3, \Delta_p\rangle = [\Delta_p - \Delta_m + 2 + \Delta_n] \, |2, \Delta_p\rangle \tag{A.17}$$

$$L_2 \, |3, \Delta_p\rangle = [\Delta_p - \Delta_m + 1 + 2\Delta_n] \, |1, \Delta_p\rangle \tag{A.18}$$

$$L_3 \, |3, \Delta_p\rangle = [\Delta_p - \Delta_m + 3\Delta_n] \, |\Delta_p\rangle \tag{A.19}$$

and $|3, \Delta_p\rangle$ is given by:

$$|3, \Delta_p\rangle = (\alpha_4 L_{-1}^3 + \alpha_5 L_{-1} L_{-2} + \alpha_6 L_{-3}) \, |\Delta_p\rangle \tag{A.20}$$

where α_4, α_5 and α_6 satisfy the following system of equations:

$$\alpha_4(24\Delta_p+6)+\alpha_5\left\{\left(4\Delta_p+\frac{C}{2}\right)+9\right\}+5\alpha_6=\frac{\acute{B}(\Delta_p+\Delta_n-\Delta_m)}{2\Delta_p} \tag{A.21}$$

$$\alpha_4(24\Delta_p)+\alpha_5\left(4\left(4\Delta_p+\frac{c}{2}\right)\right)+\alpha_6(6\Delta_p+2c)=\acute{C} \tag{A.22}$$

$$\alpha_4(8(\Delta_p+1))+\alpha_5(7+2\Delta_p)+4\alpha_6=\acute{A}(\alpha_2+\alpha_3) \tag{A.23}$$

and α_2,α_3 are given by (A.15). The coefficients $\acute{A}$, $\acute{B}$ and $\acute{C}$ are given by:

$$\acute{A}=[\Delta_p-\Delta_m+2+\Delta_m] \tag{A.24}$$

$$\acute{B}=[\Delta_p-\Delta_m+1+2\Delta_n] \tag{A.25}$$

$$\acute{C}=[\Delta_p-\Delta_m+3\Delta_n] \tag{A.26}$$

the inverse of the matrix of coefficients is

$$M_{ij}^{-1}=\frac{1}{\Delta}B_{ij} \tag{A.27}$$

where

$$\begin{aligned}
B_{11}&=C(2\Delta_p+3)+\Delta_p(6\Delta_p-11)\\
B_{12}&=C+3\Delta_p+1/2\\
B_{13}&=1/2[C^2+C(11\Delta_p+8)+2\Delta_p(12\Delta_p-13)]\\
B_{21}&=4[C(\Delta_p+1)+3\Delta_p(\Delta_p-1)]\\
B_{22}&=2(7\Delta_p-2)\\
B_{23}&=3[C(4\Delta_p+1)+\Delta_p(12\Delta_p-7)]\\
B_{31}&=2[2C(\Delta_p+1)+5\Delta_p(2\Delta_p-1)]\\
B_{32}&=2C(\Delta_p+1)-8\Delta_p^2-38\Delta_p+15\\
B_{33}&=3[C(3\Delta_p+1)+2\Delta_p(12\Delta_p-5)]
\end{aligned}$$

and

$$\Delta=2(2c^2(\Delta_p+1)+c(-2\Delta_p^2-16\Delta_p+7)-\Delta_p(24\Delta_p^2-74\Delta_p+25)) \tag{A.28}$$

Acknowledgments

We are indebted to many of our colleagues for interesting and helpful discussions, in particular M. Alimohmmadi, F. Ardalan, H. Arfaei, S. Azakov, and A. Yu. Morozov.

References

1. A. M. Polyakov, Preprint PUPT-1341, hep-th/9209046, Princeton University, (1992).
2. A. M. Polyakov, *Nucl. Phys. B* **396** (1993), 397.
3. D. A. Lowe, *Mod. Phys. Lett. A* **8** (1993), 923.
4. G. Ferretti and Z. Yang, *Europhys. Lett.* **22**, No. 9 (1993), 639.
5. G. Falkovich and A. Hanany, WIS-92/88/NOV-PH, hep-th/9212015, 1993.
6. Y. Matsuo, *Mod. Phys. Lett. A* **8** (1993), 619.
7. B. K. Chung, S. Nam, Q. H. Park, and H. J. Shin, *Phys. Lett. B* **309** (1993), 58; Chia-Chu Chen, *Mod. Phys. Lett. A* **9** (1994), 123.
8. V. S. L'Vov, *Phys. Rep.* **207** (1991), 2.
9. A. S. Monin and A. M. Yaglom, "Statistical Fluid Mechanics," MIT Press, Cambridge, MA, 1075.
10. V. N. Tsytovich, "Theory of Turbulent Plasma," Consultants Bureau, New York, 1977.
11. R. Kraichnan, *Phys. Fluids* **10** (1967), 1417.
12. W. J. Mattheaus and S. L. Lamkin, *Phys. Fluid* **29** (1986), 2513.
13. R. H. Kraichnan and D. Montgomery *Rep. Prof. Phys.* **43** (1980), 547.
14. B. Legras, P. Santangelo, and R. Benzi, *Europhys. Lett.* **5**, No. 1 (1988), 37.
15. R. Benzi, *J. Phys. A* **17** (1984), 3521.
16. R. Kraichnan, *Phys. Rev. A* **25**, No. 6 (1982), 3281.
17. J. L. Mc Cauly, *Phys. Rep.* **189**, No. 5 (1990).
18. W. Z. Liang and P. H. Diamond, *Phys. Fluid B* **5**, No. 1 (1993), 63.
19. D. Fyfe, D. Montgomery, and G. Joyce, *J. Plasma Phys. B* **17** (1977), 67.
20. A. Pouquet, *J. Fluid Mech.* **88**, No. 1 (1978), 1.
21. D. Biskamp and H. Welter, *Phys. Fluid B* **1**, No. 10 (1989).
22. P. S. Iroshnikov, *Astron. Zh.* **40** (1963), 742; *Sov. Astron.* **7** (1964), 569.
23. R. H. Kraichnan, *Phys. Fluids* **8** (1965), 1385.
24. A. M. Polyakov, *JETP Lett.* **12** (1970), 381.
25. J. L. Cardy, *Phys. Lett.* **54** (1985), 1354.
26. B. K. Chung, S. Nam, Q. Han Park, and H. J. Shin, *Phys. Lett. B* **309** (1993), 58.
27. A. Belavin, A. Polyakov, and A. Zamolodchikov, *Nucl. Phys. B* **241** (1984), 33.
28. P. Goddard and D. Olive (Eds.), "Kac–Moody and Virasoro Algebra," World Scientific, Singapore, 1988.
29. D. Biskamp, H. Welter, and M. Walter, *Phys. Fluid B* **2**, No. 12 (1990).
30. S. I. Pai, "Magnetogasdynamics and Plasma Dynamics," Springer-Verlag, New York/Berlin, 1963.
31. S. Chandrasekhar, *Ann. Phys. (N.Y.)* **2** (1957), 615.